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# OPTIMAL CONTROL STRATEGIES TO ATTRACT STUDENTS BETWEEN LOCAL COLLEGES

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**ABSTRACT.** In Orangeburg, South Carolina there are two prominent Historically Black Colleges and Universities (HBCU), which are Claflin University and South Carolina State University. The competition between the two colleges for attracting and retaining students has been a long time. In this paper, we study and model the competition between two colleges, and discover the optimal competition strategy for each competitor. According to the properties of this competition, a non-cooperative Differential Game (DG) model is set up to study the effects of competition. The model involves two competitors whose objectives are to maximize the enrollment over a finite period of time. The constraints are the dynamics of the population of students. The controls of competitors are activities to attract students. Compared with classical optimal control models, there are a finite number of decision makers in DG models. The necessary optimality condition of optimal control model is Pontryagin's maximum principle. The solution of non-cooperative DG model is defined in terms of Nash Equilibrium. We derive and analyze optimality conditions for Nash Equilibrium of our model, and develop a numerical method to solve its optimality condition. Based on numerical results, practical guidelines for those two colleges have been given.

**AMS (MOS) Subject Classification**. 35K60, 35K57

## 1. INTRODUCTION

A dynamical system was introduced which is a concept where a "fixed" rule describes the amount of time of dependence of a point in a geometrical space. It has a state given by a set of real numbers (or vector), represented by a point in a respective state space. For a dynamic system which is continuous, $x(t)$ is used to represent the systems state at time $t$. Dynamics are used to describe the "changing" of states. The evolution rule of the dynamical system is a fixed rule that describes what future states follow from the current state. The rule is deterministic. In other words, for a given time interval only one future state follows from the current state. In most dynamic systems, differential or difference equations can be used to represent various components and the interactions between them. This paper will consider continuous dynamic system, and use ordinary differential equations to describe the dynamics. In most dynamic systems, differential equations are used to represent

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various components and interactions between them. The theory of dynamics games is concerned with multi-person decision making. The principal characteristic of a dynamic game is that involves a dynamic decision process evolving in time (continuous or discrete), with more than on decision maker, each with its own cost function and possibly having access to different information. Dynamic game theory adopts characteristics from game theory and optimal control theory, although it is much more versatile than both.

Optimal control model plays an important role in the design of dynamic system. This model consists of (1) an objective function, which usually is to maximize the return or to minimize the cost of operation of physical, social, or economic processes, or the state of the system $x(t)$ to be at some point at some specific time; or use the least time to move the system to some point; or use the smallest controls to move the state $x(t)$ to some point. (2) A constraint, which is usually a dynamics described by ordinary differential equations, where state and control variables, and necessary boundary conditions are imbedded. The optimal control model can be solved by the Pontryagin Maximum Principle. This principle states a necessary condition that must hold on an optimal trajectory. It is a powerful method for the computation of optimal controls. It is used to find the best possible control for taking a dynamical system from one state to another. The Principle is given by a system of Boundary Value Problems (BVP) or Differential Algebraic Equations (DAE). Generally it is difficult to solve BVP or DAE analytically, so numerical method is the best way to solve it.

The halcyon period of game theory began around the mid of 20th century. With the pioneering work of John von Neumann and Oskar Morgenstern, this mathematics tool became the central focus of the research efforts of the major military analysis group. This theory was then developed extensively by many scholars. According to the properties of decision variables, game models can be classified as discrete and continuous games. Discrete game, which has a finite number of players, moves, events, outcomes, etc, has was well developed by game theorists Reinhard Selten, John Harsanyi, and John Forbes Nash. For discrete game model, existence of the equilibrium has been developed [22], and Linear Programming has been used to solve for its equilibrium. Investigation of Differential Game (DG) started from Rufus Isaacs [21], which is motivated by military pursuit games in the early 1950's. DG is a dynamic game model in which the state and decision variables of the competitors develop continuously in time. According to different assumptions about competitive system, DG can be classified as different types of DG model. Following is one classical non-cooperative DG model:

$$opt_{u_i} J_i = \int_0^T f_i(x(t), u_1(t), \ldots, u_n(t), t)dt + h_i(x(T)), \quad i = 1, \ldots, n$$

$$subject\ to$$

$$\left\{ \begin{array}{ll} \frac{dx(t)}{dt} = & a(x(t), u_1(t), \ldots, u_n(t), t) \\ u_i(t) & \in \Omega,\ t \in [0, T] \\ b(x(t), u_1(t), & \ldots, u_n(t), t) \leq 0 \\ \varphi(x(0), x(t)) & \leq b \end{array} \right. \tag{1.1}$$

In this model, all competitors' decision variables are combined in the same dynamics, but each competitor tries to optimize his own objective function. The key assumption for above model is that each competitor has the same full knowledge of the dynamics, and knows the other competitors' objective functions. The solution of the above model is defined in terms of Nash Equilibrium.

**Definition 1** Let $U$ be the set of admissible controls. A set of control vectors $(u_1^*, u_2^*, \ldots, u_N^*)$ with $u_i^* \in U,\ i = 1, \ldots, N$, is defined as Nash Equilibrium if the set of inequalities which is described by the set of inequalities 1.2 is satisfied for any control $u_i \in U$. The corresponding point $(J_1, \ldots, J_N) \in R^N$ is called outcome of the game.

$$\left\{ \begin{array}{c} J_1(u_1, u_2^*, \ldots, u_N^*) \preceq J_1(u_1^*, u_2^*, \ldots, u_N^*) \\ \vdots \\ J_N(u_1^*, u_2^*, \ldots, u_N) \preceq J_N(u_1^*, u_2^*, \ldots, u_N^*) \end{array} \right. \tag{1.2}$$

This set of inequalities implies if some competitor deviates from this equilibrium, then his objective value will be hurt. The rigorous mathematical framework with existence and optimality condition for Nash Equilibrium of non-cooperative DG is due to Friedman [9], L. D. Berkovtiz [12], and W.H. Fleming [8]. The optimality condition is Pontriagon type optimality condition, which is Differential Algebraic Equation (DAE) or Boundary Value Problem (BVP), which is changeling to solve, especially when the dimension of model is high. Most recent results based on solving optimality condition for equilibrium of non-cooperative DG models include our first-order algorithm based on steepest descent method with full discretization [15]; our second-order algorithm based on Quasi-Newton method [14], [24]. This algorithm is developed by using random perturbation technique to approximate Jacobian matrix in its each iteration. This formalism is more efficient than multiple shooting methods to solve DAE or BVP, and has been further developed by us to solve Stackelberg DG models [16].

## 2. Modelling Background

In Orangeburg, SC, there exist two HBCUs, which are South Carolina State University (SCSU) and Claflin University (CU). Since the major students of these two colleges are from South Carolina, we are able to regard these two schools in a closed system, where they compete for students. The efforts of their attraction strategies are their controls. The enrollment is the state of this system, which follows some ordinary differential equations. Between those two colleges, there just exists pure competition, and no collaborations. With these assumptions, how to optimally compete for this resource can be considered as a zero-sum differential game. There are few research papers that emphasize the concerns of applications of differential game theory to the optimal management of attracting students. However, if we regard our students as a replenishable natural resource, then we are able to establish the dynamics of the system. The paper from Clark [6] is about how to optimally compete for some natural resource (fish). He stated that the relative simplicity of his model stems from its linearity in the control variable, which allows deducing the optimal policy feedback form. He also makes note of the questionability of this linearity assumption due to market imperfections in actual fish markets. Within his paper he studies the case of duopoly in which the price depends on the total biomass harvested by both producers being at the market. He analyzes a competitive open- and limited-access fishery by means of the theory of $n$-person dynamic games. He calculates the non-cooperative solution for the common property fishery model. The non-cooperative Nash solution results in the long run dissipation of economic rents.

Gordon [10] shows for a static open-access fishery that, regardless of physical constraints, the economic rents are inevitably dissipated. Clark [4] considers a firm faced by pure competition (price-taking firm), i.e., the price that it can obtain for the harvested biomass is independent of its output. Within his paper it provided a generalization of the Gordon-Schafer fishery model. The main purpose of this paper was to study the impact of different information structures (symmetric information in the Nash case and asymmetric in the Stackelberg case) on the optimal duopolistic exploitation.

In Raimo paper [20], a two-country differential game model of whaling is used for analyzing a dynamic bargaining problem. At any given initial time, the two countries may either continue on a non-cooperative of play characterized by an open loop Nash equilibrium, or negotiate a bargaining solution which is defined as the Kalia-Smorodinsky solution. The cooperative solution calls for a restraint in the whaling efforts which leave a temptation to cheat for any player. His model shows how, by announcing a credible threat, namely to make whaling an “open-access” fishery, a country can eliminate this temptation to cheat and transform the cooperative solution into an equilibrium.

Above all, those DG models in harvesting natural resource are solved by analytical method because of simplifications. This method will give out some qualitative properties about DG, but it will limit the applicability of DG model without 'exact' results. This motivates us to develop numerical method for DG models.

## 3. **Optimality Necessary Conditions**

The non-cooperative DG model is given by Nash Equilibrium (1.2). In order to solve for Nash Equilibrium, we will solve its optimality necessary condition. In a noncooperative n-player DG models, assuming that players $2, 3, \ldots, n$ players give their optimal open-loop controls strategies $u_2^*, \ldots, u_n^*$, so player 1 has the following problem:

$$
\begin{aligned}
&Min_{u_1} J_1 = \textstyle\int_{t_0}^{t_f} g_1(x(t), u_1(t), u_2^*(t), \ldots, u_n^*(t), t)dt + h_1(x(t_f))\\
&\textit{subject to}\\
&\begin{cases} \frac{dx(t)}{dt} = & a(x(t), u_1(t), u_2^*(t), \ldots, u_n^*(t), t) \\ \quad x(t_0) & = x_0 \end{cases}
\end{aligned}
\tag{3.1}
$$

Player 1's Hamiltonian is

$$
\begin{aligned}
H_1(x(t), u_1(t), u_2^*(t), \ldots, u_n^*(t), t) = g_1(x(t), u_1(t), u_2^*(t), \ldots,\\
u_n^*(t), t) + \lambda_1^T \dot{a}(x(t), u_1(t), u_2^*(t), \ldots, u_n^*(t), t)
\end{aligned}
$$

According to the definition of Nash equilibrium, the optimal control for player 1 should make this Hamiltonian to satisfy:

$$
H_1(x^*(t), u_1^*(t), u_2^*(t), \ldots, u_n^*(t), t) \leq H_1(x^*(t), u_1(t), u_2^*(t), \ldots, u_n^*(t), t)
$$

where $u_1(t)$ belongs to admissible control set of player 1.Thus, the Pontryagin optimality condition for player 1 is as follows:

$$
\begin{cases}
\frac{dx^*(t)}{dt} = \frac{\partial H_1}{\partial p}(x^*(t), u_1^*(t), u_2^*(t), \ldots, u_n^*(t), p^*(t), t) \\
\frac{dp^*(t)}{dt} = -\frac{\partial H_1}{\partial x}(x^*(t), u_1^*(t), u_2^*(t), \ldots, u_n^*(t), p^*(t), t) \\
0 = \frac{\partial H_1}{\partial u_1}(x^*(t), u_1^*(t), u_2^*(t), \ldots, u_n^*(t), p^*(t), t) \\
x * (0) = x_0, \quad p^*(t_f) = \frac{\partial h_1}{\partial x}(x^*(t_f), t_f)
\end{cases}
\tag{3.2}
$$

The status of each player is symmetric, so each player faces the same kind of optimal control problem, so we will draw $n$ necessary conditions for all players, and in each necessary condition, there are $n$ state, $n$ co-state equations and $n$ algebraic equation for controls. By combing all necessary conditions we get $n$ state equations, $n^2$ co-state equations and $n$ algebraic equations, so we get $n^2 + 2n$ equations. Further if we could solve controls explicitly in terms of state and co-state variables, then we will get a $n^2 + n$ Two Point-Boundary Value Problems.

## 4. **Modeling**

Based on the models for fishery resources by Gordon [10], and Clark [6], we develop a nonzero-sum DG between two historically black colleges and universities (Claflin University and South Carolina State University), each having restricted rights to recruit, advertise and exploit the stock of the renewable resource (students) in competition with the other player. The following notation is used:

- $i = 1, 2$: Competitor $i$
- $x(t)$: Population of students at time $t$
- $G(x) = x(a - \ln x)$: Gompertz law of population growth
- $u_i(t)$: Advertisement strategies to attract students to Claflin and SCSU, respectively
- $q_i$: Catchability coefficient for player $i$
- $p$: Price of tuition per student
- $c_i$: Marginal cost of player $i$
- $r$: Discount rate
- $w$: Weight of different objectives
- $v$: The ideal quantity of enrollment after 5 years

In this paper, a nonlinear version of Clark's model for two HBCU's is developed. One of major differences between our model and Clark's is that we assume that the quantities of students are fixed within the period of recruiting students to attend the respective universities. Nonlinearity of dynamics comes from Gompertz law of population growth and linear summation of controls on the population of students. The main purpose of the model is to study the impact of different parameters of the competitive system, which include payoffs, policies of the institutions of higher learning, and qualities that each player of the Nash equilibrium bring to attract students. The major assumption about competitors is that the control of the second player is more efficient than that of the other one. The major coefficients to describe the competitors is rate of attracting students 'effort' of competitor $i : u_i(t)$ and catchability coefficient for competitor $i : q_i$. The bigger $q_i$ implies this competitor's controls being more efficient. We will assume that the control of second competitor has bigger catchability than the first one. Another assumption imbedded in this system is that after the recruiting season is over, the quantity of students enrolling to attend the institution full time must be kept at some level, in order to have sustainable development of retention. A non-cooperative DG model is set up to investigate the optimal strategies for each competitor.

The objectives of each competitor consists of two processes: the first goal is to maximize its current benefits, which is *income* $-$ *cost*; the second objective is to keep the quantity of students to some ideal level in order to maintain and sustain individual school development, which is $(x(T)-ve^a)^2$ . The coefficient is the balance between these two objectives. The number $e^a$ is the equilibrium of the state variable (quantity of students) without controls. Thus, following objective functions for both competitors are attained.

$$Max_{u_i>0}J_i=\int_0^T e^{-rt}(pq_ix^2-c_iu_i^2)dt-w(x(T)-ve^a)^2,\ i=1,2$$

Gompertz law of population growth is adopted to model the dynamics of quantity of students without control. The dynamics of the objective functions are constructed as follows:

$$\frac{dx(t)}{dt}=x(t)(a-\ln x(t))$$

The rate of change of student quantity is assumed to be proportional to the multiplication of rate of student attraction 'effort' of competitor $i:u_i$, and catchability coefficient of player $i:q_i$, and the student quantity square, that is:

$$\frac{dx(t)}{dt}=-(q_1u_1(t)+q_2u_2(t))x(t)^2$$

Above two effects are combined in following dynamics:

$$\frac{dx(t)}{dt}=x(t)(a-\ln x(t))-(q_1u_1(t)+q_2u_2(t))x(t)^2$$

In conclusion, we derive the following as the non-cooperative DG model:

$$\begin{cases} Max_{u_1>0}J_1=\int_0^T e^{-rt}(pq_1x^2-c_1u_1^2)dt-w(x(T)-ve^a)^2 \\ Max_{u_2>0}J_2=\int_0^T e^{-rt}(pq_2x^2-c_2u_2^2)dt-w(x(T)-ve^a)^2 \end{cases}$$

(4.1) *subject to*

$$\begin{cases} \frac{dx(t)}{dt}=x(t)(a-\ln x(t))-(q_1u_1(t)+q_2u_2(t))x(t)^2 \\ x(0)=x_0 \end{cases}$$

## 5. **Development of Solution Methods**

Generally, the optimality condition for non-cooperative DG is Differential Algebraic Equations (DAE) or Boundary Value Problems (BVP). In Wan, Peng [21], an algorithm based on random perturbation is designed to solve this type of optimality condition. In this paper, an improved algorithm will be designed to solve this model. The Pontryagin's optimality condition for above DG model is as follows:

$$\begin{cases} \frac{dx(t)}{dt} &= x(t)(a-\ln x(t))-(q_1u_1(t)+q_2u_2(t))x(t)^2 \\ \frac{d\lambda_i(t)}{dt} &= 2e^{-rt}pq_ix(t)-\lambda_i[a-\ln x(t)-1-2(q_1u_1(t)+q_2u_2(t))x(t), \quad i=1,2 \\ u_1 &= \frac{\lambda_1x^2q_1}{2e^{-rt}c_1} \\ u_2 &= \frac{\lambda_2x^2q_2}{2e^{-rt}c_2} \end{cases}$$

The boundary condition is as follows:

$$\begin{cases} x(0) &= x_0 \\ \lambda(T) &= 2w(x(T) - ve^a) \end{cases}$$

Since the control for each competitor can be solved, above BVP can be simplified as following system:

$$\begin{cases} \dot{x}(t) &= a(x_i(t), m_{ij}(t), t) \\ \dot{m}_{ij}(t) &= d(x_i(t), m_{ij}(t), t) \end{cases}$$

The boundary condition is:

$$\begin{cases} x_i(0) &= x_{i0}(t) \\ \dot{m}_{ij}(T) &= h_{ij}(x_1(T), \ldots, x_n(T)) \end{cases}$$

As to the above BVP, it will be better to guess the terminal values since we just need to guess the terminal values of state variables, which is much less than guessing the initial values; furthermore, the objective function of each player can give information about terminal value. Thus, the initial value problem (IVP) associated with this BVP is

$$\begin{cases} x_i(T) &= s \\ \dot{m}_{ij}(T) &= h_{ij}(x_1(T), \ldots, x_n(T)) \end{cases}$$

where $s$ is a parameter. Medhin and Wan [13] gave out the condition for existence of solution the above BVP. The above model satisfies the existence condition in [16]. For each $s \in \mathbb{R}^n$ there is a unique solution of the above IVP, which is denoted by $x_i(t; s)$, $m_{ij}(t; s)$, then there is a unique $x_i(0; s)$. Thus, we can define a functional relationship between $x_i(T)$ and $x_i(0)$, that is, $x_i(0) = f_i(x_i(T))$, for which we cannot, in general, find analytic formula. Now suppose $x_i(t; s^*)$, $m_{ij}(t; s^*)$ are solved for from the IVP where $s = s^*$, and if $s^*$ is such that the boundary conditions are satisfied, that is $x_i(0; s^*) = x_{i0}$ or $f_i(x_i(T; s^*)) - x_{i0} = 0$, then $x_i(t; s^*)$ is the solution of the BVP. The first algorithm is designed to find such $s^*$, that is to find solution for following system:

$$F_i(x_1(T), \ldots, x_n(T)) = f_i(x_1(T), \ldots, x_n(T)) - x_i(0) = 0, \ i = 1, \ldots, n$$

The algorithm is based on Newton's method to update $x_i(T)$. As to the above functions $F_i$, using Taylor series expansion up to first-order about some estimate point $(x_1(T)^{(k)}), \ldots, x_n(T)^{(k)})$ , we have the system of equations:

$$F_i(x_1(T), \ldots, x_n(T)) = F_i(x_1(T)^{(k)}, \ldots, x_n(T)^{(k)}) + J_P\dot{(}x_i(T) - x_i(T)^{(k)}) = 0,$$

which we solve for $(x_1(T), \ldots, x_n(T))$ to get the updated vector estimate:

$$X^{(k+1)}(T) = X^{(k)}(T) - J_P^{-1}\dot{F}(X^{(k)}(T))$$

where $P = (x_1(T)^{(k)}, \ldots, x_n(T)^{(k)})$, and $J_P$ is the Jacobian matrix at $P$ . As to the initial guess of $s$, the expected fish quantity at the end of time is our reference. To obtain the Jacobian Matrix $J_P$, the following recipe is one way to approximate it:

$$\frac{\partial F_i}{\partial x_j} \cong \frac{\sum_n}{n}\left(\frac{F_i(x(k)) - F_i(x(k-1))}{x_j(k) - x_j(k-1)}\right)$$

From the above expression, $\partial F_i$ is approximated by difference of values of $F_i$ between step $k$ and step $k-1$; $\partial x_j$ is approximated by difference of values of $x_j$ between step $k$ and step $k-1$. We will pick a value for $n$ in $\frac{\sum_n}{n}$ to take average as approximation of $\frac{\partial F_i}{\partial x_j}$. Above idea is realized in the following algorithm.

**Algorithm 1**

**Step 1:** Guess two terminal values for state variables: $\widetilde{x}_0(T)$ and $\widetilde{x}_1(T)$, which are $n$ dimensional vectors.

**Step 2:** Compute terminal values for the co-state variables: $\lambda_0(T)$ and $\lambda_1(T)$ corresponding to $\widetilde{x}_0(T)$ and $\widetilde{x}_1(T)$.

**Step 3:** Solve backward the state and co-state equations by Runge-Kutta method and get $\widetilde{x}_0(0)$ and $\widetilde{x}_1(0)$ , then compute $F_0, F_1$.

**Step 4:** If $\|F_0\| < \epsilon$ or $\|F_1\| < \epsilon$, then stop, output the optimal solution; if not, $k = 1$ go to Step 5.

**Step 5:** Update $\widetilde{x}_{k+1}(T) = \widetilde{x}_k(T) - J_P^{-1} F_0(\widetilde{x}_k(T))$, where $J_k$ is approximated by Jacobian matrix.

**Step 6:** Compute terminal values for co-state variables $\lambda(T)$.

**Step 7:** If $\|F_0\| < \epsilon$, then stop, output the optimal solution; if not, $k+1$, go to Step 5.

## 6. **Numerical Results and Analysis**

In our experiment, there are two colleges in competition. The basic assumption is college 2 sustainability of control is better than college 1's. Another interested coefficient is the marginal cost of each competitor. In the following, we will analyze three cases for effects of these coefficients on Nash equilibrium based on these coefficients.

In case 1, the cost of control of college 1 is less than that of college 2. Thus, we can expect that college 1's control is bigger than the other, which can be seen from norm of control in Table (1) and the Figure (1). However, the objective value of college 1 is worse than the other, which implies that Catchability plays a leading role in determination of competitor's benefits. The values for parameters are as follows:

$$a = 1; q_1 = 0.15; q_2 = 0.18; c_1 = 1; c_2 = 1.2; w = 1; v = 0.5; p = 5; r = .03;$$

The results are as follows:

TABLE 1. Compare Controls and Objective Values in Case 1

| | Catchability | Cost of COntrol | $J_i$ | $\parallel u_i \parallel_2$ |
|---|---|---|---|---|
| College 1 | 0.15 | 1 | 0.402 | 31.20 |
| College 2 | 0.18 | 1.2 | 0.720 | 29.37 |

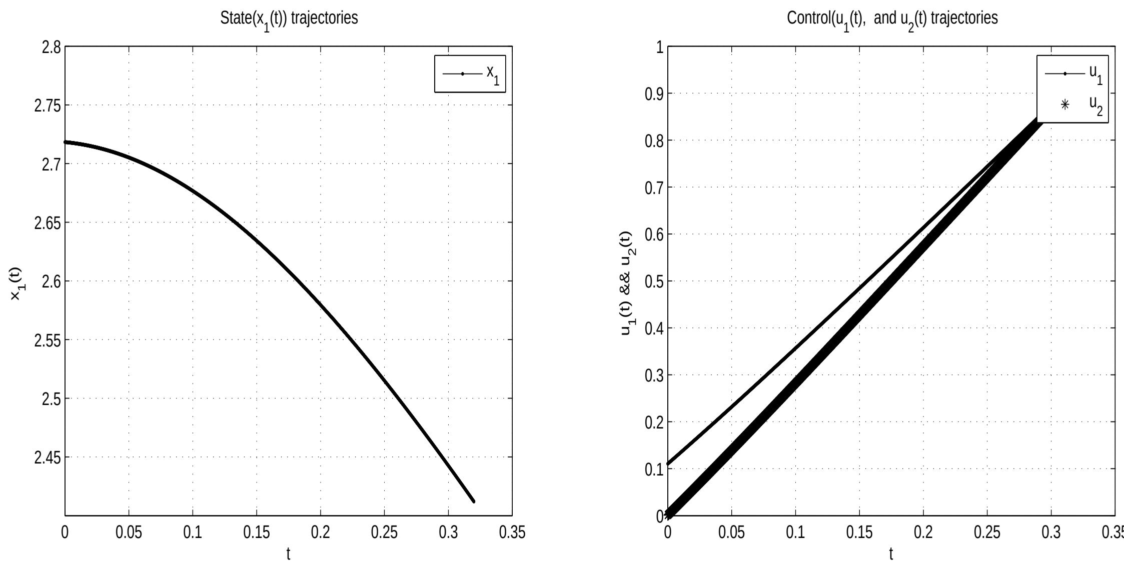


FIGURE 1. State and Control Trajectories

In Case 2, we simulate the competition with all coefficients being equal except for catchability. From Table (2) and Figure (2) we can see that college 1's objective value is much less than the other, and less than that in case 1. In this case, it is obvious that college 2 increases his control, and takes advantage of efficiency of his control to improve his objective value further. The values for parameters are as follows:

$$a = 1; q_1 = 0.15; q_2 = 0.18; c_1 = 1; c_2 = 1; w = 1; v = 0.5; p = 5; r = .03;$$

The results are as follows:

TABLE 2. Compare Controls and Objective Values in Case 2

| | Catchability | Cost of COntrol | $J_i$ | $\parallel u_i \parallel_2$ |
|---|---|---|---|---|
| College 1 | 0.15 | 1 | 0.438 | 30.13 |
| College 2 | 0.18 | 1 | 0.735 | 34.03 |

In Case 3, we simulate the situation where college 2 totally dominates college 1. From Table (3) and Figure (3) it is obvious that college 1 decrease his control level, which then decrease the total cost. However, in this case, college 1's objective value is better than case 1, but college 2 does worse than case 1, although his control cost

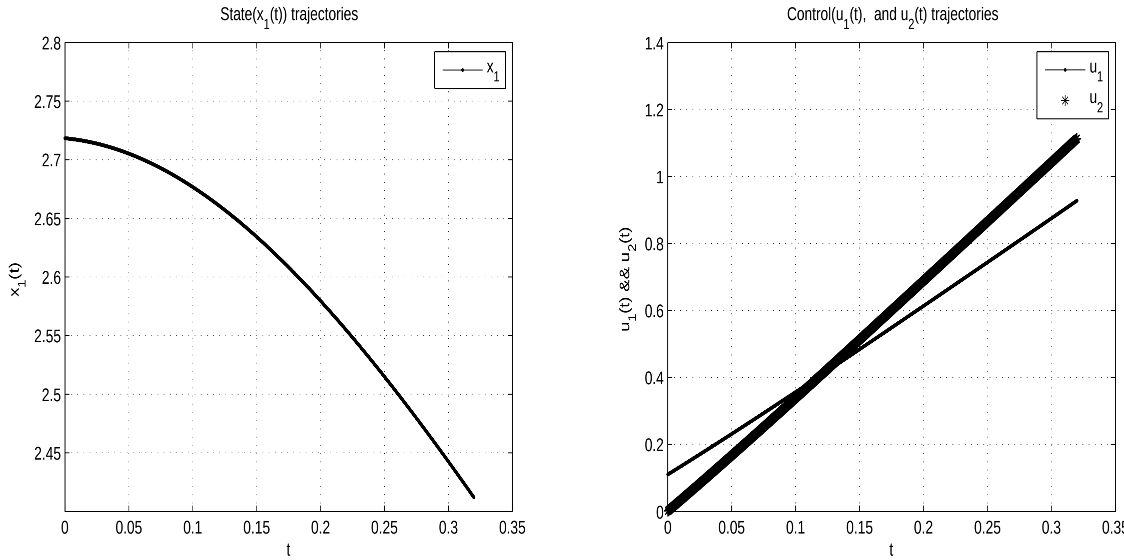


FIGURE 2. State and Control Trajectories

is less. This is because in objective function, the income from attracting students is calculated by the multiplication of fixed enrollment and number of students, and penalty for distance from ideal student enrollment.

$$a = 1; q_1 = 0.15; q2 = 0.18; c_1 = 1.2; c2 = 1; w = 1; v = 0.5; p = 5; r = .03;$$

The results are as follows:

TABLE 3. Compare Controls and Objective Values in Case 3

| | Catchability | Cost of COntrol | $J_i$ | $\parallel u_i \parallel_2$ |
|---|---|---|---|---|
| College 1 | 0.15 | 1.2 | 0.429 | 25.78 |
| College 2 | 0.18 | 1 | 0.709 | 34.95 |

Above all, from above three cases, we can conclude that in this competition, the strategy to attract students is the most important factor to affect Nash Equilibrium. Thus, it will be beneficial to use better technique to improve its strategies and techniques, even by sacrificing the cost of control.

## 7. **Conclusion**

In this paper, we studied the competition among two HBCU's in Orangeburg, SC, which are Claflin University and South Carolina State University. Based on the properties of their competition, we adopted $n$ person non-cooperative DG model to analyze this competition. In this model, we adopted open-loop controls, derived Pontryagin optimality condition for its Nash equilibrium, which is DAE system. Because of nonlinearity of DAE system, we developed numerical methods to solve it. The

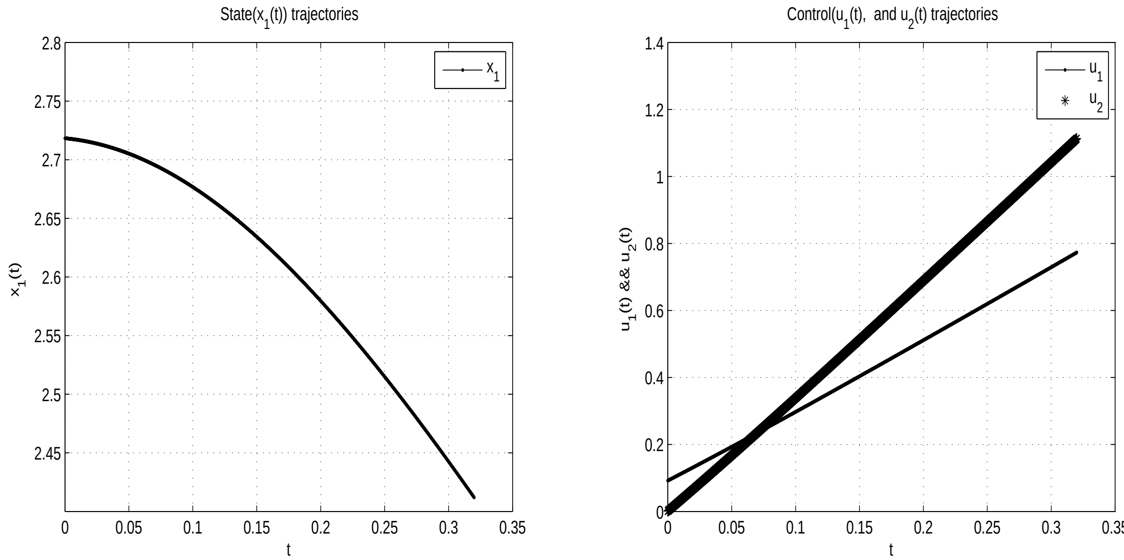


FIGURE 3. State and Control Trajectories

algorithm we designed is based on Quasi-Newton Method, which supplies the second order convergence. For the DG model in this paper, it converges with 4 iterations. From the numerical results, we are able to see that in this competition between two colleges, the effort of attracting students is the key to improve benefits; the competitor's preference between those two objectives will directly affect control trajectories. Another observation is that the optimality condition of DG is BVP or DAE, so it is difficult to solve it analytically. It will be easy to use the algorithm developed in this paper as a prototype to develop numerical methods to solve more general DG models.

## 8. **Acknowledgement**

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## REFERENCES

[1] A. W. Starr, Y. C. Ho, *Nonzero-Sum Differential Games*, Journal of Optimization Theory and Applications, 3, (1969) 184–206.

[2] Botkin, N. D, *Evaluation of Numerical Construction Error in Differential Game with Fixed Terminal Time*, Problems of Control and Information Theory,11, (1982) 283–295.

[3] Carl Chiarella, Murray C. Kemp, Ngo Nan Long, *On the Economics of International Fisheries*, International Economic Review, 25, (1984) 85–92.

[4] Clark Cohn W., *Restricted Access to Common-property Fishery Resources: A Game-Theoretic Analysis*, in: P. T. Liu, ed., Dynamic Optimization and Mathematical Economics. pp. 117–132, Plenum Press, New York.

[5] Charles Plourde, David Yeung, *Harvesting of a Transbourdary Replenishable Fish Stock: A Noncooperative Game Solution*, Marine Resource Economics, 6, (1989) 57–70.

[6] Colin W. Clark, *Restricted Access to Common-Property Fisher Resources: A Game-Theoretic Analysis*, Dynamic Optimization and mathematical economics, March 1, (1980) 117–132.

[7] E. Dockner, G. Feichtinger, A. Mehlmann, *Noncooperative solutions for a differential game model of a fishery*, Journal of Economic Dynamics and Control 13, (1989) 1–20.

[8] FLEMING, W. H., *The convergence problem for differential games*, Journal of Mathematical Analysis and Applications 3, (1961) 102–116.

[9] FRIEDMAN, A, *On the definition of differential games and the existence of value and of saddle points*, Journal of Differential Equations 7, (1970) 69–91.

[10] Gordon H. S, *The economic theory of a common property resource: the fishery*, Journal of Political Economy 62, (1954) 124–142.

[11] Leitmann George, *The Calculus of Variations and Optimal Control: An Introduction*, Plenum Press-New York and London, 1981.

[12] Leonard D. Berkovitz, *Two Person Zero Sum Differential Games: An Overview*, The Theory and Application of Differential Games-Proceedings of the NATO advanced Study Institute held at the University of Warwick, Coverntry, England, 27 August-6 Sept, 1974.

[13] Negash Medhin, Wei Wan, *Marketing Competition in the Middle of Product Life-cycle: A Differential Game Analysis*, International Journal of Pure and Applied Mathematics, 60 (2010) 119–148.

[14] Negash Medhin, Wei Wan, *Competition in the Last Stage of Product Life-cycle: A Differential Games Approach*, Communications in Applied Analysis, 12, (2008) 113–136.

[15] Negash Medhin, Wei Wan, *Leader-Follower Games in Marketing: A Differential Game Approach*, International Journal of Mathematics in Operational Research, 2, (2010) 56–71.

[16] Negash Medhin, Wei Wan, *Multi-New Product Competition in Duopoly: A Differential Game Analysis*, Dynamic Systems and Applications, 8 (2009) 161–178.

[17] Plourde Charles, Yeung David, *Harvesting of a Transboundary Replenishable Fish Stock: A Noncooperative Game Solution*, Marine Resource Economics, 6,(1989) 57–70.

[18] Pontryagin L. S, *On the Theory of Differential Games*, ussian Mathematical Surveys, 21, (1966) 193–246.

[19] Reinganum J. F., N.L. Stokey, *Oligopolistic Extraction of a Non-renewable, Common Property Resource: The Importance of the Period of Commitment in Dynamic Game*, Northwestern University, 1981.

[20] Raimo P. Hamalainen, Veijo Kaitala, Alain Haurie, *Bargaining on whales: A differential game model with pareto optimal equilibria*, OPERATIONS RESEARCH LETTERS, 3, (1981) 5–11.

[21] ISAACS Rufus, *Differential Games: A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization*, Dover Publications, 1999.

[22] S. Vajda, *The Theory of Games and Linear Programming*, METHUEN INC, Jan. 1956.

[23] Thomas R. Fischer, *A Hierarchical Game Approach to Fishery Resource Management*, IEEE Transactions on Systems, Man and Cybernetics, Vol 11, No. 3, March 1981.

[24] Wei Wan, Yuanyuan Peng, *Numerical Solution of Differential Game Models*, International Journal of Neural, Parallel, and Scientific Computations, 18, (2010), 451–460.