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DIFFERENTIAL EQUATIONS ARISING FROM BELL-CARLITZ POLYNOMIALS AND COMPUTATION OF THEIR ZEROS

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ABSTRACT. In this paper, we study differential equations arising from the generating functions of the Bell-Carlitz polynomials. We give explicit identities for the Bell-Carlitz polynomials. Finally, we investigate the zeros of the Bell-Carlitz polynomials by using computer.

Key Words Differential equations, Bell polynomials, Bell-Carlitz polynomials,

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1. Introduction

Recently, many mathematicians have studied in the area of the Bernoulli numbers, Euler numbers, Genocchi numbers, and tangent numbers (see [2, 3, 4, 6, 7, 8, 9, 10]). The Bell-Carlitz polynomials $B_n^c(x)$ $(n \geq 0)$, were introduced by Alain M. Robert (see [5]).

The Bell-Carlitz polynomials $B_n^c(x)$ are defined by the generating function:

$$F = F(t,x) = \sum_{n=0}^{\infty} B_n^c(x)\frac{t^n}{n!} = e^{(xt+e^t-1)} \text{ (see [5]).} \tag{1.1}$$

First few examples of Bell-Carlitz polynomials are

$$\begin{aligned}
&B_0^c(x) = 1, \quad B_1^c(x) = 1 + x, \quad B_2^c(x) = 2 + 2x + x^2,\\
&B_3^c(x) = 5 + 6x + 3x^2 + x^3,\\
&B_4^c(x) = 15 + 20x + 12x^2 + 4x^3 + x^4,\\
&B_5^c(x) = 52 + 75x + 50x^2 + 20x^3 + 5x^4 + x^5,\\
&B_6^c(x) = 203 + 312x + 225x^2 + 100x^3 + 30x^4 + 6x^5 + x^6,\\
&B_7^c(x) = 877 + 1421x + 1092x^2 + 525x^3 + 175x^4 + 42x^5 + 7x^6 + x^7.
\end{aligned} \tag{1.2}$$

It is well known, the Bell numbers B_n are given by the generating function

$$e^{(e^t-1)} = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!}, \text{ (see [5]).} \tag{1.3}$$

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From (1.1), we see that

$$
\begin{aligned}
\sum_{n=0}^{\infty} B_n^c(x)\frac{t^n}{n!} &= e^{(xt+e^t-1)} \\
&= e^{(e^t-1)}e^{xt} \\
&= \left(\sum_{k=0}^{\infty} B_k \frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty} x^m \frac{t^m}{m!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k}\right)\frac{t^n}{n!}.
\end{aligned} \tag{1.4}
$$

Comparing the coefficients on both sides of (1.4), we obtain

$$
B_n^c(x) = \sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k} \quad (n \geq 0). \tag{1.5}
$$

Recently, nonlinear differential equations arising from the generating functions of special polynomials are studied by many authors in order to give explicit identities for special polynomials (see [3, 4, 7]). In this paper, we study differential equations arising from the generating functions of Bell-Carlitz polynomials. We give explicit identities for the Bell-Carlitz polynomials. In addition, we investigate the zeros of the Bell-Carlitz polynomials using numerical methods.

2. Differential equations associated with Bell-Carlitz polynomials

In this section, we study linear differential equations arising from the generating functions of Bell-Carlitz polynomials.

Let

$$
F = F(t,x) = e^{(xt+e^t-1)} = \sum_{n=0}^{\infty} B_n^c(x)\frac{t^n}{n!}. \tag{2.1}
$$

Then, by (2.1), we have

$$
\begin{aligned}
F^{(1)} &= \frac{d}{dt}F(t,x) = \frac{d}{dt}\left(e^{xt+e^t-1)}\right) = e^{(xt+e^t-1)}(x+e^t) \\
&= (x+e^t)F,
\end{aligned} \tag{2.2}
$$

(2.3)

$$
\begin{aligned}
F^{(2)} &= \frac{d}{dt}F^{(1)} = e^t F + (x+e^t)F^{(1)} \\
&= e^t F + (x+e^t)^2 F = (x^2 + (2x+1)e^t + e^{2t})F,
\end{aligned}
$$

and

$$
F^{(3)} = \frac{d}{dt}F^{(2)} = \left(x^3 + (3x^2+3x+1)e^t + (2x+3)e^{2t} + e^{3t}\right)F.
$$

Continuing this process, we can guess that

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F(t,x) = \left(\sum_{i=0}^{N} a_i(N,x)e^{it}\right) F, \quad (N = 0,1,2,\ldots). \tag{2.4}$$

Differentiating (2.4) with respect to t, we have

$$\begin{aligned} F^{(N+1)} = \frac{dF^{(N)}}{dt} &= \left(\sum_{i=0}^{N} ia_i(N,x)e^{it}\right) F + \left(\sum_{i=0}^{N} a_i(N,x)e^{it}\right) F^{(1)} \\ &= \left(\sum_{i=0}^{N} ia_i(N,x)e^{it}\right) F + \left(\sum_{i=0}^{N} a_i(N,x)e^{it}\right)(x+e^t)F \\ &= \left\{\sum_{i=0}^{N}(x+i)a_i(N,x)e^{it} + \sum_{i=0}^{N} a_i(N,x)e^{(i+1)t}\right\} F \\ &= \left\{\sum_{i=0}^{N}(x+i)a_i(N,x)e^{it} + \sum_{i=1}^{N+1} a_{i-1}(N,x)e^{it}\right\} F. \end{aligned} \tag{2.5}$$

Now replacing N by $N+1$ in (2.4), we find

$$F^{(N+1)} = \left(\sum_{i=0}^{N+1} a_i(N+1,x)e^{it}\right) F. \tag{2.6}$$

Comparing the coefficients on both sides of (2.5) and (2.6), we obtain

$$a_0(N+1,x) = xa_0(N,x), \quad a_{N+1}(N+1,x) = a_N(N,x), \tag{2.7}$$

and

$$a_i(N+1,x) = a_{i-1}(N,x) + (x+i)a_i(N,x), (1 \le i \le N). \tag{2.8}$$

In addition, by (2.4), we have

$$F = F^{(0)} = a_0(0,x)F, \tag{2.9}$$

which gives

$$a_0(0,x) = 1. \tag{2.10}$$

It is not difficult to show that

$$\begin{aligned} (x+e^t)F = F^{(1)} &= \left(\sum_{i=0}^{1} a_i(1,x)e^{it}\right) F \\ &= a_0(1,x)F + a_1(1,x)e^t F. \end{aligned} \tag{2.11}$$

Thus, by (2.11), we also find

$$a_0(1,x) = x, \quad a_1(1,x) = 1. \tag{2.12}$$

From (2.7), we note that

$$a_0(N+1,x) = xa_0(N,x) = x^2 a_0(N-1,x) = \cdots = x^N a_0(1,x) = x^{N+1}, \tag{2.13}$$

and

$$a_{N+1}(N+1,x) = a_N(N,x) = a_{N-1}(N-1,x) = \cdots = a_1(1,x) = 1. \tag{2.14}$$

For $i = 1, 2, 3$ in (2.8), we have

$$a_1(N+1,x) = \sum_{k=0}^{N} (x+1)^k a_0(N-k,x), \tag{2.15}$$

$$a_2(N+1,x) = \sum_{k=0}^{N-1} (x+2)^k a_1(N-k,x), \tag{2.16}$$

and

$$a_3(N+1,x) = \sum_{k=0}^{N-2} (x+3)^k a_2(N-k,x). \tag{2.17}$$

Continuing this process, we can deduce that, for $1 \le i \le N$,

$$a_i(N+1,x) = \sum_{k=0}^{N-i+1} (x+i)^k a_{i-1}(N-k,x). \tag{2.18}$$

Note that, here the matrix $a_i(j)_{0 \le i,j \le N+1}$ is given by

$$\begin{pmatrix} 1 & x & x^2 & x^3 & \cdots & x^{N+1} \\ 0 & 1 & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 1 & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 1 & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{2.19}$$

Now, we give explicit expressions for $a_i(N+1,x)$. By (2.15), (2.16), and (2.17), we have

(2.20)

$$a_1(N+1,x) = \sum_{k_1=0}^{N} (x+1)^{k_1} a_0(N-k_1,x) = \sum_{k_1=0}^{N} (x+1)^{k_1} x^{N-k_1},$$

$$\begin{aligned} a_2(N+1,x) &= \sum_{k_2=0}^{N-1} (x+2)^{k_2} a_1(N-k_2,x) \\ &= \sum_{k_2=0}^{N-1} \sum_{k_1=0}^{N-1-k_2} (x+2)^{k_2} (x+1)^{k_1} x^{N-k_2-k_1-1}, \end{aligned}$$

and

$$\begin{aligned} a_3(N+1,x) &= \sum_{k_3=0}^{N-2} (x+3)^{k_3} a_2(N-k_3,x) \\ &= \sum_{k_3=0}^{N-2} \sum_{k_2=0}^{N-2-k_3} \sum_{k_1=0}^{N-2-k_3-k_2} (x+3)^{k_3} (x+2)^{k_2} (x+1)^{k_1} x^{N-k_3-k_2-k_1-2}. \end{aligned}$$

Continuing this process, we get

$$\begin{aligned} &a_i(N+1,x) \\ &= \sum_{k_i=0}^{N-i+1} \sum_{k_{i-1}=0}^{N-i+1-k_i} \cdots \sum_{k_1=0}^{N-i+1-k_i-\cdots-k_2} \left(\prod_{l=1}^{i} (x+l)^{k_l} \right) x^{N-i+1-\sum_{l=1}^{i} k_l}. \end{aligned} \tag{2.21}$$

Thus, by (2.21), the following theorem follows.

Theorem 1. *For $N = 0, 1, 2, \ldots$, the differential equation*

$$F^{(N)} = \left(\sum_{i=0}^{N} a_i(N,x) e^{it} \right) F$$

has a solution

$$F = F(t,x) = e^{(xt+e^t-1)},$$

where

$$\begin{aligned} &a_0(N,x) = x^N, \\ &a_i(N,x) = \sum_{k_i=0}^{N-i} \sum_{k_{i-1}=0}^{N-i-k_i} \cdots \sum_{k_1=0}^{N-i-k_i-\cdots-k_2} \left(\prod_{l=1}^{i} (x+l)^{k_l} \right) x^{N-i-\sum_{l=1}^{i} k_l}, (1 \le i \le N). \end{aligned}$$

From (2.1), we note that

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F(t,x) = \sum_{k=0}^{\infty} B^c_{k+N}(x) \frac{t^k}{k!}. \tag{2.22}$$

From Theorem 1 and (2.22), we can derive the following equation:

$$\begin{aligned} \sum_{k=0}^{\infty} B^c_{k+N}(x) \frac{t^k}{k!} &= F^{(N)} = \left(\sum_{i=0}^{N} a_i(N,x) e^{it} \right) F \\ &= \sum_{i=0}^{N} a_i(N,x) \left(\sum_{l=0}^{\infty} i^l \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} B^c_m(x) \frac{t^m}{m!} \right) \\ &= \sum_{i=0}^{N} a_i(N,x) \left(\sum_{k=0}^{\infty} \sum_{m=0}^{k} \binom{k}{m} i^{k-m} B^c_m(x) \frac{t^k}{k!} \right) \\ &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{N} \sum_{m=0}^{k} \binom{k}{m} i^{k-m} a_i(N,x) B^c_m(x) \right) \frac{t^k}{k!}. \end{aligned} \tag{2.23}$$

Now comparing the coefficients on both sides of (2.23), we obtain the following theorem.

Theorem 2. *For $k, N = 0, 1, 2, \ldots$, we have*

$$B^c_{k+N}(x) = \sum_{i=0}^{N} \sum_{m=0}^{k} \binom{k}{m} i^{k-m} a_i(N,x) B^c_m(x),$$

where

$$a_0(N,x) = x^N,$$
$$a_i(N,x) = \sum_{k_i=0}^{N-i}\sum_{k_{i-1}=0}^{N-i-k_i}\cdots\sum_{k_1=0}^{N-i-k_i-\cdots-k_2}\left(\prod_{l=1}^{i}(x+l)^{k_l}\right)x^{N-i-\sum_{l=1}^{i}k_l}, (1\le i\le N).$$

If we take $k=0$ in Theorem 2, then we have the following corollary.

Corollary 3. *For* $N = 0,1,2,\ldots,$ *we have*

$$B_N^c(x) = \sum_{i=0}^{N} a_i(N,x).$$

3. Zeros of the Bell-Carlitz polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the Bell-Carlitz polynomials $B_n^c(x)$. By using computer, the Bell-Carlitz polynomials $B_n^c(x)$ can be determined explicitly. We display the shapes of the Bell-Carlitz polynomials $B_n^c(x)$ and investigate the zeros of the Bell-Carlitz polynomials $B_n^c(x)$. For $n = 1,\ldots,10$, we can draw a plot of the Bell-Carlitz polynomials $B_n^c(x)$, respectively. This shows the ten plots combined into one. We display the shape of $B_n^c(x)$, $-5\le x\le 5$. (Figure 1).

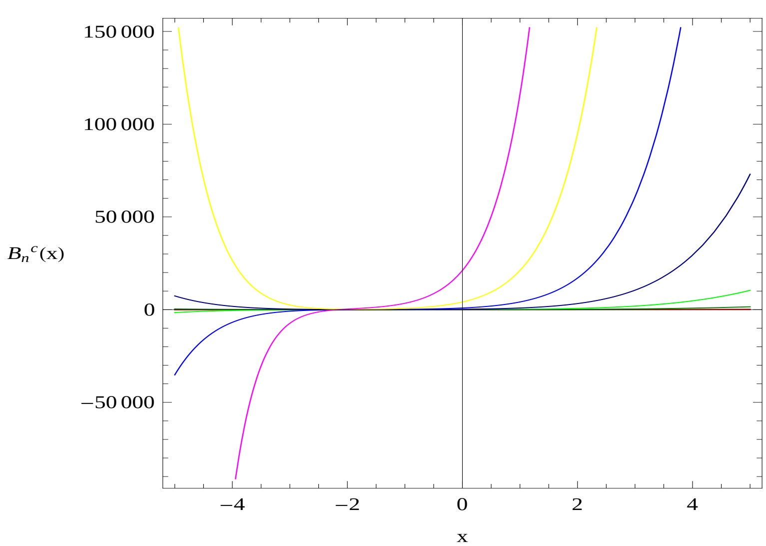

FIGURE 1. Curve of the Bell-Carlitz polynomials $B_n^c(x)$

We investigate the beautiful zeros of the Bell-Carlitz polynomials $B_n^c(x)$ by using a computer. We plot the zeros of the $B_n^c(x)$ for $n = 5, 10, 15, 20$ and $x \in \mathbb{C}$ (Figure 2). In Figure 2(top-left), we choose $n = 5$. In Figure 2(top-right), we choose $n = 10$. In Figure 2(bottom-left), we choose $n = 15$. In Figure 2(bottom-right), we choose

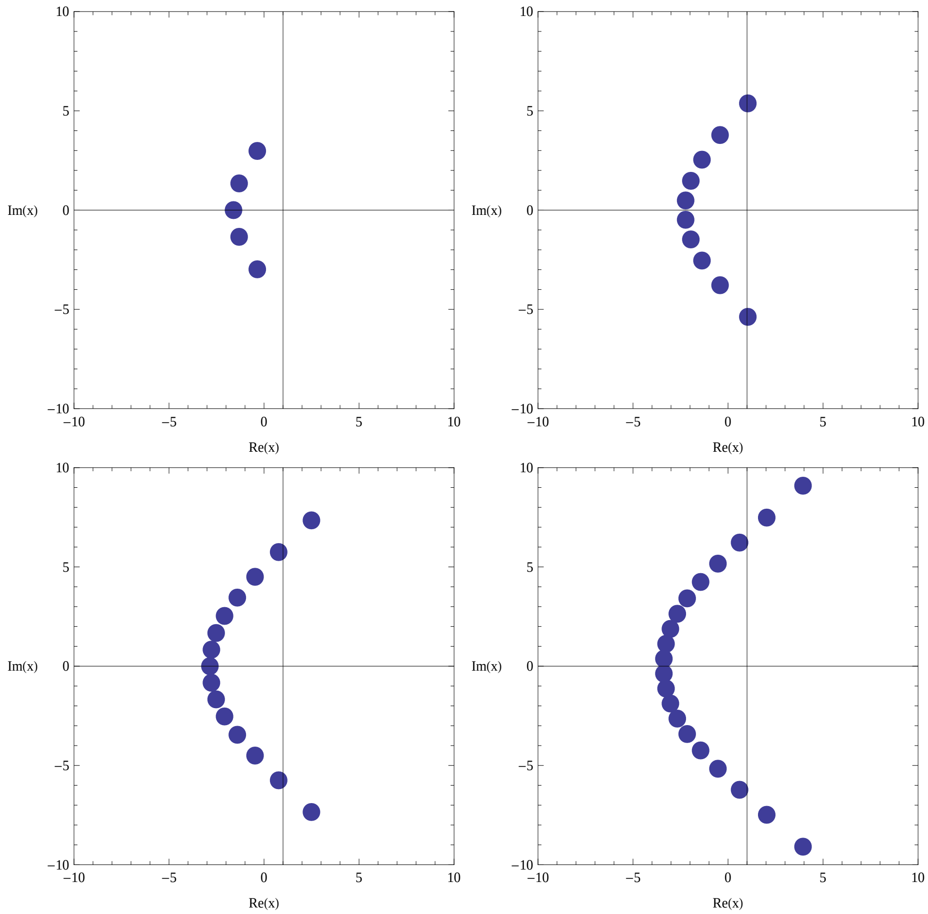

Figure 2. Zeros of $B_n^c(x)$

$n = 20$. It is expected that $B_n^c(x)$, $x \in \mathbb{C}$, has $Im(x) = 0$ reflection symmetry analytic complex functions (see Figure 2).

Stacks of zeros of the Bell-Carlitz polynomials $B_n^c(x)$ for $1 \leq n \leq 20$ from a 3-D structure are presented (Figure 3).

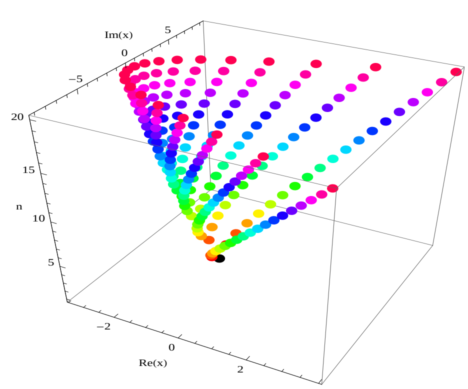

Figure 3. Stacks of zeros of $B_n^c(x), 1 \leq n \leq 20$

Our numerical results for approximate solutions of real zeros of the Bell-Carlitz polynomials $B_n^c(x)$ are displayed (Tables 1, 2).

Table 1. Numbers of real and complex zeros of $B_n^c(x)$

degree n	real zeros	complex zeros
1	1	0
2	0	2
3	1	2
4	0	4
5	1	4
6	0	6
7	1	6
8	0	8
9	1	8
10	0	10
11	1	10
12	0	12
13	1	12
14	0	14

Since n is the degree of the polynomial $B_n^c(x)$, the number of real zeros $R_{B_n^c(x)}$ lying on the real plane $Im(x) = 0$ is then $R_{B_n^c(x)} = n - C_{T_n(x)}$, where $C_{B_n^c(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{B_n^c(x)}$ and $C_{B_n^c(x)}$. We expect that the numbers of real zeros $R_{B_n^c(x)}$ of $B_n^c(x), Im(x) \neq 0$ is

$$R_{B_n^c(x)} = \begin{cases} 1, & \text{if } n = \text{ odd}, \\ 0, & \text{if } n = \text{ even}. \end{cases} \tag{3.1}$$

The plot of real zeros of $B_n^c(x)$ for $1 \leq n \leq 30$ structure are presented (Figure 4).

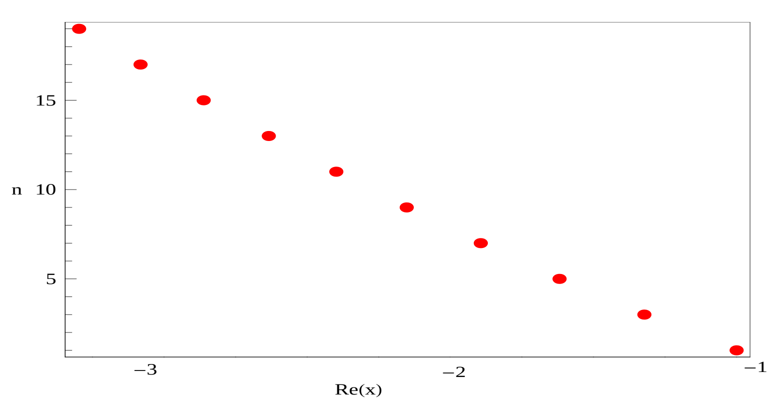

FIGURE 4. Real zeros of $B_n^c(x)$ for $1 \leq n \leq 20$

We observe a remarkable regular structure of the complex roots of the Bell-Carlitz polynomials $B_n^c(x)$. We also hope to verify a remarkable regular structure of the complex roots of the Bell-Carlitz polynomials $B_n^c(x)$ (Table 1). Next, we calculated an approximate solution satisfying $B_n^c(x) = 0, x \in \mathbb{C}$. The results are given in Table 2.

Table 2. Approximate solutions of $B_n^c(x) = 0, x \in \mathbb{C}$

degree n	x
1	-1
2	$-1.0000 - 1.0000i, -1.0000 + 1.0000i$
3	$-1.3222, \quad -0.8389 - 1.7544i, -0.8389 + 1.7544i$
4	$-1.3824 - 0.7286i, -1.3824 + 0.7286i,$ $-0.6176 - 2.4003i, -0.6176 + 2.4003i$
5	$-1.6184, -1.3232 - 1.3458i, -1.3232 + 1.3458i,$ $-0.3675 - 2.9807i, -0.3675 + 2.9807i$
6	$-1.6981 - 0.6049i, -1.6981 + 0.6049i, -1.2004 - 1.8999i,$ $-1.2004 + 1.8999i, -0.1015 - 3.5156i, -0.1015 + 3.5156i$
7	$-1.8936, -1.6881 - 1.1431i, -1.6881 + 1.1431i, -1.0391 - 2.4115i,$ $-1.0391 + 2.4115i, 0.1740 - 4.0163i, 0.1740 + 4.0163i$
8	$1.9813 - 0.5334i, -1.9813 + 0.5334i, -1.6216 - 1.6377i,$ $-1.6216 + 1.6377i, -0.8526 - 2.8915i, -0.8526 + 2.8915i,$ $0.4556 - 4.4903i, 0.4556 + 4.4903i$

For $N = 0, 1, 2, \ldots$, the functional equation

$$F^{(N)} = \left(\sum_{i=0}^{N} a_i(N, x) e^{it} \right) F \tag{3.2}$$

has a solution

$$F = F(t, x) = e^{(xt + e^t - 1)}. \tag{3.3}$$

In Figure 5(left), we plot of the surface for this solution. In Figure 5(right), we show a higher-resolution density plot of the solution.

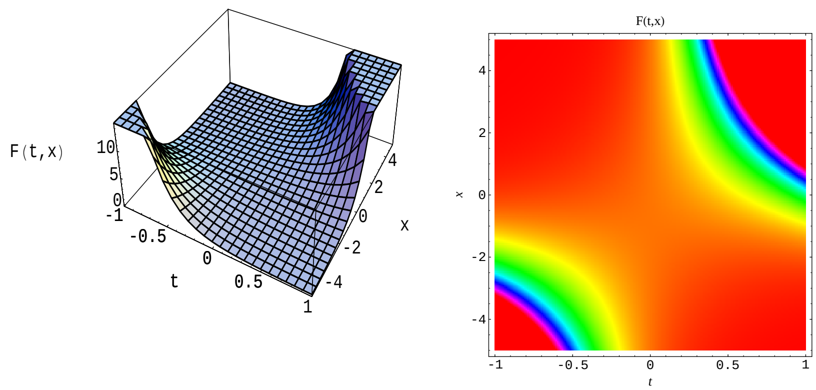

FIGURE 5. The surface for the solution $F(t,x)$

Finally, we consider the more general problems. How many zeros does $B_n^c(x)$ have? We are not able to decide if $B_n^c(x)=0$ has n distinct solutions (see Table 2). We would also like to know the number of complex zeros $C_{B_n^c(x)}$ of $B_n^c(x), Im(x)\neq 0$. Since n is the degree of the polynomial $B_n^c(x)$, the number of real zeros $R_{B_n^c(x)}$ lying on the real line $Im(x)=0$ is then $R_{B_n^c(x)}=n-C_{B_n^c(x)}$, where $C_{B_n^c(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{B_n^c(x)}$ and $C_{B_n^c(x)}$. The authors have no doubt that investigations along these lines will lead to a new approach employing numerical method in the research field of the Bell-Carlitz polynomials $B_n^c(x)$ which appear in mathematics and physics. The reader may refer to [6, 7, 8, 9] for the details.

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