Neural, Parallel, and Scientific Computations 24 (2016) 419-430

JOST SOLUTION AND THE SPECTRUM OF THE DISCRETE STURM-LIOUVILLE EQUATIONS WITH HYPERBOLIC EIGENPARAMETER

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ABSTRACT. In this paper, spectral analysis of discrete Sturm-Liouville equation with boundary condition is taken under investigation for hyperbolic eigenparameter. Introducing the Jost solution and spectrum of the problem, we established several analogies between hyperbolic eigenparameter and trigonometric eigenparameter cases in [1, 4].

AMS (MOS) Subject Classification. 39A70, 47A10, 47A75.

1. PRELIMINARIES

Spectral analysis of operators have received a lot of attention by various authors since they play a crucial role in the solutions of certain problems in engineering, quantum mechanics, economics and control theory [17–20]. In particular, spectral properties of some discrete equations have been taken into consideration in [1, 4, 5, 6, 9].

Let the discrete boundary value problem (BVP)

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in \mathbb{N} = \{1, 2, \ldots\}, \tag{1.1}$$

$$y_0 = 0, \tag{1.2}$$

where (a_n) and (b_n) are complex sequences, $a_0 \neq 0$ and λ is a spectral parameter. In [9], it has been shown that the spectrum of the BVP (1.1), (1.2) consists of a continuous spectrum, eigenvalues and spectral singularities. Also, the spectral theory of discrete operators have been studied in connection with the classical moment problem in [2, 3, 16]. It is also worth to point out here that the Jost solutions are especially useful in the study of the spectral analysis of differential and difference operators [1, 5–9]. Therefore the Jost solutions of Dirac systems, Schrödinger and discrete Sturm-Liouville equations have been obtained in [4, 12, 14, 15].

Received April 3, 2016

Let us consider the non-selfadjoint boundary value problem (BVP) for the difference equation of second order

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in \mathbb{N} = \{1, 2, \ldots\}, \tag{1.3}$$

$$(\gamma_0 + \gamma_1\lambda)y_1 + (\beta_0 + \beta_1\lambda)y_0 = 0, \quad \gamma_0\beta_1 - \gamma_1\beta_0 \neq 0, \quad \gamma_1 \neq \frac{\beta_0}{a_0}, \tag{1.4}$$

where (a_n), (b_n), $n \in \mathbb{N}$ are complex sequences, $a_n \neq 0$ for all $n \in \mathbb{N} \cup \{0\}$, γ_i, $\beta_i \in \mathbb{C}$, $i = 0, 1$. Note that, (1.3) can be rewritten in the following Sturm-Liouville form:

$$\bigtriangledown(a_n \Delta y_n) + h_n y_n = \lambda y_n, \quad n \in \mathbb{N},$$

where $h_n = a_{n-1} + a_n + b_n$, $\triangle$ is the forward difference operator, $\triangle\, y_n = y_{n+1} - y_n$ and $\bigtriangledown$ is the backward difference operator, $\bigtriangledown y_n = y_n - y_{n-1}$.

There have been many studies concerning discrete Sturm-Liouville, discrete Dirac and discrete Schrödinger problems on upper half-plane. Differently other studies in the literature, discrete Sturm-Liouville equation with boundary condition is taken under investigation for hyperbolic eigenparameter, as a result, representation of Jost solution has changed. As a consequence of this, analicity region of Jost solution and application region of Naimark and Pavlov conditions have shifted from upper half-plane to left half-plane. This new approach will provide a wide perspective on applications of these problems in physics, economics and engineering.

In this paper, we investigate the eigenvalues and spectral singularities of the BVP (1.3), (1.4) and prove that this BVP has a finite number of eigenvalues and spectral singularities with finite multiplicities if

$$\sup_{n\in\mathbb{N}} \exp(\varepsilon n^{\delta})(|1 - a_n| + |b_n|) < \infty$$

for some $\varepsilon > 0$ and $\frac{1}{2} \leq \delta \leq 1$.

2. JOST SOLUTION AND JOST FUNCTION OF (1.3), (1.4)

Suppose that complex sequences (a_n) and (b_n) satisfy

$$\sum_{n\in\mathbb{N}} n(|1 - a_n| + |b_n|) < \infty. \tag{2.1}$$

Under the condition (2.1), Eq. (1.3) has the solution

$$e_n(z) = \alpha_n e^{nz}\left(1 + \sum_{m=1}^{\infty} A_{nm} e^{mz}\right), \quad n \in \mathbb{N} \cup \{0\}, \tag{2.2}$$

for $\lambda = 2\cosh z$ where $z \in \overline{\mathbb{C}}_{left} := \{z : z \in \mathbb{C}, \operatorname{Re} z \le 0\}$ and A_{nm}, α_n are expressed in terms of (a_n) and (b_n) as

$$\begin{aligned}
\alpha_n &= \left\{ \prod_{k=n}^{\infty} a_k \right\}^{-1}, \\
A_{n1} &= -\sum_{k=n+1}^{\infty} b_k, \\
A_{n2} &= \sum_{m=1}^{\infty} \left\{ \left(1 - a_k^2\right) + b_k \sum_{s=k+1}^{\infty} b_s \right\}, \\
A_{nm} &= A_{n+1,m-2} + \sum_{k=n+1}^{\infty} \left\{ \left(1 - a_k^2\right) A_{k+1,m-2} - b_k A_{k,m-1} \right\}, \quad m = 3, 4, \ldots.
\end{aligned}$$

Moreover A_{nm} satisfy

$$|A_{nm}| \le C \sum_{k=n+[\frac{m}{2}]} \left(|1 - a_k| + |b_k|\right) \tag{2.3}$$

where $C > 0$ is constant and $[\frac{m}{2}]$ is the integer part of $\frac{m}{2}$. Hence, $e_n(z)$ is analytic with respect to z in $\mathbb{C}_{left} := \{z : z \in \mathbb{C}, \operatorname{Re} z < 0\}$ and continuous in $\operatorname{Re} z = 0$.

Using the boundary condition (1.3) and (2.2), we define the function f,

$$f(z) = (\gamma_0 + 2\gamma_1 \cosh z) e_1(z) + (\beta_0 + 2\beta_1 \cosh z) e_0(z). \tag{2.4}$$

The function f is analytic in $\mathbb{C}_{left}$, continuous in $\overline{\mathbb{C}}_{left}$ and $f(z) = f(z + 2\pi i)$.

Analogously to the Sturm-Liouville differential equation, the solution

$$e(z) = \{e_n(z)\}, \quad n \in \mathbb{N} \cup \{0\}$$

and the function f are called the Jost solution and Jost function of (1.3), (1.4), respectively ([20]).

Let $\widehat{\varphi}(\lambda) = \{\widehat{\varphi}_n(\lambda)\}$, $n \in \mathbb{N} \cup \{0\}$ be the solution of (1.3) satisfying the initial conditions

$$\widehat{\varphi}_0(\lambda) = (\gamma_0 + \gamma_1 \lambda), \quad \widehat{\varphi}_1(\lambda) = -(\beta_0 + \beta_1 \lambda).$$

If we define

$$\varphi(z) = \widehat{\varphi}(2\cosh z) = \{\widehat{\varphi}_n(2\cosh z)\}, \quad n \in \mathbb{N} \cup \{0\}$$

then φ is entire function and

$$\varphi(z) = \varphi(z + 2\pi i).$$

Let us take the semi-strip $P_0 := \left\{z : z \in \mathbb{C}, z = \xi + i\tau, -\frac{\pi}{2} \le \tau \le \frac{3\pi}{2}, \xi < 0\right\}$ and $P := P_0 \cup \left\{z : z \in \mathbb{C}, z = \xi + i\tau, -\frac{\pi}{2} \le \tau \le \frac{3\pi}{2}, \xi = 0\right\}$.

For all $z \in P$ and $f(z) \neq 0$, we define

$$G_{nk}(z) = \begin{cases} -\frac{\varphi_k(z)e_n(z)}{a_0 f(z)}, & k \leq n, \\ -\frac{\varphi_n(z)e_k(z)}{a_0 f(z)}, & k > n. \end{cases} \tag{2.5}$$

The function $G_{nk}(z)$ is called the Green function of the BVP (1.3), (1.4). It is clear that for $g = (g_k)$ and $k \in \mathbb{N} \cup \{0\}$,

$$(Rg)_n := \sum_{k=0}^{\infty} G_{nk}(z) g_k, \quad n \in \mathbb{N} \cup \{0\} \tag{2.6}$$

is the resolvent of the BVP (1.3), (1.4).

3. CONTINUOUS SPECTRUM OF (1.3) AND (1.4)

Let L denote the difference operator of second order generated in $l^2(\mathbb{N})$ by the difference expression

$$(ly)_n = a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1}, \quad n \in \mathbb{N} = \{1, 2, \ldots\},$$

where (a_n), (b_n), $n \in \mathbb{N}$ are complex sequences, $a_n \neq 0$ for all $n \in \mathbb{N} \cup \{0\}$. Note that (1.3) can be rewritten as

$$(ly)_n = \lambda y_n.$$

Suppose that L_1 and L_2 denote difference operators in $l^2(\mathbb{N})$ by

$$(l_1 y)_n := y_{n-1} + y_{n+1}, \tag{3.1}$$

$$(l_2 y)_n := (a_{n-1} - 1)y_{n-1} + (a_n - 1)y_{n+1} + b_n y_n, \quad n \in \mathbb{N}, \tag{3.2}$$

respectively. It is evident that $L = L_1 + L_2$ and $L_1 = L_1^*$. L_1 is bounded in Hilbert space $l^2(\mathbb{N})$ and it is obvious that e^{nz} and e^{-nz} are solutions of $(l_1 y)_n = \lambda y_n$ for $\lambda = 2\cosh z$ and $z \in \overline{\mathbb{C}}_{left}$.

Definition 3.1 ([1])**.** The Wronskian of two solutions $y = \{y_n(\lambda)\}$ and $u = \{u_n(\lambda)\}$ of (1.1) is defined by

$$W[y, u] = a_n[y_n(\lambda)u_{n+1}(\lambda) - y_{n+1}(\lambda)u_n(\lambda)], \quad n \in \mathbb{N} \cup \{0\}.$$

From definition (3.1), we get

$$W\left[e^{nz}, e^{-nz}\right] = a_n\left[e^{nz}e^{-(n+1)z} - e^{(n+1)z}e^{-nz}\right] = -2\sinh z, \quad z \in \overline{\mathbb{C}}_{left}.$$

For all $z \in P$ and $a_0 = 1$, we define the Green function of L_1 by

$$\varsigma_{nk}(z) = \begin{cases} \frac{e^{-kz}e^{nz}}{2\sinh z}, & k < n, \\ \frac{e^{kz}e^{-nz}}{2\sinh z}, & k \geq n. \end{cases} \tag{3.3}$$

It is easy to see that

$$(R_\lambda(L_1)\psi)_n = \sum_{k\in\mathbb{N}} \varsigma_{nk}(z)\psi_k \quad \text{for } \psi \in l^2(\mathbb{N}) \tag{3.4}$$

is resolvent of L_1.

Lemma 3.2. *For every $\vartheta > 0$, there is a number c_z such that*

$$\|R_\lambda(L_1)\| \geq \frac{c_z}{|\sinh z|\sqrt{1-e^{2\operatorname{Re} z}}} \tag{3.5}$$

for all $z \in \{z \in P_0 : |Re\ z| < \vartheta\}$.

Proof. Let us define the function

$$g_k(z) := \begin{cases} 0, & k < n \\ e^{-k\widetilde{z}}, & k \geq n \end{cases}$$

where $z \in \mathbb{C}$, $z = x + iy$ and $\widetilde{z} = -x + iy$. Thus, we obtain

$$||g_k(z)||^2 = \sum_{k=n}^{\infty} \left|e^{-k\widetilde{z}}\right|^2 = \sum_{k=n}^{\infty} \left|e^{2k\operatorname{Re} z}\right| < \infty.$$

Then $g_k(z) \in l^2(\mathbb{N})$. Using (3.3) and (3.4), we obtain

$$(R_\lambda(L_1)g)_n = \frac{e^{-nz}}{2\sinh z} ||g_k(z)||^2 .$$

Since

$$\left|e^{-nz}\right| \geq e^{n\operatorname{Re} z},$$

we get that

$$\begin{aligned} ||R_\lambda(L_1)g||^2 &= \frac{||g_k(z)||^4}{4|\sinh z|^2} \sum_{n=1}^{\infty} \left|e^{-nz}\right|^2 \\ &\geq \frac{||g_k(z)||^4}{4|\sinh z|^2} \sum_{n=k}^{\infty} (e^{n\operatorname{Re} z})^2 \\ &= \frac{||g_k(z)||^4}{4|\sinh z|^2} \sum_{n=k}^{\infty} \left(e^{2\operatorname{Re} z}\right)^n \\ &= \frac{||g_k(z)||^4 e^{2\operatorname{Re} zk}}{4|\sinh z|^2 (1-e^{2\operatorname{Re} z})} \end{aligned}$$

By choosing $c_z = \frac{||g_k(z)||e^{\operatorname{Re} zk}}{2}$, the proof is complete. □

Theorem 3.3. $\sigma(L_1) = \sigma_c(L_1) = [-2, 2]$, *where $\sigma(L_1)$ and $\sigma_c(L_1)$ denote the spectrum and continuous spectrum of the operator L_1, respectively.*

Proof. ($\Rightarrow$) It is easy to see that L_1 is self-adjoint and bounded operator. So, the spectrum of the operator L_1 consists of only its continuous spectrum. Firstly, take an arbitrary element $\lambda_0 \in \sigma_c(L_1)$. Then, limit of $\|R_{\lambda_0}(L_1)\|$ is obtained from the definition of continuous spectrum [20]. From this definition, if $\lambda_0 \in \sigma_c(L)$, then $R_{\lambda_0}(L)$ is unbounded which implies $\|R_{\lambda_0}(L_1)\| \to \infty$. From the definition of $R_\lambda(L_1)$ in (3.3) and (3.4), we get $\|R_{\lambda_0}(L_1)\| \to \infty \Rightarrow \operatorname{Re} z_0 \to 0$. If $\operatorname{Re} z_0 \to 0$ then, $\lambda_0 = 2\cosh z_0 \in [-2,2]$.

($\Leftarrow$) Now, take an arbitrary element $\lambda = 2\cosh z \in [-2,2]$. It is clear that $\lambda = 2\cosh z \in [-2,2] \Rightarrow \operatorname{Re} z = 0$. From (3.5), we find $\|R_\lambda(L_1)\| \to \infty$ for $\operatorname{Re} z \to 0$. S o, we show that $\lambda \in \sigma_c(L)$. This completes the proof. □

Theorem 3.4. *If* (2.1) *holds, then* $\sigma_c(L) = [-2,2]$.

Proof. Since L_2 is compact operator in $l^2(\mathbb{N})$, using Theorem 3.3 and Weyl's theorem of a compact perturbation [13, p. 13], we get

$$\sigma_c(L) = \sigma(L_1) = \sigma_c(L_1) = [-2,2]\,.$$

□

4. EIGENVALUES AND SPECTRAL SINGULARITIES OF (1.3) AND (1.4)

We will denote the set of all eigenvalues and spectral singularities of the BVP (1.3) and (1.4) by σ_d and σ_{ss}, respectively. From (2.5) and (2.6) and the definition of the eigenvalues and the spectral singularities, we have [20]

$$\sigma_d = \left\{\lambda : \lambda = 2\cosh z,\ z \in P_0,\ f(z) = 0\right\}, \tag{4.1}$$

$$\sigma_{ss} = \left\{\lambda : \lambda = 2\cosh z, z = \xi + i\tau, \xi = 0, \tau \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right], f(z) = 0\right\} \setminus \{0\}\,. \tag{4.2}$$

From (2.2) and (2.4), we obtain

$$\begin{aligned} f(z) &= \left[\gamma_0 + \gamma_1\left(e^z + e^{-z}\right)\right]\left[\alpha_1 e^z\left(1 + \sum_{m=1}^{\infty} A_{1m} e^{mz}\right)\right] \\ &\quad + \left[\beta_0 + \beta_1\left(e^z + e^{-z}\right)\right]\left[\alpha_0\left(1 + \sum_{m=1}^{\infty} A_{0m} e^{mz}\right)\right] \\ &= \alpha_0\beta_1 e^{-z} + \alpha_1\gamma_1 + \alpha_0\beta_0 + (\alpha_1\gamma_0 + \alpha_0\beta_1)e^z + \alpha_1\gamma_1 e^{2z} \\ &\quad + \sum_{m=1}^{\infty} \alpha_0\beta_1 A_{0m} e^{(m-1)z} + \sum_{m=1}^{\infty}\left(\alpha_1\gamma_1 A_{1m} + \alpha_0\beta_0 A_{0m}\right)e^{mz} \\ &\quad + \sum_{m=1}^{\infty}\left(\alpha_1\gamma_0 A_{1m} + \alpha_0\beta_1 A_{0m}\right)e^{(m+1)z} + \sum_{m=1}^{\infty} \alpha_1\gamma_1 A_{1m} e^{(m+2)z}. \end{aligned}$$

Let

$$F(z) := f(z)e^z; \tag{4.3}$$

then, the function F is analytic in $\mathbb{C}_{left}$, continuous in $\overline{\mathbb{C}}_{left}$,

$$F(z) = F(z + 2\pi i)$$

and

$$\begin{aligned} F(z) &= \alpha_0\beta_1 + (\alpha_1\gamma_1 + \alpha_0\beta_0)\, e^z + [\alpha_1\gamma_0 + \alpha_0\beta_1]\, e^{2z} + (\alpha_1\gamma_1)e^{3z} \\ &+ \sum_{m=1}^{\infty} \alpha_0\beta_1 A_{0m} e^{mz} + \sum_{m=1}^{\infty} (\alpha_1\gamma_1 A_{1m} + \alpha_0\beta_0 A_{0m})\, e^{(m+1)z} \\ &+ \sum_{m=1}^{\infty} (\alpha_1\gamma_0 A_{1m} + \alpha_0\beta_1 A_{0m})\, e^{(m+2)z} + \sum_{m=1}^{\infty} \alpha_1\gamma_1 A_{1m} e^{(m+3)z}. \end{aligned} \tag{4.4}$$

It follows from (4.1)–(4.3) that

$$\sigma_d = \{\lambda : \lambda = 2\cosh z, z \in P_0, F(z) = 0\}, \tag{4.5}$$

$$\sigma_{ss} = \left\{\lambda : \lambda = 2\cosh z, z = \xi + i\tau, \xi = 0, \tau \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right], F(z) = 0\right\} \setminus \{0\} \tag{4.6}$$

Definition 4.1. The multiplicity of a zero of F in P is called the multiplicity of the corresponding eigenvalue or spectral singularity of the BVP (1.3) and (1.4).

From (4.5) and (4.6) we get that, in order to investigate the quantitative properties of the sets σ_d and σ_{ss}, we need to discuss the quantitive properties of the zeros of F in P. Let us define

$$\begin{aligned} A_1 &:= \{z : z \in P_0, F(z) = 0\}, \\ A_2 &:= \left\{z : z = \xi + i\tau, \xi = 0, \tau \in \left[-\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right], F(z) = 0\right\}. \end{aligned} \tag{4.7}$$

We also denote the set of all limit points of A_1 by A_3 and the set of all zeros of F with infinite multiplicity by A_4.

From (4.5), (4.6) and (4.7) we find that

$$\sigma_d = \{\lambda : \lambda = 2\cosh z, z \in A_1\}, \quad \sigma_{ss} = \{\lambda : \lambda = 2\cosh z,\ z \in A_2\} \setminus \{0\}. \tag{4.8}$$

Theorem 4.2. *If* (2.1) *holds, then*

(i) *The set A_1 is bounded and countable.*
(ii) $A_1 \cap A_3 = \varnothing$, $A_1 \cap A_4 = \varnothing$.
(iii) *The set A_2 is compact and $\mu(A_2) = 0$, where μ is Lebesgue measure in the imaginary axis.*
(iv) $A_3 \subset A_2$, $A_4 \subset A_2$; $\mu(A_3) = \mu(A_4) = 0$.
(v) $A_3 \subset A_4$.

Proof. From (2.3) and (4.4), we have

$$F(z) = \begin{cases} \alpha_0\beta_1 + O(e^{\xi}), & \beta_1 \neq 0, z \in P, \xi \to -\infty, \\ (\alpha_1\gamma_1 + \alpha_0\beta_0)\, e^{z} + O(e^{2\xi}), & \beta_1 = 0, z \in P, \xi \to -\infty. \end{cases} \tag{4.9}$$

(4.9) shows that the set A_1 is bounded. Since F is analytic in $\mathbb{C}_{left}$ and is a $2\pi i$ periodic function, we get that A_1 has at most a countable number of elements. This proves (i). From the boundary uniqueness theorems of analytic functions, we obtain (ii)–(iv) [11]. Using the continuity of all derivatives of F on $\left\{z \in \mathbb{C} : z = \xi + i\tau, \xi = 0, \tau \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]\right\}$, we get (v). □

From Theorem 4.2 and (4.8), we have the following:

Theorem 4.3. *Under the condition* (2.1)*,*

(i) *the set of eigenvalues of the BVP* (1.3), (1.4) *is bounded, has at most a countable number of elements, and its limit points can lie only in* $[-2, 2]$*.*

(ii) $\sigma_{ss} \subset [-2, 2]$ *and* $\mu(\sigma_{ss}) = 0$*.*

Let us assume that the complex sequences (a_n) and (b_n) satisfy

$$\sup_{n\in\mathbb{N}} \left[\exp(\varepsilon n^{\delta})\left(|1 - a_n| + |b_n|\right)\right] < \infty, \tag{4.10}$$

for some $\varepsilon > 0$ and $\frac{1}{2} \leq \delta \leq 1$. Note that, for $\delta = 1$, (4.10) reduces to the condition

$$\sup_{n\in\mathbb{N}} \left[\exp(\varepsilon n)\left(|1 - a_n| + |b_n|\right)\right] < \infty. \tag{4.11}$$

Theorem 4.4. *Under the condition* (4.11)*, the BVP* (1.3), (1.4) *has a finite number of eigenvalues and spectral singularities with a finite multiplicity.*

Proof. From (2.3), we obtain that

$$|A_{nm}| \leq C \exp\left[-\frac{\varepsilon}{4}(n + m)\right], \quad n, m \in \mathbb{N}$$

where $C > 0$ is a constant. Using (4.4), we observe that the function F has an analytic continuation to the right half-plane Re $z < \frac{\varepsilon}{4}$. Since F is a $2\pi i$ periodic function, the limit points of its zeros in P cannot lie in $\left\{z \in \mathbb{C} : z = \xi + i\tau, \xi = 0, \tau \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]\right\}$. Also we obtain that the bounded sets A_1 and A_2 have a finite number of elements from Bolzano-Weierstrass Theorem and Theorem 4.2. Using the analyticity of F in Re $z < \frac{\varepsilon}{4}$, we get that all zeros of F in P have a finite multiplicity. Therefore we obtain the finiteness of the eigenvalues and the spectral singularities of the BVP (1.3) and (1.4). □

Now let us suppose that

$$\sup_{n\in\mathbb{N}} \exp(\varepsilon n^{\delta})\left(|1 - a_n| + |b_n|\right) < \infty, \quad \varepsilon > 0, \quad \frac{1}{2} \leq \delta < 1, \tag{4.12}$$

which is weaker than (4.11). It is seen that the condition (4.11) guarantees the analytic continuation of F from the imaginary axis to the right half-plane. Consequently, the finiteness of the eigenvalues and the spectral singularities of the BVP (1.3), (1.4) is achieved as a result of this analytic continuation. It is evident that, under the condition (4.12), the function F is analytic in $\mathbb{C}_{left}$ and infinitely differentiable on imaginary axis. But F does not have an analytic continuation from the imaginary axis to the right half-plane. Therefore, under the condition (4.12), the finiteness of the eigenvalues and the spectral singularities of the BVP (1.3), (1.4) cannot be shown in a way similar to Theorem 4.4.

Under the condition (4.12), to show that the eigenvalues and the spectral singularities of the BVP (1.3), (1.4) are of finite number, we will use the following:

Theorem 4.5 ([5]). *Let us assume that the $2\pi i$ periodic function g is analytic in $\mathbb{C}_{left}$, all of its derivatives are continuous in $\overline{\mathbb{C}}_{left}$, and*

$$\sup_{z\in P}\left|g^{(k)}(z)\right| \le \eta_k, \quad k \in \mathbb{N}\cup\{0\}.$$

If the set $G \subset \left\{z\in\mathbb{C} : z=\xi+i\tau, \xi=0, \tau\in\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]\right\}$ with Lebesgue measure zero is the set of all zeros the function g with infinite multiplicity in P, and if

$$\int_0^\omega \ln t(s)d\mu(G_s) = -\infty,$$

where $t(s)=\inf_k \frac{\eta_k s^k}{k!}$ and $\mu(G_s)$ is the Lebesgue measure of s-neighborhood of G and $\omega>0$ is an arbitrary constant, then $g\equiv 0$ in $\overline{\mathbb{C}}_{left}$.

It follows from (2.3) and (4.4) that

$$\left|F^{(k)}(z)\right| \le \eta_k, \quad k\in\mathbb{N}\cup\{0\},$$

where

$$\eta_k = 4^k C\sum_{m=1}^{\infty} m^k \exp(-\varepsilon m^\delta).$$

We can get the estimate for η_k:

$$\eta_k \le 4^k C\int_0^\infty x^k\exp(-\varepsilon x^\delta)dx \le Dd^k k! k^{k\frac{1-\delta}{\delta}}, \tag{4.13}$$

where D and d are constants depending on C, ε and δ.

Theorem 4.6. *If (4.12) holds, then $A_4=\varnothing$.*

Proof. Using Theorem 4.5, we obtain that the function F satisfies the condition

$$\int_0^\omega \ln t(s)d\mu(A_{4,s}) > -\infty, \tag{4.14}$$

where $t(s) = \inf\limits_k \frac{\eta_k s^k}{k!}$, $k\in\mathbb{N}\cup\{0\}$, $\mu(A_{4,s})$ is the Lebesgue measure of the s-neighborhood of A_4, and η_k is defined by (4.13).

Now we get

$$t(s) = D \inf_k [k^{k(\frac{1}{\delta}-1)}(ds)^k].$$

To be able to determine $t(s)$, we define the auxilary function $h(x) := x^{x(\frac{1}{\delta}-1)}(ds)^x$ for $x \in [0, \infty)$ and calculate the minumum of $h(x)$ as $x_0 = e^{-1}(ds)^{\frac{-\delta}{1-\delta}}$. So that we find

$$t(s) \le D \exp\left\{ -\frac{1-\delta}{\delta} e^{-1} d^{-\frac{\delta}{1-\delta}} s^{-\frac{\delta}{1-\delta}} \right\}. \tag{4.15}$$

It follows from (4.14) and (4.15) that

$$\int_0^\omega s^{-\frac{\delta}{1-\delta}} d\mu(A_{4,s}) < \infty. \tag{4.16}$$

From (4.16) and since $\frac{\delta}{1-\delta} \ge 1$, we obtain that for arbitrary s, $\mu(A_{4,s}) = 0$ or $A_4 = \varnothing$. □

Theorem 4.7. *Under the condition* (4.12)*, the BVP* (1.3), (1.4) *has a finite number of eigenvalues and spectral singularities with finite multiplicity.*

Proof. To be able to prove the theorem we have to show that the function F has a finite number of zeros with finite multiplicities in P. Using Theorem 4.2 and Theorem 4.6, we find that $A_3 = \varnothing$. So the bounded sets A_1 and A_2 have no limit points, i.e., the function F has only a finite number of zeros in P. Since $A_4 = \varnothing$, these zeros are of finite multiplicity. □

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