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PSEUDO ALMOST PERIODIC MILD SOLUTIONS OF QUASILINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH APPLICATION TO MATHEMATICAL BIOLOGY

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ABSTRACT. Using the theory of semigroup of bounded linear operators in a complex Banach space, we establish the existence and uniqueness of a pseudo almost periodic mild solution of a quasilinear functional differential equation. The main result is applied to a partial integro-differential equation modeling nonlocal reaction-diffusion problem in biology. Then we develop a numerical method based on the nonstandard finite difference discretization. We use a non-local approximation to approximate a nonlinear term to preserve the positivity of the solution. We then provide a detailed stability analysis and establish the convergence of this numerical method. We prove that this method is unconditionally stable. Finally, we present several numerical results which show the existence of positive asymptotically stable solutions of the model.

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1. Introduction

Pseudo-almost periodicity was introduced by Zhang, 1994; Zhang, 1995. The problems having almost periodic and pseudo-almost periodic solutions often require serious attention from the qualitative theory of differential equations because the outcomes are applicable in biology, economics and other domains of sciences and engineering. Some works related to pseudo-almost periodic solutions to abstract differential and partial differential equations can be found in Cuevas & Pinto, 2001; Dads & Arino, 1996; Dads et al., 1997; Diagana, 2005; and Li, 2001.

In this paper, we consider the following quasilinear functional differential equation in a complex Banach space X:

$$\frac{du(t)}{dt} + A(t, u(t))u(t) = f(t, u(t), u_t), \quad t \in \mathbb{R}, \quad u \in X, \tag{1}$$

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where $u_t(\theta) = u(t+\theta)$, $\theta \in \mathbb{R}$, $f : \mathbb{R} \times X \times X \to X$ is an L^1 function. We assume that for $(t,u) \in \mathbb{R} \times B$, $B \subset X$, $A(t,u(t))$ is a linear operator in X. We study theoretically and numerically, the existence and uniqueness of a pseudo-almost periodic mild solution of the above equation.

After the introduction of almost periodicity, there have been many generalization to this terminology. One immediate generalization is the concept of pseudo almost periodicity given by Zhang, 1994. It has several applications, for instance, in theory of partial differential equations, integral equations and functional differential equations. The existence and uniqueness of a pseudo-almost periodic solution to a differential equation has been of great interest to many mathematicians in the past few decades, see for instance, the works Amir & Maniar, 1999; Cuevas & Pinto, 2001; Diagana et al., 2006; Li et al., 2001; Zhang, 1994 and references therein.

The existence of a pseudo-almost periodic solution of an abstract differential equation has been considered by many authors when $A(t,u(t)) = A$, $A(t,u(t)) = A(t)$ and without retarded argument, see for instance, the papers Amir & Maniar, 1999; Diagana, 2007 and Li et al., 2001. In Ding et al., 2007; authors showed the existence and uniqueness of a pseudo almost periodic mild solution of the following differential equation in X:

$$\frac{du(t)}{dt} + A(t)u(t) = g(t, u(t-h)), \quad t \in \mathbb{R}, \tag{2}$$

for fixed $h \geq 0$. In Amir & Maniar, 1999 and Cuevas & Pinto, 2001, the authors have shown the existence and uniqueness of a pseudo almost periodic solution of a semilinear differential equation when A is a Hille-Yosida operator.

Our aim is to show the existence and uniqueness of a mild solution of (1). We assume that for $(t,w) \in \mathbb{R} \times B$, $B \subset X$, $A(t,w)$ is the stable infinitesimal generator of a C_0-semigroup $\{S_{t,w}(s), s, t \geq 0\}$ (cf. Pazy, 2005 for more details), in the sense that there exists constants $M \geq 1$ and w known as stability constants, such that

$$\rho(A(t,w)) \supset (w, \infty), \quad (t,w) \in \mathbb{R} \times B.$$

The rest of the paper is organized as follows. In Section 2, we given some preliminarily results concerning evolution semigroups in a Banach space. The existence of mild solutions to functional as well as integro-differential equations in Banach spaces is established in Section 3. An application problem is discussed in Section 4. In Section 5, a numerical method, based on a nonstandard finite difference discretization, is developed and analyzed for solving a delay partial differential equation. Numerical simulations are presented in Section 6. Finally, some concluding remarks are presented in Section 7.

2. Preliminaries

Let us denote $BC(\mathbb{R}, X)$ the space of all bounded continuous functions from $\mathbb{R}$ to X. It is a Banach space with the norm

$$\|u\|_\infty = \sup_{t\in\mathbb{R}} \|u(t)\|.$$

Also let $B(X, Y)$ be the set of all bounded linear operators from X to Y. It is easy to see that, $B(X, Y)$ is a Banach space with the norm

$$\|A\|_{B(X,Y)} = \sup_{x\in X, x\neq 0} \frac{\|Ax\|_Y}{\|x\|_X}.$$

Now we list some basic results which are helpful to understand the present work.

Definition 2.1. A bounded continuous function $f : \mathbb{R} \to X$ is said to be almost periodic if for each $\epsilon > 0$ there exists $l_\epsilon > 0$ such that every interval of length l_ϵ contains a number τ with the property that

$$\|f(t + \tau) - f(t)\| < \epsilon, \quad for\ all \quad t \in \mathbb{R}.$$

Definition 2.2. A continuous mapping $f : \mathbb{R} \times X \times X \to X$ is said to be almost periodic in t uniformly for $(x, \chi) \in X \times X$ if for each $\epsilon > 0$ and for each compact subset $E\times\mathcal{S}$ of $X \times X$ there exists $l_\epsilon > 0$ such that every interval of length l_ϵ contains a number τ with the property that

$$\|f(t + \tau, x, \chi) - f(t, x, \chi)\| < \epsilon, \quad for\ all \quad t \in \mathbb{R}, \quad (x, \chi) \in E \times \mathcal{S}.$$

We denote by $AP(\mathbb{R} \times X \times X, X)$ the set of all such functions.

We denote by

$$AP_0(X) = \big\{f \in BC(\mathbb{R}, X) : \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|f(\xi)\| d\xi = 0\big\},$$

and by $AP_0(\mathbb{R} \times X \times X, X)$ the set of all continuous functions $f : \mathbb{R} \times X \times X \to X$ such that $f(., u, \chi) \in AP_0(X)$ and

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|f(\xi, u, \chi)\| d\xi = 0,$$

uniformly in $(u, \chi) \in X \times X$.

Definition 2.3. A mapping $f \in BC(\mathbb{R}, X)$ is called pseudo almost periodic if it can be written as $f = f_1 + f_2$, where $f_1 \in AP(X)$ and $f_2 \in AP_0(X)$.

The functions f_1 and f_2 are called the almost periodic and the ergodic perturbation components of f, respectively. The set of all such functions will be denoted by $PAP(X)$.

Definition 2.4. A continuous mapping $f : \mathbb{R} \times X \times X \to X$ is called pseudo almost periodic in $t \in \mathbb{R}$ uniformly in $(x, \chi) \in X \times X$ if it can be written as $f = f_1 + f_2$, where $f_1 \in AP(\mathbb{R} \times X \times X, X)$ and $f_2 \in AP_0(\mathbb{R} \times X \times X, X)$.

Let us assume that $S_{t,w}(s)$, $s \geq 0$ be a C_0-semigroup generated by $A(t, w)$.

Definition 2.5. A subspace $Y \subset X$ is called $A(t, w)$-admissible if Y is an invariant subspace of $S_{t,w}(s), s \geq 0$, and the restriction of $S_{t,w}(s)$ to Y is a C_0-semigroup in Y.

In order to prove the results of our paper, we need to implement the following hypothesis on $A(t, w)$.
($\mathbf{H}_1$) There exists a subset B of X such that the family of operators $\{A(t, w), (t, w) \in \mathbb{R} \times B\}$ is stable.
($\mathbf{H}_2$) For $(t, w) \in \mathbb{R}\times B$, Y is $A(t, w)$-admissible. Moreover, the family $\{\tilde{A}(t, w), (t, w) \in \mathbb{R} \times B\}$ of parts of $A(t, w)$ of Y is stable in the space Y.
($\mathbf{H}_3$) $A(t, w)$ is a bounded linear operator for $(t, w) \in \mathbb{R} \times B$, $A(\cdot, w)$ is continuous in $B(X, Y)$, and $D(A(t, w)) \supset Y$.
($\mathbf{H}_4$) A is Lipschitz, that is there exists a constants $L_A > 0$, satisfying $\|A(t, w_1) - A(t, w_2)\| \leq L_A \|w_1 - w_2\|$.
($\mathbf{H}_5$) $U_u(t, s)Y \subset Y, \quad s, t \in \mathbb{R}, s \leq t$, for each $u \in C(\mathbb{R}, X)$, and $U_u(t, s)$ is strongly continuous in Y.

Definition 2.6. A two parameters family of bounded linear operators $U(t, s), t \geq s \geq 0$, on X is called an evolution system if

(i) $U(s, s) = I$ and $U(t, r)U(r, s) = U(t, s),\ t \geq r \geq s \geq 0$.
(ii) $(t, s) \to U(t, s)$ is strongly continuous for $t \geq s \geq 0$.

If $u \in C(\mathbb{R}, X)$ and the family of operators $\{A(t, w), (t, w) \in \mathbb{R} \times X\}$, satisfies ($\mathbf{H}_1$)–($\mathbf{H}_4$), then there exists an evolution system $U_u(t, s)$ in X satisfying the following relations:

(i) $\|U_u(t, s)\| \leq \widetilde{M} e^{\delta(t-s)}$ for $t \geq s \geq 0$, where $\widetilde{M}$ and δ are the stability constants;
(ii) $\frac{\partial^+}{\partial t} U_u(t, s)w|_{t=s} = A(s, u(s))w$ for $w \in Y$;
(iii) $\frac{\partial^+}{\partial s} U_u(t, s)w|_{t=s} = -U_u(t, s)A(s, u(s))w$ for $w \in Y$.

Also there exists a positive constant C_1 such that

$$\|U_u(t, s)y - U_v(t, s)y\| \leq C_1 \|y\|_Y \int_s^t \|u(\xi) - v(\xi)\| d\xi,$$

for every $u, v \in C(\mathbb{R}, X)$ and every $y \in Y$.

Definition 2.7. By a pseudo-almost periodic mild solution $u : \mathbb{R} \to X$ we mean that $u \in PAP(X)$, and $u(t)$ satisfies

$$u(t) = U_u(t, a)u(a) + \int_a^t U_u(t, \xi) f(\xi, u(\xi), u_\xi) d\xi, \quad t \geq a. \tag{3}$$

It is easy to see that if $U_u(t,s) \le \widetilde{M}e^{\delta(t-s)}$, then relation (3) can be replaced by

$$u(t) = -\int_t^\infty U_u(t,\xi)f(\xi,u(\xi),u_\xi)d\xi.$$

Assumptions. We require the following assumptions:

1. The function $f : \mathbb{R} \times X \times X \to X$ is Lipschitz continuous, that is, there exists a positive number L_f such that
$$\|f(t,u_1,\chi_1) - f(t,u_2,\chi_2)\| \le L_f[\|u_1 - u_2\| + \|\chi_1 - \chi_2\|_\infty],$$
for all $t \in \mathbb{R}$ and for each $(u_i,\chi_i) \in X \times X$, $i = 1,2$;
2. $A(t,u(t))$, $t \in \mathbb{R}$, satisfies all the hypothesis $(\mathbf{H}_1) - -(\mathbf{H}_5)$;
3. $U_u(t,s)$, $t \ge s$, satisfy the condition that, for each $\epsilon > 0$ there exists a number $l_\epsilon > 0$ such that each interval of length $l_\epsilon > 0$ contains a number τ with the property that
$$\|U_u(t+\tau,s+\tau) - U_u(t,s)\| < \widetilde{M}e^{\delta(t-s)}\epsilon.$$

Throughout the paper, we assume that all the above assumptions are satisfied.

3. Pseudo almost periodic mild solution

In this section, we prove the existence and uniqueness of a pseudo almost periodic mild solution of (1).

Lemma 3.1. *If $u \in PAP(X)$, then $u_t \in PAP(X)$.*

Proof. By definition $u = u_1 + u_2$, where u_1 is almost periodic and u_2 is ergodic perturbation components of u. It is not difficult to observe that $(u_1)_t$ is almost periodic. Consider

$$\begin{aligned}\frac{1}{2r}\int_{-r}^{r}\|(u_2)_\xi\|d\xi &= \frac{1}{2r}\int_{-r+\theta}^{r+\theta}\|u_2(\xi)\|d\xi \\ &= 2\frac{(r+\theta)}{2r}\frac{1}{2(r+\theta)}\int_{-r+\theta}^{r+\theta}\|u_2(\xi)\|d\xi \to 0 \quad \text{as} \quad r \to \infty.\end{aligned}$$

Hence $(u_2)_t \in AP_0(X)$. So $u_t \in PAP(X)$. This completes the proof of the lemma. □

We define the operator F on $PAP(X)$ by

$$(Fu)(t) = -\int_t^\infty U_u(t,\xi)f(\xi,u(\xi),u_\xi)d\xi, \quad u \in PAP(X), \tag{4}$$

and will show that $Fu \in PAP(X)$ for $u \in PAP(X)$. The composition theorem given by Amir & Maniar, 1999; ensure that $\phi(t) = f(t,u(t))$ is pseudo almost periodic, when u are pseudo almost periodic. This result could be easily extended for the function $f \in PAP(\mathbb{R} \times X \times X)$. We have the following result.

Lemma 3.2. *The operator F is bounded.*

Proof. By the composition theorem for pseudo-almost periodic function Amir & Maniar, 1999; we claim that $\phi(\cdot) = f(\cdot, u(\cdot), u_{(\cdot)})$ is pseudo-almost periodic. Taking the norm of the operator F, we have

$$\|(Fu)(t)\|_X \leq \int_t^\infty \widetilde{M} e^{\delta(t-\xi)} \|f(\xi, u(\xi), u_\xi)\|_X \, d\xi \leq \widetilde{M}\widetilde{N} \int_t^\infty e^{\delta(t-\xi)} d\xi \leq \frac{\widetilde{M}\widetilde{N}}{\delta} < \infty,$$

where $\widetilde{N}$ is a bound for f. The above estimate guarantee the boundedness of Fu, which completes the proof of the lemma. □

Let us define $\phi(\cdot) = f(\cdot, u(\cdot), u_{(\cdot)})$, for $u \in PAP(X)$, as mentioned above, then $\phi \in PAP(X)$. Thus $\phi = \phi_1 + \phi_2$ for $\phi_1 \in AP(X)$ and $\phi_2 \in AP_0(X)$.

Lemma 3.3. *The operator F is continuous.*

Proof. Consider the sequence $u_n \to u$. We need to prove that $Fu_n \to Fu$. Taking the norm of both side, we have

$$\begin{aligned}
&\|Fu_n(t) - Fu(t)\| \\
&\quad = \Big\| - \int_t^\infty U_{u_n}(t,\xi) f(\xi, u_n(\xi), u_{n\xi}) d\xi + \int_t^\infty U_u(t,\xi) f(\xi, u(\xi), u_\xi) d\xi \Big\|, \\
&\quad \leq \int_t^\infty \|U_{u_n}(t,\xi)\| f(\xi, u_n(\xi), u_{n\xi}) - f(\xi, u(\xi), u_\xi)\| d\xi \\
&\qquad + \int_t^\infty \|(U_{u_n}(t,\xi) - U_u(t,\xi)) f(\xi, u(\xi), u_\xi)\| d\xi, \\
&\quad \leq \int_t^\infty \widetilde{M} e^{\delta(t-\xi)} L_f (\|u_n(\xi) - u(\xi)\| + \|u_{n\xi} - u_\xi\|_\infty) d\xi + D\|u_n - u\|_\infty, \\
&\quad \leq 2\widetilde{M} L_f \Big(\int_t^\infty e^{\delta(t-\xi)} d\xi \Big) \|u_n - u\|_\infty + D\|u_n - u\|_\infty, \\
&\quad \leq \left(\frac{2\widetilde{M} L_f}{\delta} + D \right) \|u_n - u\|_\infty. \qquad (5)
\end{aligned}$$

Taking supremum on the both sides, we get

$$\|Fu_n - Fu\|_\infty \leq \Big(\frac{2\widetilde{M} L_f}{\delta} + D \Big) \|u_n - u\|_\infty,$$

which proves our theorem. □

Lemma 3.4. *The map defined by*

$$(F_1\phi_1)(t) = - \int_t^\infty U(t,\xi)\phi_1(\xi) d\xi, \quad \phi_1 \in AP(X), \tag{6}$$

is $AP(X) \to AP(X)$.

Proof. As ϕ_1 is almost periodic, we can choose the period τ such that

$$\begin{aligned}
(F_1\phi_1)(t+\tau)-(F_1\phi_1)(t) &= -\int_{t+\tau}^{\infty} U_u(t+\tau,\xi)\phi_1(\xi)d\xi + \int_t^{\infty} U_u(t,\xi)\phi_1(\xi)d\xi,\\
&= -\int_t^{\infty} U_u(t+\tau,\xi+\tau)\phi_1(\xi+\tau)d\xi + \int_t^{\infty} U_u(t,\xi)\phi_1(\xi)d\xi,\\
&= -\int_t^{\infty} (U_u(t+\tau,\xi+\tau)\phi_1(\xi+\tau) - U_u(t,\xi)\phi_1(\xi))d\xi,\\
&= -\int_t^{\infty} (U_u(t+\tau,\xi+\tau)\phi_1(\xi+\tau) - U_u(t+\tau,\xi+\tau)\phi_1(\xi))d\xi\\
&\quad -\int_t^{\infty} U_u(t+\tau,\xi+\tau)\phi_1(\xi) - U_u(t,\xi)\phi_1(\xi))d\xi. \qquad (7)
\end{aligned}$$

Now we know that for every $\epsilon > 0$, we have

$$\|\phi_1(t+\tau)-\phi_1(t)\| < \epsilon.$$

From (7), for $\tau \in P_\epsilon$, we get

$$\begin{aligned}
\|F_1\phi_1(t+\tau)-F_1\phi_1(t)\| &\le \int_t^{\infty} \|(U_u(t+\tau,\xi+\tau)\|\|\phi_1(\xi+\tau)-\phi_1(\xi))\|d\xi\\
&\quad + \int_t^{\infty} \|U_u(t+\tau,\xi+\tau)-U_u(t,\xi)\|\phi_1(\xi))d\xi,\\
&\le \frac{\widetilde{M}}{\delta}\epsilon + \frac{\widetilde{M}C}{\delta}\epsilon < \epsilon',
\end{aligned}$$

which ensure the almost periodicity of $F_1(\phi_1)$. □

Lemma 3.5. *The map defined by*

$$(F_2\phi_2)(t) = -\int_t^{\infty} U_u(t,\xi)\phi_2(\xi)d\xi, \quad \phi_2 \in AP_0(X), \qquad (8)$$

is $AP_0(X) \to AP_0(X)$.

Proof. It is easy to see that $(F_2\phi_2)(t)$ is bounded and continuous in t on $\mathbb{R}$. Consider

$$\begin{aligned}
\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\|(F_2\phi_2)(t)\|dt &\le \lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\int_{\infty}^{r}\|U_u(t,\xi)\phi_2(\xi)\|d\xi dt\\
&\quad + \lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\int_{r}^{t}\|U_u(t,\xi)\phi_2(\xi)\|d\xi dt\\
&= I_1 + I_2,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\int_{\infty}^{r}\|U_u(t,\xi)\phi_2(\xi)\|d\xi dt \le \lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\int_{\infty}^{r}\widetilde{M}e^{\delta(t-\xi)}\|\phi_2\|d\xi dt,\\
&\le \lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\widetilde{M}\|\phi_2\|e^{\delta t}dt\int_{\infty}^{r}e^{-\delta\xi}d\xi \le \lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\frac{\widetilde{M}}{\delta}e^{-\delta r}\|\phi_2\|e^{\delta t}dt, \qquad (9)
\end{aligned}$$

$$
\begin{aligned}
I_2 &= \lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\int_{r}^{t}\|U_u(t,\xi)\phi_2(\xi)\|d\xi dt \le \lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\int_{r}^{t}\widetilde{M}e^{\delta(t-\xi)}\|\phi_2\|d\xi dt,\\
&\le \lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\widetilde{M}\|\phi_2\|dt\int_{0}^{t-r}e^{\delta\xi}d\xi \le \lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\frac{-\widetilde{M}}{\delta}\|\phi_2\|(e^{-\delta(t-r)}-1)dt. \qquad (10)
\end{aligned}
$$

Adding (9) and (10), we obtain

$$
I_1 + I_2 \le \frac{\widetilde{M}}{\delta}\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\|\phi_2\|dt = 0.
$$

□

Theorem 3.6. *Assume that $f \in PAP(\mathbb{R}\times X\times X, X)$ is Lipschitz continuous. Then equation* (1) *has a unique pseudo almost periodic mild solution if*

$$
\Lambda = \frac{2\widetilde{M}L_f}{\delta} + D < 1.
$$

Proof. Using Lemmas 3.2, 3.4 and 3.5, it follows that the operator F is well-defined and is from $PAP(X)$ to $PAP(X)$. Moreover, for $u, v \in PAP(X)$, we have

$$
\begin{aligned}
\|(Fu)(t)-(Fv)(t)\| &= \left\|\int_t^{\infty} U_u(t,\xi)(f(\xi,u(\xi),u_\xi) - U_v(t,\xi)f(\xi,v(\xi),v_\xi))d\xi\right\|,\\
&\le \int_t^{\infty}\widetilde{M}e^{\delta(t-\xi)}L_f[\|u(\xi)-v(\xi)\| + \|u_\xi - v_\xi\|_\infty]d\xi\\
&\quad + \int_t^{\infty}\|U_u(t,\xi)f(\xi,v(\xi),v_\xi) - U_v(t,\xi)f(\xi,v(\xi),v_\xi)\|d\xi,\\
&\le \int_t^{\infty}\widetilde{M}e^{\delta(t-\xi)}L_f\|u(\xi)-v(\xi)\|d\xi\\
&\quad + \int_t^{\infty}\widetilde{M}e^{\delta(t-\xi)}L_f\sup_{\theta\in\mathbb{R}}\|u(\xi+\theta)-v(\xi+\theta)\|d\xi\\
&\quad + D\|u-v\|_\infty,\\
&\le \left(2\int_t^{\infty}\widetilde{M}e^{\delta(t-\xi)}L_f d\xi + D\right)\|u-v\|_\infty\\
&\le \left(\frac{2\widetilde{M}L_f}{\delta} + D\right)\|u-v\|_\infty. \qquad (11)
\end{aligned}
$$

The operator F has a unique fixed point for $\Lambda < 1$, by the Banach fixed-point theorem. Because the solution is pseudo almost periodic, this fixed point is our desired unique pseudo almost periodic mild solution of (1). □

4. Applications to partial integro-differential equations

We consider the following age-structured model from Al-Omari & Gourley, 2002:

$$\partial_t u = D\partial_{xx} u - b_0(t)u^2 + \int_{-\infty}^{+\infty} h(y-x)u(y,t-\tau)dy, \tag{12}$$

where $D > 0$ (denotes diffusion rate), b_0 is a pseudo-almost periodic function and h has the form

$$h(x) = \alpha e^{-\gamma\tau} \frac{1}{\sqrt{4\pi\beta\tau}} e^{\frac{-x^2}{4\beta\tau}},$$

for some real constants α, β and γ. The history function is given by

$$H(x,t) = (1 - \sin((t-\tau)/\tau\pi))\sin(0.5(x+L)/\pi L).$$

Note that we can rewrite this function as $H(x,t) = \zeta(t)\xi(x)$, where $\xi(x)$ is a sufficiently smooth P-periodic function on $\mathbb{R}$ and $\zeta(t)$ is a pseudo-almost periodic function on $\mathbb{R}$.

We let X to be the Banach space of all continuous P-periodic functions from $\mathbb{R}$ into $\mathbb{R}$, i.e., $\phi \in X$ if $\phi : \mathbb{R} \to \mathbb{R}$ is continuous and $\phi(x+P) = \phi(x)$ for all $x \in \mathbb{R}$, endowed with the supremum norm,

$$\|\phi\|_X = \sup_{x\in\mathbb{R}} |\phi(x)|, \quad \phi \in X.$$

For each $w \in X$, let $A(t,w) : D(A(t,w)) \subset X \to X$, be given by

$$\begin{aligned} \mathcal{D}(A(t,w)) &= \{\phi \in X : \phi,\ \phi',\ \phi'' \in X\}, \\ A(t,w)\phi(x) &= D\phi''(x) - b(t,w(x))\phi(x), \\ &\qquad\qquad \phi \in \mathcal{D}(A(t,w)),\ w \in X, \end{aligned}$$

where $b : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $b(t,r) = b_0(t)r$.

Now, to tackle the integral operator, we define $K : X \to X$, by

$$(K\phi)(x) = \int_{\mathbb{R}} h(y-x)\phi(y)dy \ \Rightarrow\ |(K\phi)(x)| \le \left(\int_{\mathbb{R}} |h(y-x)|dy\right) \|\phi\|_X.$$

Let $F : PAP(X) \to X$, defined by

$$F(\psi) = K(\psi(-\tau)), \quad \psi \in PAP(X).$$

Then replacing ψ by any $u \in PAP(X)$, so that

$$F(u_t)(x) = K(u_t(-\tau))(x) = \int_{\mathbb{R}} h(y-x)u(t-\tau)(y)dy = \int_{\mathbb{R}} h(y-x)u(y,t-\tau)dy.$$

Thus (12) can be written as

$$\frac{du(t)}{dt} + A(t,u(t))u(t) = F(u_t), \quad t \in \mathbb{R}, \quad u \in X. \tag{13}$$

The existence results for (1) can therefore be applied to (13) and hence guarantees the existence of pseudo-almost periodic mild solutions to (12).

To numerically simulate the delay partial differential equations such as (12), we design and analyze a robust numerical method in the next section.

5. **Construction and analysis of a numerical method to solve DPDEs**

We partition the domain $\overline{\Omega} = [-L, L] \times [0, T]$ through the grid points (x_m, t_n) where $x_m = m\Delta x,\ t_n = n\Delta t;\ \Delta x = h = 2L/M,\ \Delta t = k = T/N;\ m = 0, \ldots, M,$ $n = 0, \ldots, N$. Here M and N are the total number of subintervals in spatial and time directions, respectively. Furthermore, we assume that N has been chosen such that the equality $\tau = s\Delta t = sk$ is satisfied where s is any positive integer.

We discretize the problem described by equation (12) along with the conditions (35)–(36) by a Crank-Nicolson's type of scheme which reads as

$$\begin{aligned}\frac{U_m^{n+1} - U_m^n}{k} = {} & \frac{D}{2}\left(\frac{U_{m-1}^{n+1} - 2U_m^{n+1} + U_{m+1}^{n+1} + U_{m-1}^n - 2U_m^n + U_{m+1}^n}{h^2}\right) \\ & - \frac{b_0(t_n)(U_m^n)^2 + b_0(t_{n+1})(U_m^{n+1})^2}{2} \\ & + \frac{\alpha e^{-\gamma\tau}}{\sqrt{4\pi\beta\tau}} \int_{-L}^{L} e^{-\frac{(y-x_m)^2}{4\beta\tau}} \left(\frac{H_m^n + H_m^{n+1}}{2}\right) dy, \end{aligned} \tag{14}$$

where

$$U_0^n = 0;\ n = 0, \ldots, N, \tag{15}$$

$$U_{M-1}^n = 0;\ n = 0, \ldots, N, \tag{16}$$

and H_m^n denotes the delayed term $u(t_n - \tau)$ which is evaluated as

$$H_m^n = \begin{cases} \theta(x_m, t_n - \tau), & \text{if } t_n < \tau,\ m = 0, \ldots, M \\ U_m^{n-s}, & \text{if } t_n \geq \tau,\ m = 0, \ldots, M. \end{cases} \tag{17}$$

The second term on the right hand side of (14) is approximated non-locally (c.f. Mickens, 1994; Patidar, 2005) as

$$\begin{aligned}\frac{b_0(t_n)(U_m^n)^2 + b_0(t_{n+1})(U_m^{n+1})^2}{2} & \approx \frac{b_0(t_n)U_m^{n+1}U_m^{n-1} + b_0(t_{n+1})U_m^{n+1}U_m^n}{2} \\ & = \frac{b_0(t_n)U_m^{n-1} + b_0(t_{n+1})U_m^n}{2} U_m^{n+1}. \end{aligned} \tag{18}$$

Using (18) into (14), multiplying the two sides of the resulting equation by k and rearranging the terms, we obtain

$$\begin{aligned} & -\frac{\phi}{2}U_{m-1}^{n+1} + \left(1 + \phi + k\frac{b_0(t_n)U_m^{n-1} + b_0(t_{n+1})U_m^n}{2}\right) U_m^{n+1} - \frac{\phi}{2}U_{m+1}^{n+1} \\ & = \frac{\phi}{2}U_{m-1}^n + (1-\phi)\,U_m^n + \frac{\phi}{2}U_{m+1}^n + \frac{k\alpha e^{-\gamma\tau}}{\sqrt{4\pi\beta\tau}} \int_{-L}^{L} e^{-\frac{(y-x_m)^2}{4\beta\tau}} \left(\frac{H_m^n + H_m^{n+1}}{2}\right) dy, \end{aligned} \tag{19}$$

where $m = 1, 2, \ldots, M-1$; $n = 0, 1, \ldots, N-1$, and $\phi = kD/h^2$.

Equation (19) together with (15)–(17) can be written as a linear system of the form

$$(20)\quad T_L v^{n+1} = T_R v^n + \frac{k}{2}\left(F^n + F^{n+1}\right); \quad F^n = \frac{\alpha e^{-\gamma\tau}}{2\sqrt{4\pi\beta\tau}} \int_{-L}^{L} e^{-\frac{(y-x)^2}{4\beta\tau}} H^n dy \in \mathbb{R}^{M-1},$$

where $v^\ell = [U_1^\ell, \ldots, U_{M-1}^\ell]^T$ and T_L and T_R are tridiagonal matrices whose entries are given by

$$(21)\quad T_L(n,m) \;=\; \begin{cases} -\frac{\phi}{2}, & \text{if } n = m-1 \\ 1+\phi+\frac{k}{2}\left(b_0(t_n)U_m^{n-1} + b_0(t_{n+1})U_m^n\right)2, & \text{if } n = m \\ -\frac{\phi}{2}, & \text{if } n = m+1 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$(22)\qquad T_R(n,m) \;=\; \begin{cases} \frac{\phi}{2}, & \text{if } n = m-1 \\ 1-\phi, & \text{if } n = m \\ \frac{\phi}{2}, & \text{if } n = m+1 \\ 0, & \text{otherwise,} \end{cases}$$

for all $m = 1, \ldots, M-1$.

The numerical solution is obtained by solving the linear system (20) at all levels $n = 1, 2, \ldots, N$.

Now we analyze the above method for convergence. We discuss the consistency and stability of this method which will then imply the convergence through the equivalence theorem of Lax (c.f., Richtmyer & Morton, 1967).

We see that the local truncation error (LTE) at the grid point (t_n, x_m) is given by

(23)

$$\text{LTE} = -\frac{Dk^2}{12}u_{tttxx}(x_m,\xi) - \frac{Dh^2}{12}u_{xxxx}(\eta,t_n) + \frac{kb_0(t_n)u_t(x_m,\zeta_1) - k^2 b_0(t_{n+1})u_t^2(x_m,\zeta_2)}{2},$$

where ξ, ζ_1 and $\zeta_2 \in [t_n, t_n+k]$ and $\eta \in [x_m-h, x_m+h]$. Hence, $|\text{LTE}| \to 0$ as $k \to 0$ and $h \to 0$. This proves the consistency of the numerical method.

To check the stability of our method, we follow our work in Bashier & Patidar, 2011 and use the matrix method

We rewrite the linear system (20) as

$$(24)\qquad T_L U^{n+1} = T_R U^n + \frac{k}{2}\left(F^n + F^{n+1}\right).$$

Let $v^n = [u(t_n,x_1), \ldots, u(t_n,x_{M-1})]^T$ and let $e^n = U^n - v^n$ be the difference between the approximate and exact solutions at level n.

If we insert the exact solution instead of the numerical solution in equation (20), we obtain an equation of the form

$$T_L v^{n+1} = T_R v^n + \frac{k}{2}\left(f^n + f^{n+1}\right). \tag{25}$$

By subtracting equation (25) from (24), we obtain the linear system

$$T_L e^{n+1} = T_R e^n + \frac{k}{2}\left(G^n + G^{n+1}\right), \text{ where } G^n = \frac{\alpha e^{-\gamma\tau}}{2\sqrt{4\pi\beta\tau}} e^{n-s} \int_{-L}^{L} e^{-\frac{(y-x)^2}{4\beta\tau}} dy. \tag{26}$$

Since the two matrices T_L and T_R are strictly diagonally dominant, they are nonsingular. By Gershgorin disk theorem, each eigenvalue λ_m of the matrix T_L should lie in one of the Gershgorin disks

$$D_L^m\left(1+\phi+k\frac{b_0^n U_m^{n-1} + B_0^{n+1} U_m^n}{2}, \phi\right).$$

Hence, all the eigenvalues of the matrix T_L lie in $\bigcup_{m=1}^{M-1} D_L^m$, yielding that $\lambda_m > 1$ for all $m = 1, 2, \ldots, M-1$. We rearrange these eigenvalues of T_L such that

$$0 < \lambda_1 \le \cdots \le \lambda_{M-1}.$$

Similarly, we find that all the eigenvalues μ_m; $m = 1, \ldots, \mu_{M-1}$ of T_R lie in the union of the Gershgorin disks

$$\bigcup_{m=1}^{M-1} D_R^m\left(1-\phi, \frac{\phi}{2}\right).$$

It is obvious that each eigenvalue μ_m of T_R satisfies $-1 < \mu_m \le 1$. If we rearrange the eigenvalues of T_R such that $\mu_j \le \mu_m$ for $j < m$, then, the eigenvalues of the two matrices T_L and T_R satisfy the relation

$$-1 < \mu_1 \le \cdots \le \mu_{M-1} \le 1 \le \lambda_1 \le \cdots \le \lambda_{M-1}.$$

Let $B = {T_L}^{-1}$ and $\widetilde{A} = BT_R$, then system (26) can be written as

$$e^{n+1} = \widetilde{A} e^n + \frac{k}{2} B\left(G^n + G^{n+1}\right). \tag{27}$$

We would like to show that the defect vector e which propagates over time, does not increase indefinitely. To this end, we note that the eigenvalues of $\widetilde{A}$, given by $\gamma_m = \mu_m/\lambda_m$, satisfy $0 < \gamma_m < 1$ and the eigenvalues of $B = {T_L}^{-1}$, given by $\nu_m = 1/\lambda_m$, satisfy the relation $0 < \nu_m < 1$ for all $m = 1, \ldots, M-1$.

Since $\widetilde{A}$ is nonsingular (as neither of its eigenvalues is zero), it has a complete set of linearly independent eigenvectors φ_m corresponding to the eigenvalues γ_m, $m = 1, \ldots, M$–1. Then, the set ω_m is a basis for $\mathbb{R}^{M-1}$. Also, B has a complete set of linearly independent eigenvectors ϑ_m, $m = 1, \ldots, M-1$ corresponding to the eigenvalues ν_m

which forms a basis for $\mathbb{R}^{M-1}$. Note that using these two different bases φ_m and ϑ_m, the vector e^0 can have two different representations of the forms

$$e^0 = \sum_{m=1}^{M-1} \omega_m \varphi_m = \sum_{m=1}^{M-1} \delta_m \vartheta_m, \tag{28}$$

where ω_m and δ_m are constants, $m = 1, \dots, M-1$.

We consider (26) in two separate intervals, namely $[0, \tau]$ and $(\tau, T]$. In $[0, \tau]$ where $n \le s$, the history terms H^n are evaluated exactly from the given history function $\theta(t, x)$. Therefore, the quantity G^n vanishes and hence (27) reduces to

$$e^n = \widetilde{A} e^{n-1}. \tag{29}$$

Iterations on equation (29) imply

$$e^n \;=\; \widetilde{A}^n e^0 = \sum_{m=1}^{M-1} \omega_m \gamma^n \varphi_m. \tag{30}$$

On the other hand, in $(\tau, T]$, where n is strictly greater than s, the history term H^n is equal to U^{n-s} and equation (26) takes the form

$$e^n = \widetilde{A} e^{n-1} + \frac{k}{2} B \left(G^n + G^{n+1} \right). \tag{31}$$

The second term on the right hand side of equation (31) is evaluated as

$$\frac{k}{2} B \left(G^n + G^{n+1} \right) = \frac{kC}{2} \int_{-L}^{L} \left[e^{-\frac{(y-x)^2}{4\beta\tau}} \left(\sum_{m=1}^{M-1} \delta_m \nu_m^{n-s} (1 + \nu_m) \vartheta_m \right) \right] dy, \tag{32}$$

where $C = \alpha e^{-\gamma\tau} / \sqrt{4\pi\beta\tau}$.

Using equations (30) and (32), we can re-write equation (27) as

$$e^n = \sum_{m=1}^{M-1} \omega_m \gamma_m^n \varphi_m + \frac{kC}{2} \int_{-L}^{L} \left[e^{-\frac{(y-x)^2}{4\beta\tau}} \left(\sum_{m=1}^{M-1} \delta_m \nu_m^{n-s} (1 + \nu_m) \vartheta_m \right) \right] dy. \tag{33}$$

Since $0 < \gamma_m < 1$ and $0 < \nu_m < 1$, we conclude that

$$e^n \to 0 \;\text{ as }\; n \to \infty.$$

This proves that the proposed numerical method is unconditionally stable.

Using (23) and the Lax equivalence theorem, we have the following main result.

Theorem 5.1. *The numerical method* (14)–(17) *is convergent of order* $\mathcal{O}(k + h^2)$ *in the sense that*

$$\sup_{0<D\le 1} \max_{m \le M, n \le N} |u(t_n, x_m) - U_m^n| \le C(k + h^2),$$

where U *is the numerical solution and* M, N *are the total number of subintervals in the spatial and time directions, respectively.*

6. Numerical results and discussion

In this section, we consider the delay partial differential equation (12) in the following form:

$$\frac{\partial u}{\partial t}(x,x) = D\frac{\partial^2 u}{\partial x^2}(x,t) - b_0(t)u^2(x,t) + \frac{\alpha e^{-\gamma\tau}}{\sqrt{4\pi\beta\tau}}\int_{-L}^{L} e^{-\frac{(y-x)^2}{4\beta\tau}} u(y,t-\tau)dy, \tag{34}$$

where $-L < x < L$ and $t > 0$, subject to the initial data

$$u(x,t) = u^0(x,t), \quad t \in [-\tau, 0], \tag{35}$$

and homogeneous Dirichlet boundary conditions

$$u(-L,t) = u(L,t) = 0, \ t \geq 0. \tag{36}$$

In the above, $D > 0$ is the diffusion rate, $b_0(t)$ is a pseudo-almost periodic function and α, β and γ are non-negative real constants. For the numerical simulations, we consider $L = 3$, $T = 200$, $M = 200$, $N = 2000$, $D = 0.01$, $\gamma = 0.05$, $\beta = 0.1$, $a = 2$ and $u(x,0) = \sin((x+L)/(2L)\pi)$. We simulate our method for $b_0(t) = 2\sin(8\pi t/T)$, $t \in [0,T]$ and for different values of time delays. From figures 1–4, it can be seen that, regardless of the values of the time delay τ, the solutions of the model always tend to be asymptotically stable and is often pseudo almost periodic.

7. Concluding remarks

In this paper, we established the existence and uniqueness of a pseudo almost periodic mild solution of a quasilinear functional differential equation. We applied this result to a partial integro-differential equation modelling nonlocal reaction-diffusion problems in biology. In order to substantiate this theoretical work, we developed a numerical methods and produced some results confirming theory. This method was also analyzed for convergence and we found that it is unconditionally stable. Our numerical simulations revealed the existence of positive asymptotically stable solutions of the model.

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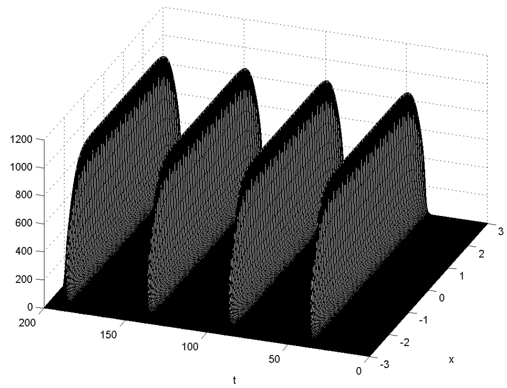

FIGURE 1. Profile of $u(x, t)$ for $\tau = 1$.

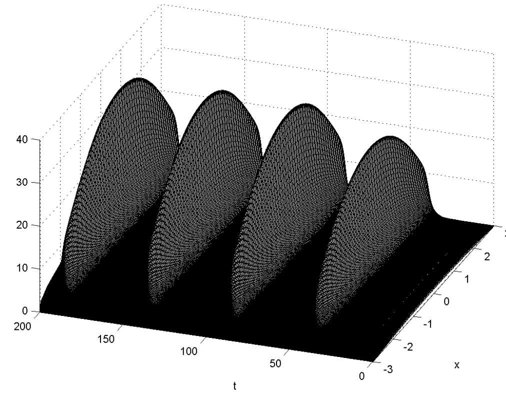

FIGURE 2. Profile of $u(x, t)$ for $\tau = 10$.

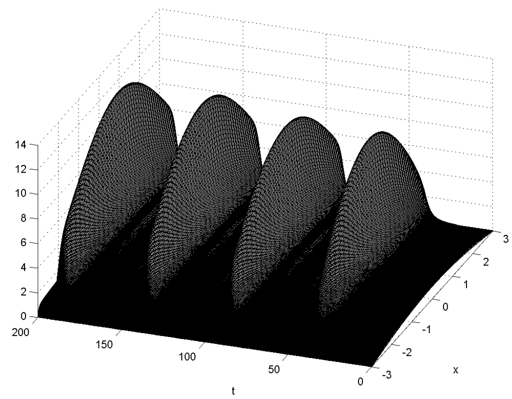

FIGURE 3. Profile of $u(x, t)$ for $\tau = 20$.

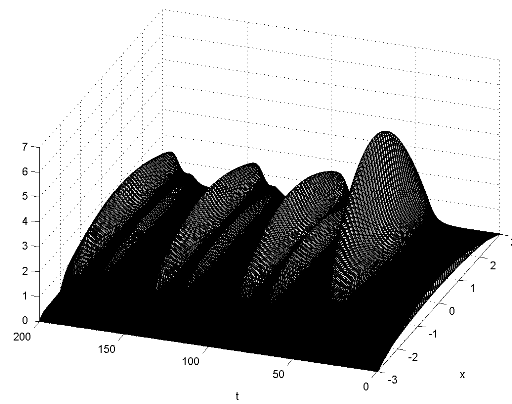

FIGURE 4. Profile of $u(x, t)$ for $\tau = 40$.

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