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A CLASS OF METHODS FOR THE NUMERICAL SOLUTIONS OF 3D MULTI-HARMONIC ELLIPTIC EQUATIONS ON A GRADED MESH

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ABSTRACT. In this paper, we report new numerical methods of order two and three for the solution of 3D multi-harmonic elliptic equations on a graded mesh. We first develop numerical methods for 3D Poisson equation on a graded mesh, then we extend our methods to solve 3D bi-harmonic and tri-harmonic equations. Numerical methods for $(\partial u/\partial n)$ are also developed, which are quite often of interest in many areas of Science and Engineering. Numerical results demonstrating the usefulness of proposed methods are presented.

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1. INTRODUCTION

In the past various numerical methods for the solution of 3D linear and non-linear elliptic equations have been discussed in the literature [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] on uniform mesh. Even methods for bi-harmonic and tri-harmonic equations for 3D elliptic equations have also been developed (see [15, 16, 17, 18, 19, 20, 21]) on constant mesh. To the author's knowledge, no numerical methods of order two and three have been developed for multi-harmonic equations on non uniform mesh so far. In this paper, we propose compact methods of order two and three for the solution of multi-harmonic elliptic equations on a graded mesh. We have also developed numerical methods of order three for the estimates of normal derivatives which help in solving many applied mathematics problems. In all the cases we use only nineteen point compact cell. In case of bi- and tri-harmonic problems, we do not discretize the boundary conditions. Numerical solutions of Laplacian and bi-Laplacian are obtained as by-product of the methods. In these cases also we use

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nineteen point compact cell. Our paper is arranged as follows. In section 2, we give mathematical derivation of the method for 3D Poisson equation. In Section 3, we discuss stability analysis of a model problem. In Section 4, we extend our technique to derive methods for 3D bi- and tri-harmonic problems. In Section 5, we present numerical results to illustrate the utility of methods developed. In Section 6, we give concluding remarks of our work.

2. VARIABLE MESH METHODS FOR 3D POISSON'S EQUATION

We consider the 3D Poisson equation of the form

$$\nabla^2 u(x,y,z) \equiv u_{xx} + u_{yy} + u_{zz} = f(x,y,z), \quad (x,y,z) \in \Omega, \tag{2.1}$$

subject to Dirichlet boundary conditions:

$$u(x,y,z) = g(x,y,z), \quad (x,y,z) \in \partial\Omega, \tag{2.2}$$

where $\Omega = \{(x,y,z) \mid 0 < x,y,z < 1\}$ is the solution domain with boundary $\partial\Omega$ and $\nabla^2 u(x,y,z)$ represents the 3D Laplacian of the function $u(x,y,z)$. We assume that the solution $u(x,y,z)$ is smooth enough to maintain the order and accuracy of the scheme as high as possible. We discretize the solution domain Ω by a set of grid points (x_l, y_m, z_n), where $0 = x_0 < x_1 < \cdots < x_{N+1} = 1$, $0 = y_0 < y_1 < \cdots < y_{N+1} = 1$ and $0 = z_0 < z_1 < \cdots < z_{N+1} = 1$, N being a positive integer with variable mesh spacing $\Delta x_l = x_l - x_{l-1} > 0$, $\Delta y_m = y_m - y_{m-1} > 0$, $\Delta z_n = z_n - z_{n-1} > 0$, $l,m,n = 1,2,\ldots,N+1$. Let $\alpha \neq 1$, $\beta \neq 1$ and $\gamma \neq 1$ are mesh ratio parameters in x-, y- and z-directions respectively, defined $\Delta x_{l+1} = \alpha \Delta x_l$, $\Delta y_{m+1} = \beta \Delta y_m$ and $\Delta z_{n+1} = \gamma \Delta z_n$. For $\alpha = \beta = \gamma = 1$, the variable mesh reduces to uniform mesh.

Let $u_{l,m,n}$ and $U_{l,m,n}$ be the approximate and exact solutions of $u(x,y,z)$ at the grid point (x_l, y_m, z_n) respectively.

The value of $f(x,y,z)$ at the grid point (x_l, y_m, z_n) is defined by $f_{l,m,n}$.

We denote:

(2.3)

$$\alpha_0 = (\alpha-1)/3,\ \alpha_1 = 1/\alpha(\alpha+1),\ \alpha_2 = (1-\alpha+\alpha^2)/12,\ \alpha_3 = 1-\alpha^2,\ \alpha_4 = 1+\alpha,$$

$$\beta_0 = (\beta-1)/3,\ \beta_1 = 1/\beta(\beta+1),\ \beta_2 = (1-\beta+\beta^2)/12,\ \beta_3 = 1-\beta^2,\ \beta_4 = 1+\beta,$$

$$\gamma_0 = (\gamma-1)/3,\ \gamma_1 = 1/\gamma(\gamma+1),\ \gamma_2 = (1-\gamma+\gamma^2)/12,\ \gamma_3 = 1-\gamma^2,\ \gamma_4 = 1+\gamma.$$

At the grid point (x_l, y_m, z_n), we define:

$$U_{ijk} = \frac{\partial^{i+j+k} u}{\partial x^i \partial y^j \partial z^k}, \quad f_{ijk} = \frac{\partial^{i+j+k} f}{\partial x^i \partial y^j \partial z^k}; \quad i,j,k = 0,1,2,. \tag{2.4}$$

We need the following approximations:

$$\overline{U}_{x_{l,m,n}} = \alpha_1 (U_{l+1,m,n} - \alpha_3 U_{l,m,n} - \alpha^2 U_{l-1,m,n})/\Delta x_l = U_{xl,m,n} + O(\Delta x_l^2) \tag{2.5a}$$

$$\overline{U}_{y_{l,m,n}} = \beta_1(U_{l,m+1,n} - \beta_3 U_{l,m,n} - \beta^2 U_{l,m-1,n})/\Delta y_m = U_{yl,m,n} + O(\Delta y_m^2) \tag{2.5b}$$

$$\overline{U}_{z_{l,m,n}} = \gamma_1(U_{l,m,n+1} - \gamma_3 U_{l,m,n} - \gamma^2 U_{l,m,n-1})/\Delta z_n = U_{zl,m,n} + O(\Delta z_n^2) \tag{2.5c}$$

$$\begin{aligned} \overline{U}_{xxl,m,n} &= 2\alpha_1(U_{l+1,m,n} - \alpha_4 U_{l,m,n} + \alpha U_{l-1,m,n})/(\Delta x_l^2) \\ &= U_{xx_{l,m,n}} + \alpha_0 \Delta x_l U_{300} + O(\Delta x_l^2) \end{aligned} \tag{2.5d}$$

$$\begin{aligned} \overline{U}_{yy_{l,m,n}} &= 2\beta_1(U_{l,m+1,n} - \beta_4 U_{l,m,n} + \beta U_{l,m-1,n})/(\Delta y_m^2) \\ &= U_{yy_{l,m,n}} + \beta_0 \Delta y_m U_{030} + O(\Delta y_m^2) \end{aligned} \tag{2.5e}$$

$$\begin{aligned} \overline{U}_{zzl,m,n} &= 2\gamma_1(U_{l,m,n+1} - \gamma_4 U_{l,m,n} + \gamma U_{l,m,n-1})/(\Delta z_n^2) \\ &= U_{zz_{l,m,n}} + \beta_0 \Delta z_n U_{003} + O(\Delta z_n^2) \end{aligned} \tag{2.5f}$$

$$\begin{aligned} \overline{U}_{xyl,m,n} &= \alpha_1(\overline{U}_{y_{l+1,m,n}} - \alpha_3 \overline{U}_{y_{l,m,n}} - \alpha^2 \overline{U}_{y_{l-1,m,n}})/\Delta x_l \\ &= U_{xyl,m,n} + O(\Delta x_l^2 + \Delta y_m^2 + \Delta z_n^2) \end{aligned} \tag{2.5g}$$

$$\begin{aligned} \overline{U}_{yzl,m,n} &= \beta_1(\overline{U}_{z_{l,m+1,n}} - \beta_3 \overline{U}_{z_{l,m,n}} - \beta^2 \overline{U}_{z_{l,m-1,n}})/\Delta y_m \\ &= U_{yzl,m,n} + O(\Delta x_l^2 + \Delta y_m^2 + \Delta z_n^2) \end{aligned} \tag{2.5h}$$

$$\begin{aligned} \overline{U}_{zxl,m,n} &= \gamma_1(\overline{U}_{x_{l,m,n+1}} - \gamma_3 \overline{U}_{x_{l,m,n}} - \gamma^2 \overline{U}_{z_{l,m,n-1}})/\Delta z_n \\ &= U_{zxl,m,n} + O(\Delta x_l^2 + \Delta y_m^2 + \Delta z_n^2) \end{aligned} \tag{2.5i}$$

$$\begin{aligned} \overline{U}_{xxyl,m,n} &= 2\alpha_1(\overline{U}_{y_{l+1,m,n}}, -\alpha_4 \overline{U}_{y_{l,m,n}} + \alpha \overline{U}_{y_{l-1,m,n}})/(\Delta x_l^2) \\ &= U_{xxy_{l,m,n}} + \alpha_0 \Delta x_l U_{310} + O(\Delta x_l^2) \end{aligned} \tag{2.5j}$$

$$\begin{aligned} \overline{U}_{yyzl,m,n} &= 2\beta_1(\overline{U}_{z_{l,m+1,n}}, -\beta_4 \overline{U}_{z_{l,m,n}} + \beta \overline{U}_{z_{l,m-1,n}})/(\Delta y_m^2) \\ &= U_{yyz_{l,m,n}} + \beta_0 \Delta y_m U_{031} + O(\Delta y_m^2) \end{aligned} \tag{2.5k}$$

$$\begin{aligned} \overline{U}_{zzxl,m,n} &= 2\gamma_1(\overline{U}_{x_{l,m,n+1}}, -\gamma_4 \overline{U}_{x_{l,m,n}} + \gamma \overline{U}_{x_{l,m,n-1}})/(\Delta z_n^2) \\ &= U_{zzx_{l,m,n}} + \gamma_0 \Delta z_n U_{103} + O(\Delta z_n^2) \end{aligned} \tag{2.5l}$$

$$\begin{aligned} \overline{U}_{xyy_{l,m,n}} &= 2\beta_1(\overline{U}_{x_{l,m+1,n}} - \beta_4 \overline{U}_{x_{l,m,n}} + \beta \overline{U}_{x_{l,m-1,n}})/(\Delta y_m^2) \\ &= U_{xyy_{l,m,n}} + \beta_0 \Delta y_m U_{130} + O(\Delta y_m^2) \end{aligned} \tag{2.5m}$$

$$\begin{aligned} \overline{U}_{yzz_{l,m,n}} &= 2\gamma_1(\overline{U}_{y_{l,m,n+1}} - \gamma_4 \overline{U}_{y_{l,m,n}} + \gamma \overline{U}_{y_{l,m,n-1}})/(\Delta z_n^2) \\ &= U_{yzz_{l,m,n}} + \gamma_0 \Delta z_n U_{013} + O(\Delta z_n^2) \end{aligned} \tag{2.5n}$$

$$\begin{aligned} \overline{U}_{zxx_{l,m,n}} &= 2\alpha_1(\overline{U}_{z_{l+1,m,n}} - \alpha_4 \overline{U}_{z_{l,m,n}} + \alpha \overline{U}_{z_{l-1,m,n}})/(\Delta x_l^2) \\ &= U_{zxx_{l,m,n}} + \alpha_0 \Delta x_l U_{301} + O(\Delta x_l^2), \end{aligned} \tag{2.5o}$$

$$\begin{aligned} \overline{U}_{xxyyl,m} &= 2\alpha_1(\overline{U}_{yy_{l+1,m,n}} - \alpha_4 \overline{U}_{yy_{l,m,n}} + \alpha \overline{U}_{yy_{l-1,m,n}})/(\Delta x_l^2) \\ &= U_{xxyy_{l,m,n}} + \alpha_0 \Delta x_l U_{320} + \beta_0 \Delta y_m U_{230} + O(\Delta x_l^2 + \Delta y_m^2 + \Delta z_n^2), \end{aligned} \tag{2.5p}$$

$$\begin{aligned} \overline{U}_{yyzzl,m} &= 2\beta_1(\overline{U}_{zz_{l,m+1,n}} - \beta_4 \overline{U}_{zz_{l,m,n}} + \beta \overline{U}_{zz_{l,m-1,n}})/(\Delta y_m^2) \\ &= U_{yyzz_{l,m,n}} + \beta_0 \Delta y_m U_{032} + \gamma_0 \Delta z_n U_{023} + O(\Delta x_l^2 + \Delta y_m^2 + \Delta z_n^2), \end{aligned} \tag{2.5q}$$

$$\overline{U}_{zzxxl,m,n} = 2\gamma_1(\overline{U}_{xx_{l,m,n+1}} - \gamma_4 \overline{U}_{xx_{l,m,n}} + \gamma \overline{U}_{xx_{l,m,n-1}})/(\Delta z_n^2) \tag{2.5r}$$

$$= U_{zzxx_{l,m,n}} + \gamma_0 \Delta z_n U_{203} + \alpha_0 \Delta x_l U_{302} + O(\Delta x_l^2 + \Delta y_m^2 + \Delta z_n^2).$$

Similarly, replacing U by V and W, we can define the approximations $\overline{V}_{x_{l,m,n}}$, $\overline{W}_{x_{l,m,n}}$, …etc.

Now we consider the linear combination $f_{000}+p_0\Delta x_l f_{100}+q_0\Delta y_m f_{010}+r_0\Delta z_n f_{001}$, where $f_{000} = U_{200} + U_{020} + U_{002}$, $f_{100} = U_{300} + U_{120} + U_{102}$, $f_{010} = U_{210} + U_{030} + U_{012}$, and $f_{001} = U_{201} + U_{021} + U_{003}$ and p_0, q_0, r_0 are parameters to be determined. By the help of (2.5a)–(2.5r), we obtain

(2.6)

$$\begin{aligned} &f_{000} + p_0\Delta x_l f_{100} + q_0\Delta y_m f_{010} + r_0\Delta z_m f_{001} \\ &\quad= \left[\overline{U}_{xx_{l,m,n}} - \alpha_0\Delta x_l U_{300}\right] + \left[\overline{U}_{yy_{l,m,n}} - \beta_0\Delta y_m U_{030}\right] + \left[\overline{U}_{zz_{l,m,n}} - \gamma_0\Delta z_n U_{003}\right] \\ &\qquad + p_0\Delta x_l(U_{300} + \overline{U}_{xyy_{l,m,n}} + \overline{U}_{xzz_{l,m,n}}) + q_0\Delta y_m(U_{030} + \overline{U}_{xxy_{l,m,n}} + \overline{U}_{zzy_{l,m,n}}) \\ &\qquad + r_0\Delta z_n(U_{003} + \overline{U}_{xxz_{l,m,n}} + \overline{U}_{zyy_{l,m,n}}) + O(\Delta x_l^2 + \Delta y_m^2 + \Delta z_n^2). \end{aligned}$$

Equating to zero the coefficients of Δx_l, Δy_m and Δz_n in (2.6), we get $p_0 = \alpha_0$, $q_0 = \beta_0$ and $r_0 = \gamma_0$. Thus the numerical method of order two for the differential equation (2.1) on the variable mesh is given by

$$\begin{aligned} (2.7)\quad SL_U \equiv\ &\overline{U}_{xx_{l,m,n}} + \overline{U}_{yy_{l,m,n}} + \overline{U}_{zz_{l,m,n}} + \alpha_0\Delta x_l(\overline{U}_{xyy_{l,m,n}} + \overline{U}_{xzz_{l,m,n}}) \\ &+ \beta_0\Delta y_m(\overline{U}_{xxy_{l,m}} + \overline{U}_{zzy_{l,m,n}}) + \gamma_0\Delta z_n(\overline{U}_{zyy_{l,m}} + \overline{U}_{zxx_{l,m,n}}) \\ &= f_{000} + \alpha_0\Delta x_l f_{100} + \beta_0\Delta y_m f_{010} + \gamma_0\Delta z_n f_{001} + O(\Delta x_l^2 + \Delta y_m^2 + \Delta z_n^2) \\ &\equiv SRH_f + O(\Delta x_l^2 + \Delta y_m^2 + \Delta z_n^2). \end{aligned}$$

Similarly, a numerical method of order three for the equation (2.1) on the variable mesh may be written as

(2.8)

$$\begin{aligned} TL_U \equiv\ &\overline{U}_{xx_{l,m,n}} + \overline{U}_{yy_{l,m,n}} + \overline{U}_{zz_{l,m,n}} + \alpha_0\Delta x_l(\overline{U}_{xyy_{l,m,n}} + \overline{U}_{xzz_{l,m,n}}) \\ &+ \beta_0\Delta y_m(\overline{U}_{xxy_{l,m}} + \overline{U}_{yzz_{l,m,n}}) + \gamma_0\Delta z_n(\overline{U}_{zyy_{l,m,n}} + \overline{U}_{zxx_{l,m,n}}) \\ &+ (\alpha_2\Delta x_l^2 + \beta_2\Delta y_m^2)\overline{U}_{xxyy_{l,m,n}} + (\beta_2\Delta y_m^2 + \gamma_2\Delta z_n^2)\overline{U}_{yyzz_{l,m,n}} \\ &+ (\gamma_2\Delta z_n^2 + \alpha_2\Delta x_l^2)\overline{U}_{zzxx_{l,m,n}} \\ = f_{000} &+ \alpha_0\Delta x_l f_{100} + \beta_0\Delta y_m f_{010} + \gamma_0\Delta z_n f_{001} + \alpha_2\Delta x_l^2 f_{200} + \beta_2\Delta y_m^2 f_{020} \\ &+ \gamma_2\Delta z_n^2 f_{002} + \alpha_0\beta_0\Delta x_l\Delta y_m f_{110} + \beta_0\gamma_0\Delta y_m\Delta z_n f_{011} + \gamma_0\alpha_0\Delta x_l\Delta z_n f_{101} \\ &+ O(\Delta x_l^3 + \Delta y_m^3 + \Delta z_n^3). \end{aligned}$$

Note that, replacing U by V and W, we can write the expressions for SL_V, SL_W, TL_V and TL_W.

Once the numerical solution of the Poisson equation has been obtained, one may compute the values of the first order derivatives (normal derivatives) using the standard differences.

$$U_{x_{l,m}} = \alpha_1[U_{l+1,m,n} - \alpha_3 U_{l,m,n} - \alpha^2 U_{l-1,m,n}]/\Delta x_l + O(\Delta x_l^2), \tag{2.9a}$$

$$U_{y_{l,m,n}} = \beta_1[U_{l,m+1,n} - \beta_3 U_{l,m,n} - \beta^2 U_{l,m-1,n}]/\Delta y_m + O(\Delta y_m^2), \tag{2.9b}$$

$$U_{z_{l,m,n}} = \gamma_1[U_{l,m,n+1} - \gamma_3 U_{l,m,n} - \gamma^2 U_{l,m,n-1}]/\Delta z_n + O(\Delta z_n^2). \tag{2.9c}$$

It is found that the formulas (2.9a)–(2.9c) yield second-order accurate results irrespective of whether equation (2.7) or (2.8) is used to solve the equation (2.1). A new difference method for computing the numerical values of these first order partial derivatives is discussed. These approximations found to yield $O(\Delta x_l^3 + \Delta y_m^3 + \Delta z_n^3)$-accuracy, when used in conjunction with nineteen point formula (2.8). By the help of the approximations (2.5a)–(2.5r), we may obtain the third order difference schemes for the numerical solution of $(\partial u/\partial x)$, $(\partial u/\partial y)$ and $(\partial u/\partial z)$ as:

$$\begin{aligned} U_{xl,m,n} = \overline{U}_{x_{l,m,n}} + \frac{\alpha \Delta x_l^2}{6}\left[\overline{U}_{xyy_{l,m,n}} + \overline{U}_{xzz_{l,m,n}} - f_{100}\right] \\ + O(\Delta x_l^3 + \Delta y_m^3 + \Delta z_n^3), \end{aligned} \tag{2.10a}$$

$$\begin{aligned} U_{yl,m,n} = \overline{U}_{y_{l,m,n}} + \frac{\beta \Delta y_m^2}{6}\left[\overline{U}_{xxy_{l,m}} + \overline{U}_{zzy_{l,m}} - f_{010}\right] \\ + O(\Delta x_l^3 + \Delta y_m^3 + \Delta z_n^3), \end{aligned} \tag{2.10b}$$

$$\begin{aligned} U_{zl,m,n} = \overline{U}_{z_{l,m,n}} + \frac{\gamma \Delta z_n^2}{6}\left[\overline{U}_{xxz_{l,m}} + \overline{U}_{yyz_{l,m}} - f_{001}\right] \\ + O(\Delta x_l^3 + \Delta y_m^3 + \Delta z_n^3). \end{aligned} \tag{2.10c}$$

3. STABILITY ANALYSIS

Consider 3-D Laplace equation

$$\nabla^2 u(x,y,z) \equiv u_{xx} + u_{yy} + u_{zz} = 0, \quad (x,y,z) \in \Omega\,. \tag{3.1}$$

subject to Dirichlet boundary conditions prescribed by (2.2).

For convenience, let us denote:

$$\begin{aligned} &U_0 = U_{l,m,n},\ U_1 = U_{l+1,m,n},\ U_2 = U_{l-1,m,n},\ U_3 = U_{l,m+1,n}, \\ &U_4 = U_{l,m-1,n},\ U_5 = U_{l,m,n+1},\ U_6 = U_{l,m,n-1},\ U_7 = U_{l+1,m+1,n}, \\ &U_8 = U_{l+1,m-1,n},\ U_9 = U_{l-1,m+1,n},\ U_{10} = U_{l-1,m-1,n},\ U_{11} = U_{l+1,m,n+1}, \\ &U_{12} = U_{l+1,m,n-1},\ U_{13} = U_{l-1,m,n+1},\ U_{14} = U_{l-1,m,n-1},\ U_{15} = U_{l,m+1,n+1}, \\ &U_{16} = U_{l,m+1,n-1},\ U_{17} = U_{l,m-1,n+1},\ U_{18} = U_{l,m-1,n-1}, \end{aligned}$$

$$H_1 = \frac{1}{\Delta x_l^2}, \quad H_2 = \frac{1}{\Delta y_m^2}, \quad H_3 = \frac{1}{\Delta z_n^2}. \tag{3.2}$$

Applying the difference scheme (2.7) to the equation (3.1) and using the notations in (3.2), we get

$$C_0U_0 + C_1U_1 + C_2U_2 + C_3U_3 + C_4U_4 + C_5U_5 + C_6U_6 + C_7U_7 + C_8U_8 + C_9U_9 + C_{10}U_{10} + C_{11}U_{11} + C_{12}U_{12} + C_{13}U_{13} + C_{14}U_{14} + C_{15}U_{15} + C_{16}U_{16} + C_{17}U_{17} + C_{18}U_{18} = T_{l,m,n} \tag{3.3}$$

where,

(3.4)

$$C_0 = 2(-\alpha_4\alpha_1H_1 - \beta_4\beta_1H_2 - \gamma_4\gamma_1H_3 + \alpha_0\alpha_1\alpha_3\beta_1\beta_4H_2 + \alpha_0\alpha_1\alpha_3\gamma_1\gamma_4H_3 + \beta_0\beta_1\beta_3\alpha_1\alpha_4H_1 + \beta_0\beta_1\beta_3\gamma_1\gamma_4H_3 + \gamma_0\gamma_1\gamma_3\alpha_1\alpha_4H_1 + \gamma_0\gamma_1\gamma_3\beta_1\beta_4H_2),$$

$$C_1 = 2(\alpha_1H_1 - \alpha_0\alpha_1\beta_1\beta_4H_2 - \alpha_0\alpha_1\gamma_1\gamma_4H_3),$$

$$C_2 = 2(\alpha\alpha_1H_1 + \alpha^2\alpha_0\alpha_1\beta_1\beta_4H_2 + \alpha^2\alpha_0\alpha_1\gamma_1\gamma_4H_3 - \alpha_1\beta_0\beta_1\beta_3H_1 - \alpha\alpha_1\beta_0\beta_1\beta_3H_1),$$

$$C_3 = 2(\beta_1H_2 - \alpha_0\alpha_1\alpha_3\beta_1H_2 - \alpha_1\alpha_4\beta_0\beta_1H_1 - \beta_0\beta_1\gamma_1\gamma_4H_3 - \gamma_0\gamma_1\gamma_3\beta_1H_2),$$

$$C_4 = 2(\beta\beta_1H_2 - \alpha_0\alpha_1\alpha_3\beta\beta_1H_2 + \beta_0\beta_1\beta^2\alpha_4H_1 - \gamma_0\gamma_1\gamma_3\beta\beta_1H_2 + \beta_0\beta_1\beta_2\gamma_1\gamma_4H_3),$$

$$C_5 = 2(\gamma_1H_3 - \alpha_0\alpha_1\alpha_3\gamma_1H_3 - \beta_0\beta_1\beta_3\gamma_1H_3 - \gamma_0\gamma_1\alpha_1\alpha_4H_1 - \gamma_0\gamma_1\beta_1\beta_4H_2),$$

$$C_6 = 2(\gamma\gamma_1H_3 - \alpha_0\alpha_1\alpha_3\gamma_1\gamma H_3 - \beta_0\beta_1\beta_3\gamma_1\gamma H_3 + \gamma_0\gamma_1\gamma^2\alpha_1\alpha_4H_1 + \gamma_0\gamma_1\gamma^2\beta_1\beta_4H_2),$$

$$C_7 = 2(\alpha_0\alpha_1\beta_1H_2 + \beta_0\beta_1\alpha_1H_1),$$

$$C_8 = 2(\alpha_0\alpha_1\beta\beta_1H_2 - \beta_0\beta_1\beta^2\alpha_1H_1),$$

$$C_9 = 2(\beta_0\beta_1\alpha_1\alpha H_1 - \alpha_0\alpha_1\alpha^2\beta_1H_2),$$

$$C_{10} = 2(\alpha_0\alpha_1\alpha^2\beta_1\beta H_2 - \beta_0\beta_1\beta^2\alpha_1\alpha H_1),$$

$$C_{11} = 2(\alpha_0\alpha_1\gamma_1H_3 + \gamma_0\gamma_1\alpha_1H_1),$$

$$C_{12} = 2(\alpha_0\alpha_1\gamma\gamma_1H_3 - \gamma_0\gamma_1\gamma^2\alpha_1H_1),$$

$$C_{13} = 2(\gamma_0\gamma_1\alpha_1\alpha H_1 - \alpha_0\alpha_1\alpha^2\gamma_1H_3),$$

$$C_{14} = -2(\gamma_1\gamma\alpha_0\alpha_1\alpha^2H_3 + \gamma_0\gamma_1\gamma^2\alpha_1\alpha H_1),$$

$$C_{15} = 2(\beta_0\beta_1\gamma_1H_3 + \gamma_0\gamma_1\beta_1H_2),$$

$$C_{16} = 2(\beta_0\beta_1\gamma\gamma_1H_3 - \gamma_0\gamma_1\gamma^2\beta_1H_2),$$

$$C_{17} = 2(\gamma_0\gamma_1\beta_1\beta H_2 - \beta_0\beta_1\beta^2\gamma_1H_3),$$

$$C_{18} = -2(\beta_0\beta_1\beta^2\gamma\gamma_1H_3 + \gamma_0\gamma_1\gamma^2\beta\beta_1H_2),$$

$$T_{l,m,n} = O(\Delta x_l^2 + \Delta y_m^2 + \Delta z_n^2)$$

From equation (3.3), we get

$$U_0 = A_1U_1 + A - 2U_2 + A_3U_3 + A_4U_4 + A_5U_5 + A_6U_6 + A_7U_7 + A_8U_8 + A_9U_9 + A_{10}U_{10} + A_{11}U_{11} + A_{12}U_{12} + A_{13}U_{13} + A_{14}U_{14} + A_{15}U_{15} + A_{16}U_{16} + A_{17}U_{17} + A_{18}U_{18} + S_{l,m,n}\,. \tag{3.5}$$

where,

$$A_i = -\frac{C_i}{C_0},\ i = 1(1)18, \quad C_0 \neq 0, \quad S_{l,m,n} = \frac{T_{l,m,n}}{C_0} = O(\Delta x_l^2 + \Delta y_m^2 + \Delta z_n^2). \tag{3.6}$$

Neglecting the error term from (3.5), we may obtain

$$\begin{aligned} u_0 = {} & A_1 u_1 + A_2 u_2 + A_3 u_3 + A_4 u_4 + A_5 u_5 + A_6 u_6 + A_7 u_7 + A_8 u_8 \\ & + A_9 u_9 + A_{10} u_{10} + A_{11} u_{11} + A_{12} u_{12} + A_{13} u_{13} + A_{14} u_{14} \\ & + A_{15} u_{15} + A_{16} u_{16} + A_{17} u_{17} + A_{18} u_{18} \end{aligned} \tag{3.7}$$

where,

$$u_0 = u_{l,m,n}, \quad u_1 = u_{l+1,m,n}, \ldots \text{etc.} \tag{3.8}$$

Assume that an error of the form $\varepsilon_{l,m,n} = u_{l,m,n} - U_{l,m,n}$ exist at each grid point (x_l, y_m, z_n), then on subtracting (3.5) from (3.7), we get the error equation as follows

$$\begin{aligned} \varepsilon_{l,m,n} = {} & A_1 \varepsilon_{l+1,m,n} + A_2 \varepsilon_{l-1,m,n} + A_3 \varepsilon_{l,m+1,n} + A_4 \varepsilon_{l,m-1,n} + A_5 \varepsilon_{l,m,n+1} + A_6 \varepsilon_{l,m,n-1} \\ & + A_7 \varepsilon_{l+1,m+1,n} + A_8 \varepsilon_{l+1,m-1,n} + A_9 \varepsilon_{l-1,m+1,n} + A_{10} \varepsilon_{l-1,m-1,n} + A_{11} \varepsilon_{l+1,m,n+1} \\ & + A_{12} \varepsilon_{l+1,m,n-1} + A_{13} \varepsilon_{l-1,m,n+1} + A_{14} \varepsilon_{l-1,m,n-1} + A_{15} \varepsilon_{l,m+1,n+1} \\ & + A_{16} \varepsilon_{l,m+1,n-1} + A_{17} \varepsilon_{l,m-1,n+1} + A_{18} \varepsilon_{l,m-1,n-1} + S_{l,m,n}. \end{aligned}$$

The above is a system of N^3 number of linear equations in N^3 unknowns, which may be expressed in the matrix form. Now applying the Jacobi iteration to the above system of equations and neglecting the error term, we obtain

(3.9)

$$\begin{aligned} \varepsilon_{l,m,n}^{(s+1)} = {} & A_1 \varepsilon_{l+1,m,n}^{(s)} + A_2 \varepsilon_{l-1,m,n}^{(s)} + A_3 \varepsilon_{l,m+1,n}^{(s)} + A_4 \varepsilon_{l,m-1,n}^{(s)} + A_5 \varepsilon_{l,m,n+1}^{(s)} + A_6 \varepsilon_{l,m,n-1}^{(s)} \\ & + A_7 \varepsilon_{l+1,m+1,n}^{(s)} + A_8 \varepsilon_{l+1,m-1,n}^{(s)} + A_9 \varepsilon_{l-1,m+1,n}^{(s)} + A_{10} \varepsilon_{l-1,m-1,n}^{(s)} + A_{11} \varepsilon_{l+1,m,n+1}^{(s)} \\ & + A_{12} \varepsilon_{l+1,m,n-1}^{(s)} + A_{13} \varepsilon_{l-1,m,n+1}^{(s)} + A_{14} \varepsilon_{l-1,m,n-1}^{(s)} + A_{15} \varepsilon_{l,m+1,n+1}^{(s)} \\ & + A_{16} \varepsilon_{l,m+1,n-1}^{(s)} + A_{17} \varepsilon_{l,m-1,n+1}^{(s)} + A_{18} \varepsilon_{l,m-1,n-1}^{(s)} \end{aligned}$$

where $\varepsilon_{l,m,n}^{(s)}$, is the error at each grid point (l, m, n), at the sth iteration.

We analyse the behavior of $\varepsilon_{l,m,n}^{(s)}$, by assuming it to be of the form:

$$\varepsilon_{l,m,n}^{(s)} = \mu^s a^l b^m c^n \sin\left(\frac{p\pi l}{N+1}\right) \sin\left(\frac{q\pi m}{N+1}\right) \sin\left(\frac{r\pi n}{N+1}\right), \quad 1 \leq p, q, r \leq N \tag{3.10}$$

where a and b are constants to be determined and μ is the propagating factor for the Jacobi iteration method which determines the rate of growth of the errors. The necessary and sufficient condition for the iterative method to be stable is

$$|\mu| \leq 1\,.$$

For convenience, let us put

$$\phi = \frac{\pi}{N+1}$$

Now substituting (3.10) in (3.9) we get

(3.11)

$$
\begin{aligned}
&\mu \sin(pl\phi)\sin(qm\phi)\sin(rn\phi)\\
&= A_1[a\sin(p(l+1)\phi)\sin(qm\phi)\sin(rn\phi)] + A_2[a^{-1}\sin(p(l-1)\phi)\sin(qm\phi)\sin(rn\phi)]\\
&\quad + A_3[b\sin(pl\phi)\sin(q(m+1)\phi)\sin(rn\phi)] + A_4[b^{-1}\sin(pl\phi)\sin(q(m-1)\phi)\sin(rn\phi)]\\
&\quad + A_5[c\sin(pl\phi)\sin(qm\phi)\sin(r(n+1)\phi)] + A_6[c^{-1}\sin(pl\phi)\sin(qm\phi)\sin(r(n-1)\phi)]\\
&\quad + A_7[ab\sin(p(l+1)\phi)\sin(q(m+1)\phi)\sin(rn\phi)]\\
&\quad + A_8[ab^{-1}\sin(p(l+1)\phi)\sin(q(m-1)\phi)\sin(rn\phi)]\\
&\quad + A_9[a^{-1}b\sin(p(l-1)\phi)\sin(q(m+1)\phi)\sin(rn\phi)]\\
&\quad + A_{10}[a^{-1}b^{-1}\sin(p(l-1)\phi)\sin(q(m-1)\phi)\sin(rn\phi)]\\
&\quad + A_{11}[ac\sin(p(l+1)\phi)\sin(qm\phi)\sin(r(n+1)\phi)]\\
&\quad + A_{12}[ac^{-1}\sin(p(l+1)\phi)\sin(qm\phi)\sin(r(n-1)\phi)]\\
&\quad + A_{13}[a^{-1}c\sin(p(l-1)\phi)\sin(qm\phi)\sin(r(n+1)\phi)]\\
&\quad + A_{14}[a^{-1}c^{-1}\sin(p(l-1)\phi)\sin(qm\phi)\sin(r(n-1)\phi)]\\
&\quad + A_{15}[bc\sin(pl\phi)\sin(q(m+1)\phi)\sin(r(n+1)\phi)]\\
&\quad + A_{16}[bc^{-1}\sin(pl\phi)\sin(q(m+1)\phi)\sin(r(n-1)\phi)]\\
&\quad + A_{17}[b^{-1}c\sin(pl\phi)\sin(q(m-1)\phi)\sin(r(n+1)\phi)]\\
&\quad + A_{18}[b^{-1}c^{-1}\sin(pl\phi)\sin(q(m-1)\phi)\sin(r(n-1)\phi)].
\end{aligned}
$$

Equation (3.11) is difficult to solve, therefore for simplicity, we consider $\alpha = \beta = \gamma = 1$ and $\Delta x_l = \Delta y_m = \Delta z_n = h$. Simplifying (3.11), we get

(3.12)

$$
\begin{aligned}
&\mu \sin(pl\phi)\sin(qm\phi)\sin(rn\phi)\\
&\quad = \frac{1}{6}[a\sin(p(l+1)\phi)\sin(qm\phi)\sin(rn\phi) + a^{-1}\sin(p(l-1)\phi)\sin(qm\phi)\sin(rn\phi)\\
&\qquad + b\sin(pl\phi)\sin(q(m+1)\phi)\sin(rn\phi) + b^{-1}\sin(pl\phi)\sin(q(m-1)\phi)\sin(rn\phi)\\
&\qquad + c\sin(pl\phi)\sin(qm\phi)\sin(r(n+1)\phi) + c^{-1}\sin(pl\phi)\sin(qm\phi)\sin(r(n-1)\phi)]
\end{aligned}
$$

(3.13)

$$
\begin{aligned}
&\mu \sin(pl\phi)\sin(qm\phi)\sin(rn\phi)\\
&= \frac{1}{6}[\sin(pl\phi)\sin(qm\phi)\sin(rn\phi)\{(a+a^{-1})\cos(p\phi) + (b+b^{-1})\cos(q\phi) + (c+c^{-1})\cos(r\phi)\}\\
&\qquad + \cos(pl\phi)\sin(qm\phi)\sin(rn\phi)\{(a-a^{-1})\sin(p\phi)\}\\
&\qquad + \sin(pl\phi)\cos(qm\phi)\sin(rn\phi)\{(b-b^{-1})\sin(q\phi)\}\\
&\qquad + \sin(pl\phi)\sin(qm\phi)\cos(rn\phi)\{(c-c^{-1})\sin(r\phi)\}].
\end{aligned}
$$

On comparing the coefficients in (3.13), we get

(3.14a) $$\mu = \frac{1}{6}[(a+a^{-1})\cos(p\phi) + (b+b^{-1})\cos(q\phi) + (c+c^{-1})\cos(r\phi)]$$

(3.14b) $$a - a^{-1} = b - b^{-1} = c - c^{-1} = 0$$

On solving (3.14b), we get

(3.15) $$a = b = c = 1$$

Using (3.15) in (3.14a) and putting back the value $\phi = \frac{\pi}{N+1}$, we get

(3.16) $$\mu = \frac{\cos(\frac{p\pi}{N+1}) + \cos(\frac{q\pi}{N+1}) + \cos(\frac{r\pi}{N+1})}{3}$$

The necessary and sufficient condition for the iteration method to be stable is given as

(3.17) $$\mu^* = \max|\mu| \le 1$$

The maximum value of $\cos(\frac{p\pi}{N+1})$, $\cos(\frac{q\pi}{N+1})$, $\cos(\frac{r\pi}{N+1})$, occur when $p = q = r = 1$ or $p = q = r = N$.

For $p = q = r = 1$, we have

(3.18) $$\mu^* = \max|\mu| = \cos\left(\frac{\pi}{N+1}\right) < 1$$

4. APPLICATION TO BI- AND TRI-HARMONIC PROBLEMS

We consider the 3D biharmonic elliptic partial differential equation with a forcing function of the form

(4.1) $$\nabla^4 u(x,y,z) \equiv u_{xxxx} + u_{yyyy} + u_{zzzz} + 2(u_{xxyy} + u_{yyzz} + u_{zzxx}) = f(x,y,z), \quad (x,y,z) \in \Omega$$

The values of u, u_{xx} are prescribed on the boundary $x = 0$, $x = 1$, the values of u, u_{yy} are prescribed on the boundary $y = 0$, $y = 1$ and the values of u, u_{zz} are prescribed on the boundary $z = 0$, $z = 1$. As the grid lines are parallel to coordinate axes and the values of u are exactly known on the boundary, this implies, the successive tangential partial derivatives of u are known exactly on the boundary. For example, on the line $y = 0$, the values of $u(x,0)$, $u_{yy}(x,0)$ and $u_{zz}(x,0)$ are known, i.e., the values of $u_x(x,0)$, $u_{xx}(x,0)$,…etc. are known on the line $y = 0$. This implies the values of $u(x,0)$ and $\nabla^2 u(x,0) \equiv u_{xx}(x,0) + u_{yy}(x,0) + u_{zz}(x,0)$ are known on the line $y = 0$. Similarly, the values of u and $\nabla^2 u$ are known on all sides of the cubic region Ω.

Let us denote $\nabla^2 u = v$. Then we can rewrite the equation (4.1) in a coupled manner as

(4.2a) $$\nabla^2 u \equiv u_{xx} + u_{yy} + u_{zz} = v(x,y,z), \quad (x,y,z) \in \Omega,$$

$$\nabla^2 v \equiv v_{xx} + v_{yy} + v_{zz} = f(x,y,z), \quad (x,y,z) \in \Omega, \tag{4.2b}$$

In this case, the values of u and v are exactly known on the boundary of Ω.

Applying the methods (2.7) and (2.8) to the system of equations (4.2a)-(4.2b), a numerical method of order two on the variable mesh for the solution of bi-harmonic equation (4.1) can be written as

$$\begin{aligned} SL_U &= V_{l,m,n} + \alpha_0 \Delta x_l \overline{V}_{x_{l,m,n}} + \beta_0 \Delta y_m \overline{V}_{y_{l,m,n}} + \gamma_0 \Delta z_n \overline{V}_{z_{l,m,n}} \\ &\quad + O(\Delta x_l^2 + \Delta y_m^2 + \Delta z_n^2) \\ &\equiv SR_V + O(\Delta x_l^2 + \Delta y_m^2 + \Delta z_n^2), \end{aligned} \tag{4.3a}$$

$$SLV = SRH_f + O(\Delta x_l^2 + \Delta y_m^2 + \Delta z_n^2). \tag{4.3b}$$

and a numerical method of order three for the equation (4.1) on the variable mesh may be written as

$$\begin{aligned} TL_U &= V_{l,m,n} + \alpha_0 \Delta x_l \overline{V}_{x_{l,m,n}} + \beta_0 \Delta y_m \overline{V}_{y_{l,m,n}} + \gamma_0 \Delta z_n \overline{V}_{z_{l,m,n}} \\ &\quad + \alpha_2 \Delta x_l^2 \overline{V}_{xx_{l,m,n}} + \beta_2 \Delta y_m^2 \overline{V}_{yy_{l,m,n}} + \gamma_2 \Delta z_n^2 \overline{V}_{zz_{l,m,n}} \\ &\quad + \alpha_0 \beta_0 \Delta x_l \Delta y_m \overline{V}_{xy_{l,m,n}} + \beta_0 \gamma_0 \Delta y_m \Delta z_n \overline{V}_{yz_{l,m,n}} \\ &\quad + \gamma_0 \alpha_0 \Delta z_n \Delta x_l \overline{V}_{zx_{l,m,n}} + O(\Delta x_l^3 + \Delta y_m^3 + \Delta z_n^3) \\ &\equiv TR_V + O(\Delta x_l^3 + \Delta y_m^3 + \Delta z_n^3), \end{aligned} \tag{4.4a}$$

$$TLV = TRH_f + O(\Delta x_l^3 + \Delta y_m^3 + \Delta z_n^3). \tag{4.4b}$$

Note that, replacing V by W, we can write the expression for SR_W and TR_W.

Next we consider the tri-harmonic equation with a forcing function of the form

$$\begin{aligned} \nabla^6 u(x,y,z) &\equiv u_{xxxxxx} + u_{yyyyyy} + u_{zzzzzz} + 6u_{xxyyzz} + 3(u_{xxxxyy} \\ &\quad + u_{xxyyyy} + u_{yyyyzz} + u_{yyzzzz} + u_{zzzzxx} + u_{zzxxxx}) \\ &= f(x,y,z), \quad (x,y,z) \in \Omega \end{aligned} \tag{4.5}$$

For this equation, the boundary values of u, u_{xx}, u_{xxxx} are prescribed on the line $x = 0$, $x = 1$; the boundary values u, u_{yy}, u_{yyyy} are prescribed on the line $y = 0$, $y = 1$ and the boundary values of u, u_{zz}, u_{zzzz} are prescribed on the line $z = 0$, $z = 1$. As discussed in bi-harmonic case, the values of u, $\nabla^2 u$ and $\nabla^4 u$ are known on all sides of the cubic region Ω.

Let $\nabla^2 u = v$ and $\nabla^2 v = w$. Then, we rewrite the equation (4.5) in a system of three Poisson equations of the form

$$\nabla^2 u \equiv u_{xx} + u_{yy} + u_{zz} = v(x,y,z), \ (x,y,z) \in \Omega, \tag{4.6a}$$

$$\nabla^2 v \equiv v_{xx} + v_{yy} + v_{zz} = w(x,y,z), \ (x,y,z) \in \Omega, \tag{4.6b}$$

$$\nabla^2 w \equiv w_{xx} + w_{yy} + w_{zz} = f(x,y,z), \ (x,y,z) \in \Omega. \tag{4.6c}$$

Applying the methods (2.7) and (2.8) to the system of equations (4.6a)-(4.6c), a numerical method of order two on the variable mesh for the solution of tri-harmonic equation (4.5) can be written as

(4.7a) $$SL_U = SR_V + O(\Delta x_l^2 + \Delta y_m^2 + \Delta z_n^2),$$

(4.7b) $$SL_V = SR_W + O(\Delta x_l^2 + \Delta y_m^2 + \Delta z_n^2),$$

(4.7c) $$SL_W = SRH_f + O(\Delta x_l^2 + \Delta y_m^2 + \Delta z_n^2).$$

and a numerical method of order three for the tri-harmonic equation (4.5) on the variable mesh may be written as

(4.8a) $$TL_U = TR_V + O(\Delta x_l^3 + \Delta y_m^3 + \Delta z_n^3),$$

(4.8b) $$TL_V = TR_W + O(\Delta x_l^3 + \Delta y_m^3 + \Delta z_n^3),$$

(4.8c) $$TL_W = TRH_f + O(\Delta x_l^3 + \Delta y_m^3 + \Delta z_n^3).$$

By the help of boundary values, writing all methods at every interior grid points, one obtains sparse systems of linear algebraic equations for the solution of multi-harmonic equations (2.1), (4.1) and (4.5). Direct solution of these linear systems is impractical because of the large size of the coefficient matrix and enormous storage requirements even for moderate values of grid size. Classical iterative methods such as Gauss-Seidel and successive over relaxation are attractive for their low storage requirements as long as convergence is guaranteed.

5. **NUMERICAL ILLUSTRATIONS**

In this section, we have solved the equations (2.1), (4.1) and (4.5) subject to prescribed appropriate boundary conditions, using the methods described in the sections 2 and 3. The exact solutions are provided in each case. The right-hand side homogeneous functions and boundary conditions may be obtained using the exact solution as a test procedure. The linear difference equations have been solved using Gauss-Seidel iterative methods ([22, 23, 24, 25]). For iteration method, we have chosen zero vector as the initial guess and the iterations were stopped when the error tolerance $\leq 10^{-12}$ was achieved. All computations were carried out using double precision arithmetic. All computations were done using MATLAB codes.

The starting values of first step lengths in x-, y- and z-directions are given by

(5.1a) $$\Delta x_1 = (1-\alpha)/(1-\alpha^{N+1}), \quad \alpha \neq 1,$$

(5.1b) $$\Delta y_1 = (1-\beta)/(1-\beta^{N+1}), \quad \beta \neq 1,$$

(5.1c) $$\Delta z_1 = (1-\gamma)/(1-\gamma^{N+1}), \quad \gamma \neq 1.$$

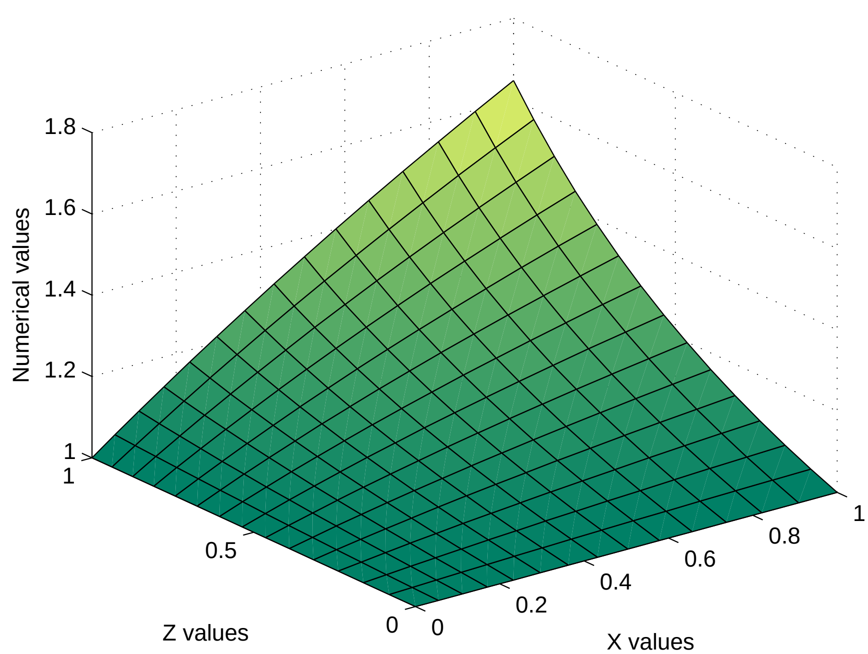
Hence, by prescribing the total number of grid points $N+2$ in x-direction, $N+2$ in y-direction and $N+2$ in z-direction we can compute the value of Δx_1, Δy_1, and Δz_1

from (5.1). The remaining mesh is determined by $\Delta x_{l+1} = \alpha \Delta x_l$, $\Delta y_{m+1} = \beta \Delta y_m$ and $\Delta z_{n+1} = \gamma \Delta z_n$; $l, m, n = 1, 2, \ldots, N$.

Test Example 1. The equation (2.1) is to be solved with the exact solution $u = e^{xyz}$. The maximum absolute errors are tabulated in Table 1 for various values of α, β and γ. Figures 1 and 2 give the plot of numerical and exact solutions at $y = 0.5$ respectively for $\alpha = 1.04$, $\beta = .96$, $\gamma = .98$, $N = 12$.

TABLE 1

Test Example 1							
$N+1$		$\nabla^2 u = f$, $u = e^{xyz}$					
		$\alpha = .95$, $\beta = .97$, $\gamma = .98$		$\alpha = 1.01$, $\beta = 1.02$, $\gamma = 1.03$		$\alpha = 1.04$, $\beta = .96$, $\gamma = .98$	
		Second order	Third order	Second order	Third order	Second order	Third order
10	u	6.1332(-6)	1.2422(-6)	9.3213(-6)	1.7945(-6)	7.7204(-6)	7.9728(-7)
	u_x	1.4327(-3)	1.4228(-3)	1.5889(-3)	1.6224(-3)	2.5129(-3)	2.5334(-3)
	u_y	1.4112(-3)	1.4250(-3)	1.7662(-3)	1.7971(-3)	1.1788(-3)	1.764(-3)
	u_z	1.5653(-3)	1.5927(-3)	1.9570(-3)	1.9856(-3)	1.3317(-3)	1.3644(-3)
15	u	2.9198(-6)	4.7790(-7)	4.8904(-6)	5.4539(-7)	3.9793(-6)	2.9485(-7)
	u_x	1.0007(-3)	9.9618(-4)	1.0154(-3)	1.0387(-3)	1.8364(-3)	1.8466(-3)
	u_y	7.9825(-4)	7.9782(-4)	1.1840(-3)	1.2049(-3)	7.8816(-4)	7.8310(-4)
	u_z	8.6297(-4)	8.7336(-4)	1.3721(-3)	1.3908(-3)	7.4094(-4)	7.4753(-4)
20	u	1.6666(-6)	2.4121(-7)	3.0941(-6)	2.4842(-7)	2.4460(-6)	1.6257(-7)
	u_x	7.8279(-4)	7.8030(-4)	7.0527(-4)	7.2188(-4)	1.4466(-3)	1.4519(-3)
	u_y	5.6730(-4)	5.6548(-1)	8.6146(-4)	8.7594(-3)	5.9304(-4)	5.8997(-4)
	u_z	5.2368(-4)	5.2315(-4)	1.0417(-3)	1.0544(-3)	4.5447(-4)	4.5282(-4)



Test Example 2. The equation (4.1) is to be solved with the exact solution $u = (1 - \cos 2\pi x)(1 - \cos 2\pi y)(1 - \cos 2\pi z)$. The maximum absolute errors are tabulated in Table 2 for various values of α, β and γ. Figures 3 and 4 give the plot of numerical and exact solution at $y = 0.5$ respectively for $\alpha = 1.1$, $\beta = .9$, $\gamma = .8$, $N = 16$.

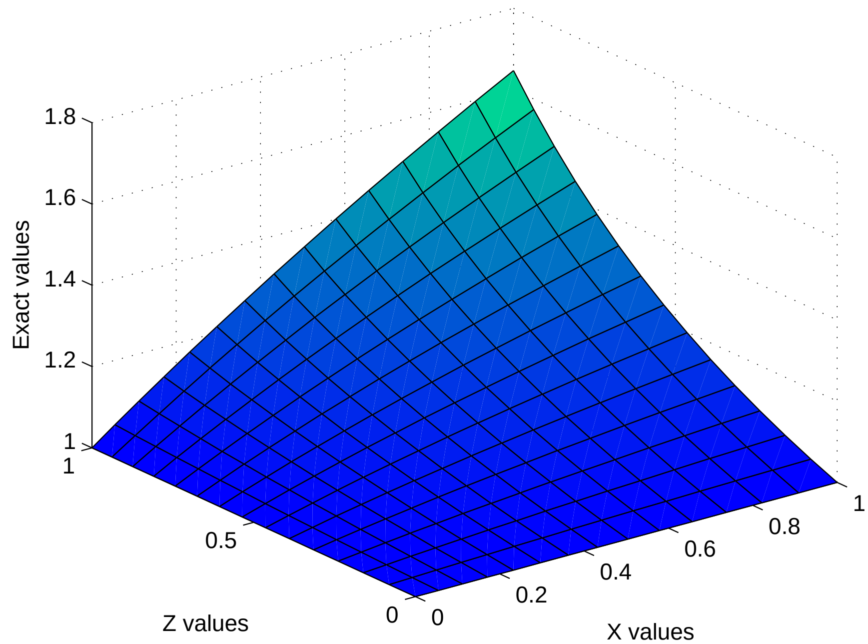
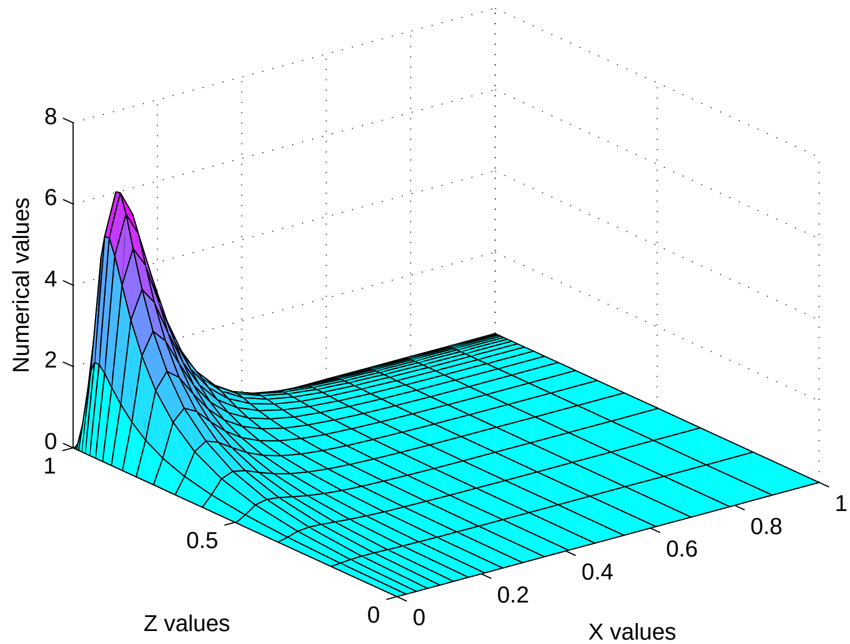


TABLE 2

Test Example 2							
$N+1$		$\nabla^2 u = f$, $u = (1-\cos 2\pi x)(1-\cos 2\pi y)(1-\cos 2\pi z)$					
		$\alpha = .89$, $\beta = .95$, $\gamma = .97$		$\alpha = 1.05$, $\beta = 1.06$, $\gamma = 1.07$		$\alpha = 1.1$, $\beta = .9$, $\gamma = .8$	
		Second order	Third order	Second order	Third order	Second order	Third order
10	u	6.1917(-2)	4.5236(-3)	5.7852(-2)	3.3405(-3)	8.2898(-2)	1.7442(-2)
	$\nabla^2 u$	1.6680(00)	2.3426(-1)	1.5500(00)	1.6990(-1)	2.1942(00)	9.7146(-1)
15	u	3.1970(-2)	1.7981(-3)	2.9404(-2)	1.0922(-3)	5.4776(-2)	1.1289(-2)
	$\nabla^2 u$	8.6924(-1)	1.0354(-1)	8.0377(-1)	5.4840(-2)	1.4676(00)	6.8143(-1)
20	u	2.1347(-2)	1.1084(-3)	1.8540(-2)	5.5491(-4)	4.4913(-2)	9.5090(-3)
	$\nabla^2 u$	5.8836(-1)	6.7305(-2)	5.1119(-1)	2.7816(-2)	1.2020(00)	6.0176(-1)

Test Example 3. The equation (4.5) is to be solved with the exact solution $u = \sin(\pi x)\sin(\pi y)\sin(\pi z)$. The maximum absolute errors are tabulated in Table 3 for various values of α, β and γ. Figures 5 and 6 give the plot of numerical and exact solution at $x = 0.5$ respectively for $\alpha = 1.1$, $\beta = .9$, $\gamma = .8$, $N = 10$.



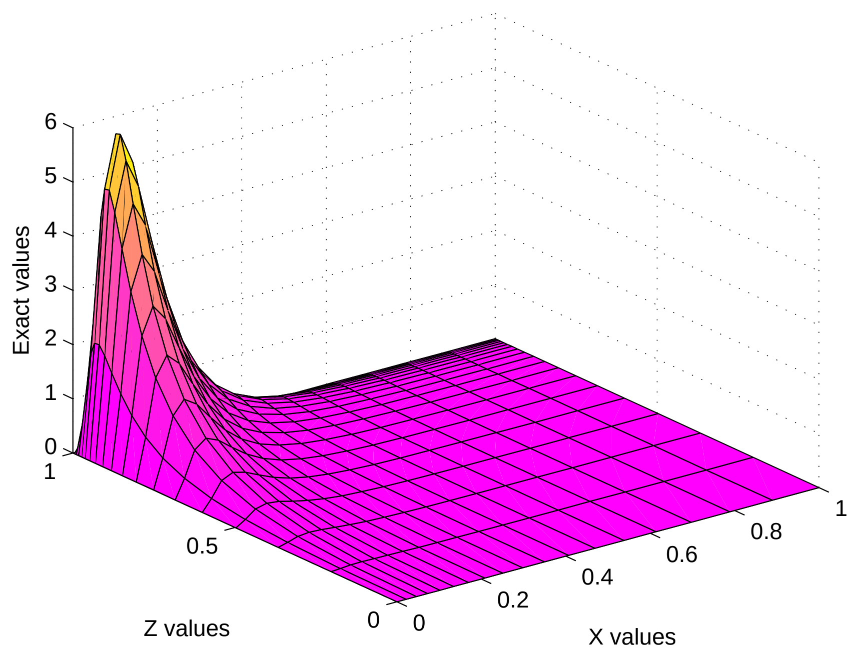
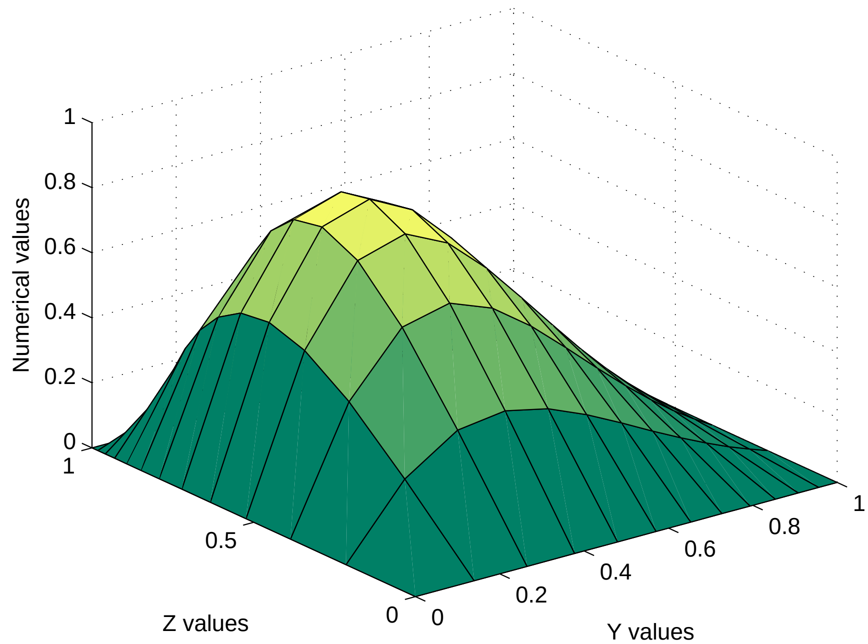
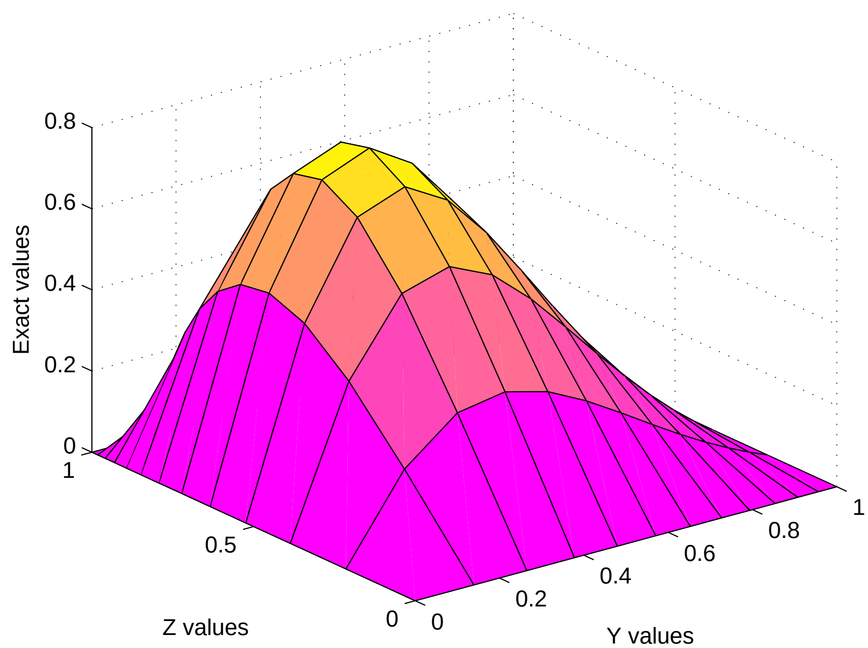


TABLE 3

Test Example 2							
$N+1$		$\nabla^6 u = f$, $u = \sin(\pi x)\sin(\pi y)\sin(\pi z)$					
		$\alpha = .96$, $\beta = .98$, $\gamma = .99$		$\alpha = 1.03$, $\beta = 1.04$, $\gamma = 1.05$		$\alpha = 1.1$, $\beta = .9$, $\gamma = .8$	
		Second order	Third order	Second order	Third order	Second order	Third order
10	u	2.0845(-3)	3.7073(-5)	2.1301(-3)	4.1255(-5)	3.3559(-3)	1.8839(-4)
	$\nabla^2 u$	4.0989(-2)	9.1941(-4)	4.1822(-2)	1.0492(-3)	6.5602(-2)	5.8500(-3)
	$\nabla^4 u$	6.0430(-1)	2.1839(-2)	6.1531(-1)	2.6151(-2)	9.4352(-1)	2.2403(-1)
15	u	9.9679(-4)	9.1920(-6)	1.0454(-3)	1.3100(-5)	2.3107(-3)	1.1092(-4)
	$\nabla^2 u$	1.9668(-2)	2.3291(-4)	2.0589(-2)	2.9931(-4)	4.5430(-2)	3.7603(-3)
	$\nabla^4 u$	2.9107(-1)	5.9208(-3)	3.0454(-1)	7.9429(-3)	6.5397(-1)	1.6000(-1)
20	u	5.9777(-4)	3.5345(-6)	6.4003(-4)	4.8541(-6)	1.9629(-3)	9.0544(-5)
	$\nabla^2 u$	1.1791(-2)	9.2535(-5)	1.2634(-2)	1.3219(-4)	3.8729(-2)	3.1394(-3)
	$\nabla^4 u$	1.7439(-1)	2.4961(-3)	1.8707(-1)	3.6550(-3)	5.5822(-1)	1.4202(-1)





6. **Conclusions**

Using 19-grid points and a single computational cell on a variable mesh, we have derived numerical methods of order two and three for the solution of multi-harmonic elliptic partial differential equations. We do not need to discretize the boundary conditions and boundary values were exactly used in the difference schemes. Numerical solutions of Laplacian and bi-Laplacian are obtained as by-product of the methods discussed in this paper. Further, using same 19-grid points and variable mesh, we have also discussed numerical methods of order two and three for the estimates of $(\partial u/\partial n)$. Numerical results confirm the utility of the proposed methods on the variable mesh.

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