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NUMERICAL SOLUTION OF CAUCHY SINGULAR INTEGRAL EQUATION WITH AN APPLICATION TO A CRACK PROBLEM

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ABSTRACT. In this paper, a numerical method has been proposed to find an approximate solution of Cauchy type singular integral equations of first kind. Legendre polynomials have been used as basis functions. The effectiveness of the method is shown with the help of various test examples. Moreover, we have shown the application of our proposed method to solve a fracture mechanics problem which occurs in an infinite isotropic elastic medium with constant load σ along its four branches. The obtained results are in good agreement with those already present in literature.

AMS (MOS) Subject Classification. 41A10, 45A05, 45E05.

1. INTRODUCTION

The Cauchy type singular integral equation of first kind is given by

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\Psi(t)}{t-x}dt+\frac{1}{\pi}\int_{-1}^{1}k(x,t)\Psi(t)dt=h(x), \quad -1<x<1, \tag{1.1}$$

where $h(x)$ is known real function defined on the interval $[-1,1]$ and $k(x,t)$ is a known real function defined on the $[-1.1]\times[-1,1]$ and $\psi(x)$ is an unknown function on the interval $[-1,1]$.

The Cauchy type singular integral equations are naturally occurring in many of branches of science which includes cruciform crack problem in fracture mechanics [1], oscillating airfoils problem in aerodynamics [2], problem of electric current in a semiconductor film placed in a magnetic field in electrical science and scattering of surface water waves in hydrodynamics [3]. Due to its wide application area and valuable practical applications, researchers interest has been continuously increasing in Cauchy type singular integral equations. As a result new methods have continuously been evolved. The few among these are cubic spline [4], Gauss quadrature [5], collocation [6], iterative [7] and generalized inverse methods [8] etc.

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The equation (1.1) with $k(x,t) = 0$ is famously known as airfoil equation in aerodynamics [2] and can be stated as

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\Psi(t)}{t-x}dt = h(x), \quad -1 < x < 1, \tag{1.2}$$

The singular integral in both the equations (1.1) and (1.2) are understood in the sense of Cauchy principal value which have been stated in the section entitled preliminaries.

The complete analytical solution of (1.2) for various cases are described in [9]. And, for the case when the solution is unbounded at both the end points $x = \pm 1$, it is given by

$$\Psi(x) = -\frac{1}{\pi^2\sqrt{1-x^2}}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}h(t)dt + \frac{C}{\sqrt{1-x^2}}, \tag{1.3}$$

where

$$\int_{-1}^{1}\Psi(t)dt = C. \tag{1.4}$$

2. PRELIMINARIES

Some of important basic definitions from functional analysis [10, 11] which are frequently used in this paper.

2.1. **Hölder continuous function:**

A function $\mathcal{H}(x)$ is said to be a Hölder continous if the following condition is true:

$$|\mathcal{H}(x) - \mathcal{H}(y)| \le d|x_1 - x_2|^{\alpha}, \quad \forall x_1, x_2 \in D(\mathcal{H}), \tag{2.1}$$

where d is a non-negative real constant and α is an exponent of Hölder condition such that $0 < \alpha \le 1$, $D(\mathcal{H})$ denotes the domain of the function $\mathcal{H}$.

2.1.1. *Cauchy principal value* ($\mathcal{CPV}$)*:* If the function $\mathcal{H}(t) \in C^{0,\alpha}$, then

$$\mathcal{CPV}\int_{-1}^{1}\frac{\mathcal{H}(t)}{t-x}dt = \lim_{\epsilon\to 0}\left[\int_{-1}^{x-\epsilon}\frac{\mathcal{H}(t)}{t-x}dt + \int_{x+\epsilon}^{1}\frac{\mathcal{H}(t)}{t-x}dt\right], \tag{2.2}$$

where $C^{0,\alpha}$ is the space of functions which are Hölder continuous on the interval $(-1,1)$ with the exponent $0 < \alpha \le 1$.

2.2. **Legendre polynomial:**

The n^{th} degree Legendre polynomial over the interval $[-1,1]$ is defined as

$$\mathcal{P}_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \tag{2.3}$$

where n is a nonnegative integer.

The orthogonal property of Legendre polynomials over the interval $[-1,1]$ is significant in the process of approximation of functions defined over the interval $[-1,1]$ and can be stated as follows:

2.2.1. *Orthogonal property of Legendre polynomials:* Let $\mathcal{P}_m(x)$and $\mathcal{P}_n(x)$ are Legendre polynomials of degree m and n respectively, then the orthogonal property for Legendre polynomials is:

$$\int_{-1}^{1} \mathcal{P}_m(x)\mathcal{P}_n(x)dx = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{2}{2n+1} & \text{if } m = n, \end{cases} \tag{2.4}$$

where m, n are nonnegative integers.

3. METHODOLOGY

One can approximate the unknown function $\Psi(x)$ in (1.1) as

$$\Psi(x) = \frac{1}{\sqrt{1-x^2}}\phi(x) \approx \frac{1}{\sqrt{1-x^2}} \sum_{i=0}^{n} a_i e_i(x), \tag{3.1}$$

where $\phi(x)$ is a well defined function on the interval $[-1,1]$, $\{e_i(x)\}_{i=0}^{n}$ denotes the orthonormalized Legendre polynomials and a_i; $i = 0, 1, \ldots, n$ are unknown real coefficients.

3.1. **Galerkin Method:** On using the above approximation for $\Psi(x)$ in equation (1.1) or (1.2), some residual error $E(x, a_0, a_1, a_2, \ldots, a_n)$ is always expected. In Galerkin's method, this residual error $E(x, a_0, a_1, a_2, \ldots, a_n)$ is assumed to be orthogonal to the space spanned by orthonormalized Legendre polynomials $\{e_i(x)\}_{i=1}^{n}$ i.e.

$$\langle E(x, a_0, a_1, a_2, \ldots, a_n), e_i \rangle_{L_2} = 0, \quad \forall i = 0, 2, \ldots, n. \tag{3.2}$$

$L^2[-1,1] = \{u : [-1,1] \to \mathbb{R} : \int_{-1}^{1}[u(x)]^2 dx < \infty\}$ is a Hilbert space of all real valued functions which are square integrable in the interval $[-1,1]$.

Finally, using equation (3.2) after substituting equation (3.1) in equation (1.1) or (1.2) one needs to solve system of $(n+1)$ linear equations in $(n+1)$ unknowns in order to get the values of unknown coefficients $a_i's$. And, substituting the values of $a_i's$ in equation (3.1), one can get the approximate solution of equation (1.1) or (1.2). There is one more equation we will get by using equation (3.1) in equation (1.4) which gives the value of a_0

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} \sum_{i=0}^{n} a_i e_i(t) = 0. \tag{3.3}$$

4. Illustration

In this section, we have implemented the method of solution through test examples.

Example 1 Consider the following Cauchy type singular integral equation [4]:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\Psi(t)}{t-x}dt+\frac{1}{\pi}\int_{-1}^{1}\sin(t-x)\Psi(t)dt=J_1(1)\cos(x)+1,\quad -1<x<1, \tag{4.1}$$

where $J_1(1)$ is the Bessel function of the first kind of order 1. The solution $\Psi(x)$ is required to satisfy the compatibility condition $\int_{-1}^{1}\Psi(t)dt=0$.

The exact solution in this example is known and is given by

$$\Psi(x)=\frac{x}{\sqrt{1-x^2}}. \tag{4.2}$$

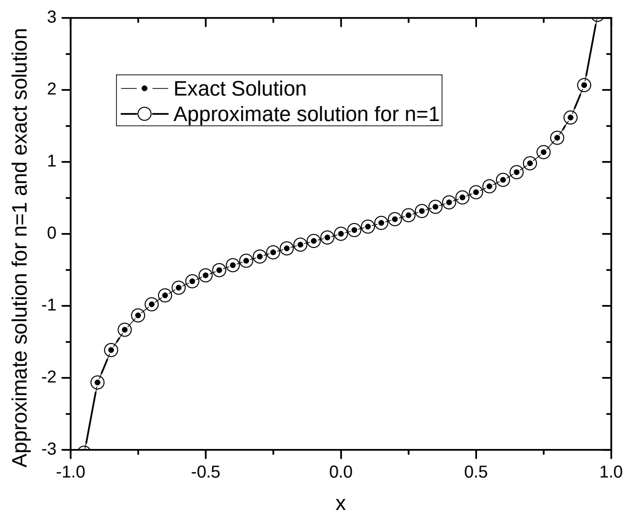

FIGURE 1. Comparison between exact and approximate solution

Jen and Srivastav [4] used cubic spline method for this example. When $n^*=7$ (where n^* is number of nodes taken in the interval $[0,1)$), the computed solution was accurate to the limits of single-precision computation (8 digits) but in our present method, the exact solution is obtained on using first degree Legendre polynomial as shown in Figure 1.

Example 2 Consider the following Cauchy type singular integral equation from [12]:

$$\int_{-1}^{1}\frac{\Psi(t)}{t-x}dt=x^4+5x^3+2x^2+x-\frac{11}{8},\quad -1<x<1, \tag{4.3}$$

The solution $\Psi(x)$ is required to satisfy the compatibility condition $\int_{-1}^{1}\Psi(t)dt=0$.

The exact solution in this example is $\Psi(x)=\frac{1}{\pi\sqrt{1-x^2}}\big(x^5+5x^4+\frac{3}{2}(x^3-x^2)-\frac{5}{2}x-\frac{9}{8}\big)$. It is shown in Figure 2 that the approximate solution is the exact solution for $n=4$.

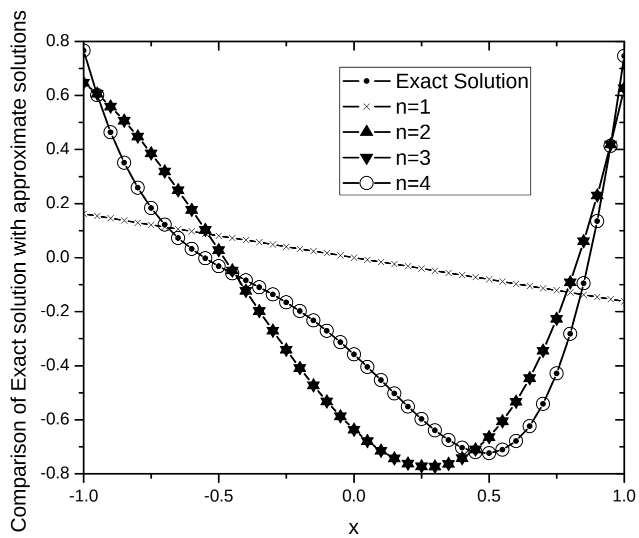

FIGURE 2. Comparison between exact and approximate solutions

Example 3 Consider the following Cauchy Singular Integral equation from [13] which appears when an infinite isotropic elastic medium under constant load σ along its four branches is considered and known as cruciform crack problem in the field of fracture mechanics :

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\Psi(t)}{t-x}dt+\frac{1}{\pi}\int_{-1}^{1}\frac{t(t^2-x^2)}{(t^2+x^2)^2}\Psi(t)dt=\sigma, \quad -1<x<1, \tag{4.4}$$

The exact solution in this example is not known. But 0.8636 is the value calculated by [13] and generally accepted for $\psi(1)$. It is shown in Table 1. the solution obtained by proposed method is identical to value of $\psi(1)$.

TABLE 1. It compares the values of $\psi(1)$ by present method with the one already present in literature

n	Method [14]	Present Method
3	0.8364	0.8892
4	0.8388	0.8892
5	0.8629	0.8570
6	0.8638	0.8570
7	0.8653	0.8653
8	0.8628	0.8653
9	0.8650	0.8631
10	0.8628	0.8631
11	0.8650	0.8636

5. CONCLUSION

Cauchy singular integral equations have been numerically solved with the help of Galerkin method by using Legendre polynomials as basis functions. This method converts the integral equation into a system of linear equations which are easily solvable and it converges to the exact solution rapidly as shown in example 1. Moreover, it is shown with the help of test example that whenever the known function is of polynomial form the proposed method gives the exact solution. One of the application of the proposed method is also shown by solving cruciform crack problem in the field of fracture mechanics. The obtained results are also compared with those already present in literature and are in good agreement.

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