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SYMBOLIC ITERATIVE SOLUTION OF VOLTERRA INTEGRAL EQUATIONS

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ABSTRACT. In this work we show how auxiliary variables can be used to give an efficient and widely applicable method involving symbolic manipulation and Picard iteration for approximating solutions of certain Volterra integral equations.

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1. Introduction and Preliminaries

Volterra integral equations of the second kind,

$$y(t) = \varphi(t) + \int_a^t K(t, s, y(s))\, ds, \tag{1.1}$$

lend themselves to solution by successive approximation using Picard iteration, although the process can break down when quadratures that cannot be performed in closed form arise. In this article we offer a method for introducing auxiliary variables in (1.1) in the case that K factors as $K(t, s, z) = f(t)k(s, z)$ in such a way that (1.1) embeds in a vector-valued polynomial Volterra integral equation, thus extending the method of auxiliary variables, as expounded in [6] by Parker and Sochacki in the case of initial value problems, to the setting of integral equations. We thereby obtain a computationally efficient method of *symbolic* rather than numerical computation for closely approximating solutions of (1.1). Of course the problem of impossible integrations could also be addressed by replacing φ, f, and k by initial segments of their power series expansions about a; the method presented here seems to be an attractive alternative in some situations, among others those that involve denominators, like Example 3.1 below, or those that involve powers of functions, like Example 3.2

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below. It is equally easy to apply when the unknown function $y(t)$ appears in the argument of a transcendental function, as in Example 3.4.

For reference we state the following generalization to the vector-valued case of Theorem 2.1.1 of [4]. The proof in [4] goes through with the obvious modifications. We only note that the proof is based on an application of the Contraction Mapping Theorem. By way of notation, for a subset S of $\mathbb{R}^m$ we let $C(S, \mathbb{R}^n)$ denoted the set of continuous mappings from S into $\mathbb{R}^n$.

Theorem 1.1. *Let* $I = [a, b] \subset \mathbb{R}$ *and* $J = \{(x, y) : x \in I,\ y \in [a, x]\} \subset I \times I$. *Suppose* $\boldsymbol{\varphi} \in C(I, \mathbb{R}^n)$ *and* $K \in C(J \times \mathbb{R}^n, \mathbb{R}^n)$ *and that* K *is Lipschitz in the last variable: there exists* $L \in \mathbb{R}$ *such that*

$$|K(x, y, \mathbf{z}) - K(x, y, \mathbf{z}')|_{\text{sum}} \leqslant L|\mathbf{z} - \mathbf{z}'|_{\text{sum}}$$

for all $(x, y) \in J$ *and all* $\mathbf{z}, \mathbf{z}' \in \mathbb{R}^n$. *Then the integral equation*

$$\mathbf{y}(t) = \boldsymbol{\varphi}(t) + \int_a^t K(t, s, \mathbf{y}(s))\, ds \tag{1.2}$$

has a unique solution $\mathbf{y}(t) \in C(I, \mathbb{R}^n)$.

Because the theorem was proved by means of the Contraction Mapping Theorem we immediately obtain the following result.

Theorem 1.2. *Under the hypotheses of Theorem* 1.1, *for any choice of the initial mapping* $\mathbf{y}^{[0]}(t)$ *the sequence of Picard iterates*

$$\mathbf{y}^{[k+1]}(t) = \boldsymbol{\varphi}(t) + \int_a^t K(t, s, \mathbf{y}^{[k]}(s))\, ds$$

converges to the unique solution of the integral equation (1.2).

2. **The Method**

Now let a Volterra integral equation

$$y(t) = \varphi(t) + \int_a^t f(t) k(s, y(s))\, ds \tag{2.1}$$

be given, where $\varphi \in C([a, b], \mathbb{R})$, $f \in C([a, b], \mathbb{R})$, $k \in C([a, b] \times [a, b], \mathbb{R})$, and k satisfies a Lipschitz condition in y. Introduce auxiliary variables $v_1, \ldots, v_r$ in such a way that $\varphi = P(v_1, \ldots, v_r)$, $f = Q(v_1, \ldots, v_r)$, and $k = R(y, v_1, \ldots, v_r)$ (i.e., $\varphi(t) = P(v_1(t), \ldots, v_r(t))$, and so on), where P, Q, and R are polynomials and the variables $v_1, \ldots, v_r$ satisfy a system of first order polynomial ordinary differential equations

$$\begin{aligned} v_1' &= P_1(v_1, \ldots, v_r) \\ &\ \vdots \\ v_r' &= P_r(v_1, \ldots, v_r). \end{aligned} \tag{2.2}$$

To illustrate, suppose we wish to approximate the solution of $y(t) = 1 - \int_0^t \sin y(s)\,ds$. The integrand indicates introducing $v_1 = v_1(t) = \sin y(t)$. Since $v_1'(t) = \cos t\, y'(t)$ we are then led to introduce $v_2 = v_2(t) = \cos y(t)$, for which $v_2' = -v_1\, y'$. From the integral equation itself we have $y'(t) = -\sin y(t) = -v_1(t)$, so no additional auxiliary variables are needed; (2.2) is $v_1' = -v_2 v_1$ and $v_2' = v_2^2$, and the integral equation is $y(t) = 1 - \int_0^t v_1(s)\,ds$. (In more complicated situations some ingenuity can be required for this step. There are no known cases for which it has proved impossible when the functions involved are analytic. See [2] for a fuller discussion.)

The initial value problem obtained by adjoining to (2.2) the initial conditions given by the values of v_1 through v_r at $t = a$ has a unique solution, which is the unique solution of the vector-valued Volterra integral equation

$$\tag{2.3} \begin{aligned} v_1 &= v_1(a) + \int_a^t P_1(v_1(s), \ldots, v_r(s))\,ds \\ &\ \ \vdots \\ v_r &= v_r(a) + \int_a^t P_r(v_1(s), \ldots, v_r(s))\,ds, \end{aligned}$$

gotten simply by applying the Fundamental Theorem of Calculus to (2.2). Adjoin to (2.3) the original Volterra equation in the form

$$y(t) = P(v_1(t), \ldots, v_r(t)) + \int_a^t Q(v_1(s), \ldots, v_r(s)) R(y(s), v_1(s), \ldots, v_r(s))\,ds$$

to obtain

$$\tag{2.4} \begin{aligned} y &= P(v_1(t), \ldots, v_r(t)) + \int_a^t Q(v_1(s), \ldots, v_r(s)) R(y(s), v_1(s), \ldots, v_r(s))\,ds \\ v_1 &= v_1(a) + \int_a^t P_1(v_1(s), \ldots, v_r(s))\,ds \\ &\ \ \vdots \\ v_r &= v_r(a) + \int_a^t P_1(v_1(s), \ldots, v_r(s))\,ds. \end{aligned}$$

System (2.4) satisfies the hypotheses of Theorem (1.1), hence has a unique solution, as does the original Volterra integral equation. Since $v_1, \ldots, v_r$ are completely specified by (2.2) and (2.3), the y component of the solution of the augmented Volterra integral equation (2.4) must be the solution of (2.1). But by Theorem 1.2 the Picard iteration scheme applied to (2.4), say with $y^{[0]}(t) \equiv \varphi(a)$ and $v_j^{[0]}(t) \equiv v_j(a)$, converges and is computationally feasible, so we obtain a computable approximation to the solution of (2.1).

3. **Examples**

In this section we illustrate the method by means of several examples which include both linear and nonlinear Volterra integral equations.

Example 3.1. In [3] Effati and Skandari introduced the linear Volterra integral equation of the second kind

$$y(t) = e^t \sin t + \int_0^t \frac{2+\cos t}{2+\cos s}\, y(s)\, ds. \tag{3.1}$$

The form of $\varphi(t)$ leads us to introduce $v_1 = e^t$ and $v_2 = \cos t$, and since $v_2' = -\sin t$, also $v_3 = \sin t$. The integrand is then $(1+v_2(t))(1+v_2(s))^{-1}y(s)$; the denominator is the issue. To express the integrand as a polynomial function of several variables, name the denominator $v_4 = 1 + v_2$ and its reciprocal $v_5 = 1/v_4$ so that the integral equation is $y(t) = v_1(t)v_2(t) + v_4(t)\int_0^t v_5(s)y(s)\, ds$. Taking the derivatives of the auxiliary variables introduced so far shows that no more are needed, so one appropriate choice of auxiliary variables is

$$v_1 = e^t, \quad v_2 = \cos t, \quad v_3 = \sin t, \quad v_4 = 2 + v_2, \quad v_5 = \frac{1}{v_4},$$

which satisfy the system of first order ordinary differential equations

$$v_1' = v_1, \quad v_2' = -v_3, \quad v_3' = v_2, \quad v_4' = v_2' = -v_3, \quad v_5' = \frac{-v_4'}{v_4^2} = v_3 v_5^2,$$

which in turn is equivalent to

$$\begin{aligned}
v_1(t) &= v_1(0) + \int_0^t v_1(s)\, ds\\
v_2(t) &= v_2(0) - \int_0^t v_3(s)\, ds\\
v_3(t) &= v_3(0) + \int_0^t v_2(s)\, ds\\
v_4(t) &= v_4(0) - \int_0^t v_3(s)\, ds\\
v_5(t) &= v_5(0) + \int_0^t v_3(s) v_5^2\, ds.
\end{aligned}$$

The initial values of the auxiliary variables are determined by their definition. The initial value $y(0)$ of the solution of the integral equation (3.1) is found simply by evaluating that equation at $t = 0$ to obtain $y(0) = 0$. Thus the iteration scheme is

$$\begin{aligned}
y^{[k+1]}(t) &= v_1^{[k]} v_3^{[k]} + v_4^{[k]} \int_0^t v_5^{[k]} y^{[k]} ds\\
v_1^{[k+1]}(t) &= 1 + \int_0^t v_1^{[k]} ds
\end{aligned}$$

$$v_2^{[k+1]}(t) = 1 - \int_0^t v_3^{[k]} ds$$
$$v_3^{[k+1]}(t) = \int_0^t v_2^{[k]} ds$$
$$v_4^{[k+1]}(t) = 3 - \int_0^t v_3^{[k]} ds$$
$$v_5^{[k+1]}(t) = \frac{1}{3} + \int_0^t v_3^{[k]} (v_5^{[k]})^2 ds$$

We can initialize as we please, but it is reasonable to choose $y^{[0]}(t) \equiv y(0)$ and $v_j^{[0]}(t) \equiv v_j(0)$, i.e., $(y^{[0]}, v_1^{[0]}, v_2^{[0]}, v_3^{[0]}, v_4^{[0]}, v_5^{[0]})(t) \equiv (0, 1, 1, 0, 3, \frac{1}{3})$.

The exact solution of (3.1) is

$$y(t) = e^t \sin t + e^t \Big(2 + \cos t\Big) \Big(\ln 3 - \ln \big(2 + \cos t\big) \Big),$$

whose Maclaurin series, with its coefficients rounded to five decimal places, begins

$$y(t) = 1.00000t + 1.50000t^2 + 0.83333t^3 + 0.16667t^4 - 0.03333t^5 - 0.02593t^6 - 0.00529t^7 + O(t^8).$$

The eighth Picard iterate with its coefficients rounded to five decimal places is

$$y^{[8]}(t) = 1.00000t + 1.50000t^2 + 0.83333t^3 + 0.16667t^4 - 0.03333t^5 - 0.02593t^6 - 0.00529t^7 + O(t^8).$$

The absolute value of the error in the approximation of the exact solution by $y^{[8]}(t)$ is practically zero up to about $t = 0.4$, then increases monotonically to about 0.00057 at $t = 1$.

Example 3.2. In [1] Biazar and Eslami introduced the nonlinear Volterra integral equation of the second kind

$$y(t) = \frac{1}{2} \sin 2t + \int_0^t \frac{3}{2} y(s)^2 \cos(s - t)\, ds. \tag{3.2}$$

To fit this into the framework of (2.1) we begin by applying the cosine difference identity $\cos(t - s) = \cos s \cos t + \sin s \sin t$, obtaining

$$y(t) = \frac{1}{2} \sin 2t + \frac{3}{2} \Big(\cos t \int_0^t y(s)^2 \cos s\, ds + \sin t \int_0^t y(s)^2 \sin s\, ds \Big).$$

Introducing the auxiliary variables $v = \cos t$ and $w = \sin t$, which solve the system

$$v' = -w, \quad w' = v,$$

upon integration we obtain the equivalent system of integral equations

$$v(t) = v(0) - \int_0^t w(s)\, ds$$

$$w(t) = w(0) + \int_0^t v(s)\, ds.$$

The initial values of the auxiliary variables are determined by their definition. The initial value $y(0)$ of the solution of the integral equation (3.2) is found simply by evaluating that equation at $t = 0$ to obtain $y(0) = 0$. Thus the iteration scheme is

$$y^{[k+1]}(t) = w^{[k]}v^{[k]} + \frac{3}{2}\left(v^{[k]}(t)\int_0^t v^{[k]}(s)(y^{[k]})^2(s)\, ds + w^{[k]}(t)\int_0^t w^{[k]}(s)(y^{[k]})^2(s)\, ds\right)$$

$$w^{[k+1]}(t) = 0 + \int_0^t v^{[k]}(s)\, ds$$

$$v^{[k+1]}(t) = 1 - \int_0^t w^{[k]}(s)\, ds.$$

We initialize with

$$y^{[0]}(t) \equiv y(0) = 0$$
$$w^{[0]}(t) \equiv \sin 0 = 0$$
$$v^{[0]}(t) \equiv \cos 0 = 1.$$

The exact solution of (3.2) is $y(t) = \sin t$, whose Maclaurin series, with its coefficients rounded to five decimal places, begins

$$y(t) = 1.00000\, t - 0.16667\, t^3 + 0.00833\, t^5 - 0.00020\, t^7 + O(t^9).$$

The eighth Picard iterate with its coefficients rounded to five decimal places is

$$y^{[8]}(t) = 1.00000\, t - 0.16667\, t^3 + 0.008333\, t^5 + 0.00000\, t^7 + O(t^9).$$

The absolute value of the error in the approximation of the exact solution by $y^{[8]}(t)$ is practically zero up to about $t = 0.4$, then increases monotonically to about 0.001 at $t = 1$.

Example 3.3. As a somewhat more elaborate example consider the nonlinear Volterra integral equation of the second kind given by

$$y(t) = \tan t - \frac{1}{4}\sin 2t - \frac{1}{2}t + \int_0^t \frac{1}{1 + y^2(s)}\, ds. \tag{3.3}$$

This is a corrected version of an integral equation given by Kamyad *et al* in [5]. Because the integral part is independent of t, (3.3) is equivalent to an initial value problem, namely

$$y'(t) = \sec^2 t - \frac{1}{2}\cos 2t - \frac{1}{2} + \frac{1}{1 + y^2(t)}, \quad y(0) = 0.$$

Of course by means of the identity $\cos^2 t = \frac{1}{2}(1 + \cos 2t)$ the differential equation can be more compactly expressed as

$$y'(t) = \sec^2 t - \cos^2 t + \frac{1}{1 + y^2(t)}, \tag{3.4}$$

which will be important later.

To approximate the unique solution of (3.3) we introduce the auxiliary variables

$$v_1(t) = \sin t, \quad v_2(t) = \cos t, \quad v_3(t) = \frac{1}{v_2}, \quad v_4(t) = 1 + y^2, \quad v_5(t) = \frac{1}{v_4}.$$

Note that in contrast with the previous examples the unknown function $y(t)$ figures into the definition of some of these variables, but in a polynomial way. Thus when we compute their derivatives y also appears. Thanks to (3.4), it does so in a polynomial way, since by that identity $y' = v_3^2 - v_2^2 + v_5$ and we have additionally

$$v_1' = v_2, \quad v_2' = -v_1, \quad v_3' = v_1 v_3^2, \quad v_4' = 2y(v_3^2 - v_2^2 + v_5), \quad v_5' = -2y v_5^2(v_3^2 - v_2^2 + v_5).$$

This system of ordinary differential equations, together with the equation satisfied by y' and the known initial values of all the variables involved, is equivalent to the system of integral equations

$$\begin{aligned}
y(t) &= v_1 v_3 - \frac{1}{2} v_1 v_2 - \frac{1}{2} t + \int_0^t v_5(s)\, ds \\
v_1(t) &= \int_0^t v_2(s)\, ds \\
v_2(t) &= 1 - \int_0^t v_1(s)\, ds \\
v_3(t) &= 1 + \int_0^t v_1(s) v_3^2(s)\, ds \\
v_4(t) &= 1 + 2\int_0^t y(s)(v_3^2(s) - v_2^2(s) + v_5(s))\, ds \\
v_5(t) &= 1 - 2\int_0^t y(s) v_5^2(s)(v_3^2(s) - v_2^2(s) + v_5(s))\, ds.
\end{aligned}$$

Setting up the obvious iteration scheme based on these integral equations, and initializing with the constant functions $y(t) \equiv y(0)$ and $v_j(t) \equiv v_j(0)$, the Picard iterate $y^{[12]}(t)$ with coefficients rounded to five decimal places is

$$\begin{aligned}
y^{[12]}(t) = 1.00000\, t + 0.33333\, t^3 + 0.13333\, t^5 + 0.05397\, t^7 + 0.02187\, t^9 \\
+ 0.00886\, t^{11} + O(t^{13}).
\end{aligned}$$

The exact solution is $y(t) = \tan t$, whose Maclaurin series, with coefficients rounded to five decimal places is

$$\begin{aligned}
y(t) = 1.00000\, t + 0.33333\, t^3 + 0.13333\, t^5 + 0.05397\, t^7 + 0.02187\, t^9 \\
+ 0.00886\, t^{11} + O(t^{13}).
\end{aligned}$$

On the interval $[0, 0.10]$ the error in the approximation of the exact solution by $y^{[28]}(t)$ increases monotonically from zero to about 3.5×10^{-14}.

Finally, we consider the Volterra equation already looked at in Section 2, showing how the method applies easily even when the unknown function is in the argument of a transcendental function.

Example 3.4. The Volterra equation

$$y = 1 - \int_0^t \sin y(s)\, ds \tag{3.5}$$

has solution $y(t) = 2\text{arccot}(\cot(\frac{1}{2})e^t)$ whose Maclaurin series with coefficients rounded to five decimal places is

$$\begin{aligned} y(t) = 1.00000 - 0.84147\,t + 0.22732\,t^2 + 0.05836\,t^3 - 0.06154\,t^4 + 0.00791\,t^5 \\ + 0.01180\,t^6 - 0.00629\,t^7 - 0.00078\,t^8 + 0.00202\,t^9 + O\left(t^{10}\right). \end{aligned}$$

Introducing auxiliary variables as described for this example in Section 2 we obtain the recursion

$$\begin{aligned} y^{[k+1]}(t) &= 1 - \int_0^t v_1^{[k]}(s)\, ds \\ v_1^{[k+1]}(t) &= \sin 1 - \int_0^t v_2^{[k]}(s) v_1^{[k]}(s)\, ds \\ v_2^{[k+1]}(t) &= \cos 1 + \int_0^t (v_1^{[k]})^2(s)\, ds. \end{aligned}$$

The tenth Picard iterate with its coefficients rounded to five decimal places is

$$\begin{aligned} y^{[10]}(t) = 1.00000 - 0.84147\,t + 0.22732\,t^2 + 0.05836\,t^3 - 0.06154\,t^4 + 0.00791\,t^5 \\ + 0.01180\,t^6 - 0.00629\,t^7 - 0.00078\,t^8 + 0.00202\,t^9 + O\left(t^{10}\right). \end{aligned}$$

The absolute value of the error in the approximation of the exact solution by $y^{[10]}(t)$ is practically zero up to about $t = 0.5$, then increases monotonically to about 6.5×10^{-4} at $t = 1$.

4. **Conclusion**

After noting the extension to the vector-valued case of a well-known theorem on existence of solutions of Volterra equations of the second kind, we have observed that the method of proof by means of the Contraction Mapping Theorem guarantees that Picard iterates will converge to the solution. We have then described a method for introducing auxiliary variables into Volterra equations of the form

$$y(t) = \varphi(t) + \int_a^t f(t) k(s, y(s))\, ds,$$

in such a way that such an equation embeds in a vector-valued polynomial Volterra integral equation. We have thus extended the method of auxiliary variables for surmounting the obstacle of impossible quadratures that can arise in Picard iteration, well known in the case of initial value problems, to the setting of integral equations. We have thereby obtained a computationally efficient method of symbolic rather than numerical computation for closely approximating solutions of Volterra equations of this type, whether linear or nonlinear in the unknown solution y, and even when y appears in the argument of transcendental functions. We have illustrated the ease of use, broad applicability, and efficiency of the method with examples.

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