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A PROBABILISTIC CONSIDERATION ON ONE DIMENSIONAL KELLER SEGEL SYSTEM

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ABSTRACT. In this paper, we perform a consideration on partial differential equations known as the Keller Segel system (KS) through both probabilistic and numerical methods. Stochastic differential equation (SDE), having a correspondence with (KS) is the main tool of our probabilistic consideration. In our main theorem, by using probabilistic expressions of (u, v) with $u = u(x, t)$, $v = v(x, t)$, the solution of (KS), by means of expectation of functionals of the SDE, we derive a set of bounds for (u, v) which gives a good estimate of (u, v) around time zero.

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1. Introduction

The Keller Segel system is a biologcal model which is proposed by Keller and Segel [3] in 1970's. Here, we deal with the following one-dimensional Keller Segel system (1.1), (1.2) with Neuman boundary conditions (1.3).

$$\text{(KS)}\begin{cases} u_t = u_{xx} - a(uv_x)_x & (x,t) \in I \times (0,\infty), \quad (1.1) \\ v_t = v_{xx} - \gamma v + \alpha u & (x,t) \in I \times (0,\infty), \quad (1.2) \\ u_x(L_1,t) = u_x(L_2,t) = v_x(L_1,t) = v_x(L_2,t) = 0 & t \in (0,\infty), \quad (1.3) \\ u(x,0) = \overline{u}(x),\ v(x,0) = \overline{v}(x) & x \in I, \end{cases}$$

where $I = (L_1, L_2)$ with some L_1 and L_2 such that $-\infty < L_1 < L_2 < \infty$, is a bounded open interval, and a, α, γ are some positive constants. The solutions $u = u(x, t)$ and $v = v(x, t)$ represent the cell density of the cellular slime mold and the cell concentration of the chemical substance that released by the cellular slime mold in $I \times (0, \infty)$, respectively.

There exists an intensive consideration on the existence of unique solution of the Keller Segel system (KS) and its asymptotic behavior. In fact, it is known that (KS) has a time global unique classical solution (u, v) under suitable initial conditions (Osaki and Yagi [5]). Nevertheless, the behavior of the solution near to time zero

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has not been investigated in detail. Here we consider such behavior of the solution through an analytical approach with a help of numerical technique.

Our analytical method for the consideration of the solution (u, v) is a stochastic analysis, by using the stochastic differential equation (SDE) driven by a standard Brownian motion.

In Theorem 4.2, by using an expression of (u, v) by means of expectation of functionals of the SDE, we derive bounds for (u, v) which gives a good estimate of (u, v) around time zero. This theorem, however, it would been an application or modification of maximal principle in the usual analysis, can be proved easily through the stochastic analytic methods.

Figures 4.5 and 4.6 which are composed by using finite defference method, are visualzations of the results of Theorem 4.2.

The contents of this paper is the following: in section 2, we give a short review on the correspondence with the Keller Segel system and the original biological phenomenon. Also we summarize a correspondence with heat equation and the standard Brownian motion, the results of which are basic and important throughout this paper. Section 3 gives two results introduced by [5] and Bensoussan and Lions [1] by which we derive our main theorem. [5] guarantees the uniqueness and existence of the classical solution of (KS). Section 4 is devoted to state our main result, and the final section 5 is the detailed proof of the propositions given in section 4.

2. Preliminaries

Figure 2.1.1 The life cycle of the cellular slime mold
(From the homepage of Japanese Society for the Study of Cellular Slime Molds)

2.1. **The Keller Segel system as the biological model.** The cellular slime mold forms the structure like the plant called a fruit body. Then the spore released from a fruit body germinates, and increases in the state of the amoeba. After the cell created from the spore eats whole of feed of bacteria in the surrounding area, it falls into starvation. Then it begins to release a chemical substance which attracts other cells. Hence they are gathering. And a cell body moves to the lightning place, and it grows to a fruit body. (See Figure 2.1.1.) The Keller Segel system is the biological model which describes the movement until a cellular slime mold falls in the hunger state and forms an aggregate.

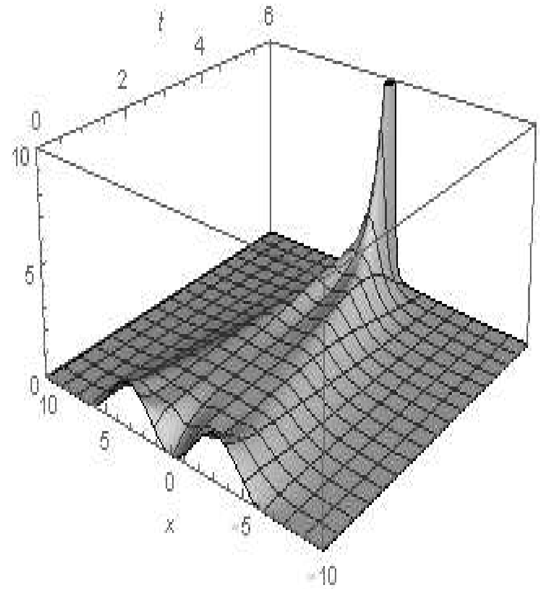

Figure 2.1.2 Result of the nummerical computation

We introduce the following example. Let parameters $a, \alpha, \gamma, L_1, L_2$ and initial functions $\overline{u}, \overline{v}$ in (KS) be as follows: $a = 3$, $\alpha = \gamma = 1$, $L_1 = -10$, $L_2 = 10$,

$$\overline{u}(x) = \begin{cases} \cos(x+\pi)+1 \ (-2\pi \le x \le 2\pi), \\ 0 \ (-10 < x < -2\pi,\ 2\pi < x < 10), \end{cases}$$

$$\overline{v}(x) = \begin{cases} \cos x+1 \ (-\pi \le x \le \pi), \\ 0 \ (-10 < x < -\pi,\ \pi < x < 10). \end{cases}$$

Then we have the above graph of $u(x,t)$ by a direct, numerical computation. If we interpret Figure 2.1.2 as the biological model, it shows that two groups of the cells form an aggregate as time passes.

2.2. **Brownian motion and heat equation.** The heat equation is given by

$$u_t = ku_{xx},$$

with $u = u(x,t)$ and a positive constant k. Example 2.1, given below, shows a correspondence between the heat equation and the standard Brownian motion.

Example 2.1. We consider the following example with the initial conditon of the heat equation:

$$\text{(H)} \begin{cases} u_t = \dfrac{1}{2} u_{xx} & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = \overline{u}(x) & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $u = u(x,t)$ is the temperature of the object in the location x and at time t. The fundamental solution $K(x,t)$ of (2.1) is given by

$$K(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}. \tag{2.2}$$

The solution of the initial problem of the heat equation (H) is given by the convolution of $K(x,t)$ and the initial function $\overline{u}$ as follows:

$$u(x,t) = \int_{-\infty}^{\infty} K(x-y,t)\, \overline{u}(y)\, dy. \tag{2.3}$$

In fact, the relation $K_t = \frac{1}{2} K_{xx}$ can be certified through a direct calculation with (2.2). By this and by performing differentiations for (2.3) (noting that for $t > 0$ the kernel K is smooth with respect to both variables), we obviously see that $u(x,t)$ satisfies (H). On the other hand, we define a sequence of random variables $\{X_n\}$ as follows: Suppose that we throw one coin repeatedly. At the k-th trial, if the coin is head then we define the random variable $X_k = 1$, and if it is tail then we set $X_k = -1$. Let S_n and $B_n(x,t)$ be as follows:

$$S_n = \sum_{k=1}^{n} X_k, \quad B_n(x,t) = \frac{S_{[nt]}}{\sqrt{n}} + x.$$

Then from the central limit theorem, $B_n(x,t)$ converges to a certain stochastic process $B(x,t)$ that follows the normal distribution with mean x and variance t. That is, there exists a Brownian motion $B(x,t)$ which holds the following equation:

$$P(a \leq B(x,t) \leq b) = \int_a^b K(x-y,t) dy,$$

where $B(x,0) = x$, and P is a probability measure on a measurable space of continuous path $C([0,\infty);\mathbb{R})$. Let E denote the expectation by means of the probablistic measure P. In the sequel, we adopt the similar notations (cf. Proposition 4.1). Then $u(x,t)$ given by (2.3) can be expressed the following equation:

$$u(x,t) = E[\, \overline{u}(B(x,t)) \mid B(x,0) = x].$$

The standard Brownian motion is defined as a Markov process from the mathematical viewpoint in the probability theory. The standard Brownian motion is a continuous process, and $B(0,1)$ has the normal distribution with mean 0 and variance 1. In this paper, we define stochastic processes $X(s)$ and $Y(s)$ which are independent standard Brownian motions each other and also give probabilistic expressions to the solution (u,v) of (KS).

3. Existing results

3.1. **The existance and uniqueness of the solution of the Keller Segel system.** It is known that (KS) has an unique time global classical solution (u,v) under suitable conditions. (See Theorem 4.2 and section 7 in [5].)

Proposition 3.1 (Osaki, Yagi [5]). *Suppose that the initial functions $\overline{u},\overline{v}$ satisfy the following conditions,*

$$\inf_{x\in I}\overline{u}>0,\quad \inf_{x\in I}\overline{v}>0,\quad \overline{u}\in H^2_N(I),\quad \overline{v}\in H^3_N(I).$$

Then, there exists a unique time global solution (u,v) of (KS) such that

$$u\in C^1([0,\infty);L^2(I))\cap C^1((0,\infty);H^1(I))\cap C([0,\infty);H^2_N(I))\cap C((0,\infty);H^3_N(I)), \tag{3.1}$$

$$v\in C^1([0,\infty);H^1(I))\cap C([0,\infty);H^3_N(I))\cap C^1((0,\infty);H^2_N(I))\cap C^1((0,\infty);H^4_{N^2}(I)), \tag{3.2}$$

where

$$H^k_N(I):=\left\{u\in H^k(I)\mid \frac{du}{dx}(L_1)=\frac{du}{dx}(L_2)=0\right\}\quad (k=2,3),$$

$$H^4_{N^2}(I):=\left\{u\in H^4(I)\mid \frac{du}{dx}(L_1)=\frac{du}{dx}(L_2)=0,\frac{d^3u}{dx^3}(L_1)=\frac{du^3}{dx^3}(L_2)=0\right\}.$$

Moreover, by Sobolev's embedded theorem, from (3.1) and (3.2) we have

$$u\in C^1([0,\infty);L^2(I))\cap C((0,\infty);C^2_N(\overline{I})), \tag{3.3}$$

$$v\in C^1([0,\infty);C(\overline{I}))\cap C([0,\infty);C^2_N(\overline{I})), \tag{3.4}$$

where

$$C^2_N(\overline{I}):=\left\{u\in C^2(\overline{I})\mid \frac{du}{dx}(L_1)=\frac{du}{dx}(L_2)=0\right\}.$$

In this paper, our object is to give a probablistic expression to the time global solution (u,v) of (KS), and to consider the properties of the solution. To do so, before proceeding to the main section of the present paper, we recall the fundamental formula in stochastic analysis.

3.2. **Itô's formula.** As we see in Example 2.1, the solution of the initial problem of the heat equation is expressed by means of the expectation with the standard Brownian motion. K. Itô led to the following famous Itô's formula to correspondence with stochastic differential equations and diffusion equations.

Proposition 3.2 (Theorem 7.4, Bensoussan, Lions [1])**.** *Let $u \in C^2(\mathbb{R})$. For any $x \in \mathbb{R}$, $t \in [0,\infty)$, let $X(s)$, $t \le s < \infty$, be the stochastic process defined by the following stochastic differential equation:*

$$\begin{cases} dX(s) = b(X(s)ds + \sigma(X(s))\, dB_s, \\ X(t) = x, \end{cases}$$

where B_s is the standard Brownian motion defined on a complete probability space $(\Omega, F, P; F_t)$, with a filtration $(F_t)_{t\ge 0}$ and $b \in C^{1,1}(\mathbb{R} \times [0,\infty))$, $\sigma \in C^1(\mathbb{R})$. In addition, if we assume that there exist a positive constant M such that $\sigma(y) \ge M$ for any $y \in \mathbb{R}$, then the following Itô's formula holds:

$$\begin{cases} du(X(t)) = u'(X(t))(b(X(t))\, dt + \sigma(X(t))dB + \frac{1}{2}\sigma^2(X(t)) \cdot u''(X(t))\, dt, \\ u(X(t)) = u(x). \end{cases}$$

The above Itô's formula is extended to general semi-martingale (cf. section II-4 of [2]), known as generalized Itô's formula. The first and the second equasions (1.1) and (1.2) of (KS) are diffusion equations. We give the probablistic expressions to the solution of backward equations of (KS) by using stochastic differential equations.

4. Main result

As has seen in Proposition 3.1, (KS) has a unique time global classical solution (u, v) under the initial conditions given in the same proposition. In our main theorem, Theorem 4.2, we give bounds for the solution (u, v) of (KS). We prepare the backward equations of the Keller Segel system (KS) which is replaced t by $T - t$ for any $T > 0$.

$$(\mathrm{KS})^* \begin{cases} -\tilde{u}_t = \tilde{u}_{xx} - a(\tilde{u}\tilde{v}_x)_x & (x,t) \in I \times (0,T), & (4.1) \\ -\tilde{v}_t = \tilde{v}_{xx} - \gamma\tilde{v} + \alpha\tilde{u} & (x,t) \in I \times (0,T), & (4.2) \\ \tilde{u}_x(L_1,t) = \tilde{u}_x(L_2,t) = \tilde{v}_x(L_1,t) = \tilde{v}_x(L_2,t) = 0 & t \in (0,T), & (4.3) \\ \tilde{u}(x,T) = \overline{u}(x),\ \tilde{v}(x,T) = \overline{v}(x) & x \in I, \end{cases}$$

Note that there exists a unique solution $(\tilde{u}, \tilde{v})$ of $(\mathrm{KS})^*$ that possesses sufficient regularities by using Proposition 3.1. For the solution $\tilde{u}(x,t)$ of the backward equations that has sufficient regularity (cf. (3.3), (3.4)), we can apply Itô's formula to the stochastic process $\{\tilde{u}(X(t),t)\}_{t\ge 0}$ composed by $\tilde{u}$ with some Itô process $(X(t))_{t\ge 0}$, and can clearly derive an expression of $\tilde{u}$ by means of an expectation of a process through the standard discussion given e.g. chapter VIII of [1]. We can derive the following results for the solution $(\tilde{u}, \tilde{v})$ of our partial differential equations (cf. Pardoux and Pengor [6], precisely see (4.8) below and its proof). We can also treat a solution of the forward equations (KS) and discuss a probabilistic expression of it (as was seen in Example 2.1), but in order to do so, we have to pass through another careful discussion on interchanging of semi-group and its generator corresponding to the Itô process, namely, we need to pass through a discussion on identifications between a solution of a stochastic defferential equation and a diffusion process defined through Markow semi-group cf. e.g., chapter IV of Ikeda and Watanabe [2] (cf. also Ma and Röckner [4]).

Proposition 4.1. *Suppose that the conditions given by Propositon* 3.1 *are satisfied. Let* $(\tilde{u}, \tilde{v})$ *be a classical solution of* $(KS)^*$ *in* $I \times (0,T)$*, and let the stochastic processes* $X(s), Y(s)$ *be the solutions of the following stochastic differential equations* (4.4), (4.5) *respectively:*

$$\begin{cases} dX(s) = -a\tilde{v}_x(X(s), s)\chi_I(X(s))\, ds + \sqrt{2}\, \chi_I(X(s))\, dB_s + d\phi_1(s), \\ X(t) = x. \end{cases} \tag{4.4}$$

$$\begin{cases} dY(s) = \sqrt{2}\, \chi_I(Y(s))\, dB_s + d\phi_2(s), \\ Y(t) = x, \end{cases} \tag{4.5}$$

where $\chi_I(\cdot)$ *is the indicator function,* $\chi_I(z) = \begin{cases} 1 & (z \in I) \\ 0 & \textit{(otherwise)}, \end{cases}$ $\phi_1(s)$ *and* $\phi_2(s)$ *are the local time of the process* $\{X(s)\}$ *and* $\{Y(s)\}$ *respectively by which the boundary points* L_1 *and* L_2 *become reflection boundaries* (*cf. section IV of* [2]*, and* [7] *and references therein*)*. The conditions* $X(t) = x$ *and* $Y(t) = x$ *imply that the position of stochastic processes* $X(s)$ *and* $Y(s)$ *at time* t *are* x*. Then we have the following probabilistic expressions:*

$$\tilde{u}(x,t) = E\left[\overline{u}(X(T))e^{-\int_t^T a\tilde{v}_{xx}(X(\tau),\tau)\, d\tau} \middle| X(t) = x\right]. \tag{4.6}$$

(4.7)
$$\tilde{v}(x,t) = E\left[\overline{v}(Y(T))e^{-\gamma(T-t)} \mid Y(t) = x\right] + E\left[\int_t^T \alpha\tilde{u}(Y(s), s)e^{-\gamma(s-t)} ds \mid Y(t) = x\right],$$

where B_s *is the standard Brownian motion, and* E *is the expectation depended on expectation measure* P*.*

Proof of Proposition 4.1. We consider the following functional with stochastic process $X(s)$

$$\tilde{u}(X(s), s)e^{-\int_t^s a\tilde{v}_{xx}(X(\tau),\tau)\, d\tau} \quad (s \geq t).$$

From Proposition 3.2, in particular by the generalized Itô's formula, and (4.1), we have

$$d\left(\tilde{u}(X(s),s)e^{-\int_t^s a\tilde{v}_{xx}(X(\tau),\tau)\,d\tau}\right)=\sqrt{2}\,\tilde{u}_x(X(s),s)\,dB_s\cdot e^{-\int_t^s a\tilde{v}_{xx}(X(\tau),\tau)\,d\tau}. \tag{4.8}$$

(Refer to Appendix 5.1 for more detailed proof.)

We integrate both sides of (4.8) from t to T, and take the expectation E, then we have the following equation:

$$\begin{aligned}&E[\tilde{u}(X(T),T)e^{-\int_t^T a\tilde{v}_{xx}(X(\tau),\tau)d\tau}-\tilde{u}(X(t),t)\mid X(t)=x]\\&\qquad=E\left[\sqrt{2}\int_t^T e^{-\int_t^s a\tilde{v}_{xx}(X(\tau),\tau)d\tau}\tilde{u}_x(X(s),s)dB_s\mid X(t)=x\right].\end{aligned} \tag{4.9}$$

Since the right side of (4.9) equals to zero and $\tilde{u}(x,T)$ equals to $\overline{u}(x)$, we have the equation (4.6). By considering the functional $\tilde{v}(Y(s),s)e^{-\gamma\int_t^s\,d\tau}$, we obtain (4.7) in a similar fashion. (Refer to Appendix 5.2 for the proof.) □

By taking s as the time to go, for the backward equations (KS)*, from (4.6) we have

$$\tilde{u}(x,T-s)=E\left[\overline{u}(X(T))e^{-\int_{T-s}^T a\tilde{v}_{xx}(X(\tau),\tau)\,d\tau}|X(T-s)=x\right]. \tag{4.10}$$

Noting the regularity given by Proposition 4.1, for each $T>0$, by setting

$$m:=\inf_{(x,t)\in I\times[0,T)}v_{xx}(x,t),\quad M:=\sup_{(x,t)\in I\times[0,T)}v_{xx}(x,t),$$

from (4.10) we immediately have

$$\begin{aligned}E\left[\overline{u}(X(T))\mid X(T-s)=x\right]\cdot e^{-saM}&\le\tilde{u}(x,T-s)\\&\le E\left[\overline{u}(X(T))\mid X(T-s)=x\right]\cdot e^{-sam}.\end{aligned}$$

Therefore, we have the following inequality:

$$\inf_{x\in I}\overline{u}(x)\cdot e^{-saM}\le\tilde{u}(x,T-s)\le\sup_{x\in I}\overline{u}(x)\cdot e^{-sam}. \tag{4.11}$$

Note that because of the Neuman boundary condition for v, $M>0$ and $m<0$ hold.

For the solution $\tilde{v}$ of the backward equation system (KS)*, from (4.7) we have

$$\begin{aligned}\tilde{v}(x,T-s)&=E[\overline{v}(Y(T))e^{-\gamma s}\mid Y(T-s)=x]\\&\quad+E\left[\int_{T-s}^T\alpha\tilde{u}(Y(\tau),\tau)e^{-\gamma(\tau-T+s)}d\tau\mid Y(T-s)=x\right]:=I+II.\end{aligned}$$

By (4.5), since the process Y is a Brownian motion with reflection boundaries, we have the following equation:

$$\begin{aligned}I&=e^{-\gamma s}E[\overline{v}(Y(T))\mid Y(T-s)=x]\\&=e^{-\gamma s}\left(\frac{A_0}{2}+\sum_{n=1}^{\infty}A_n e^{-(\frac{n\pi}{L_2-L_1})^2 s}\cos\frac{n\pi}{L_2-L_1}(x-L_1)\right),\end{aligned}$$

where $A_n = \frac{2}{L_2-L_1}\int_{L_1}^{L_2}\overline{v}(x)\cos\frac{n\pi}{L_2-L_1}(x-L_1)dx$ $(n=0,1,2,\dots)$. For the integrand of II, from (4.11) it holds that

$$\tilde{u}(Y(\tau),\tau) = \tilde{u}(Y(\tau),T-(T-\tau)) \le \sup_{x\in I}\overline{u}\cdot e^{-(T-\tau)am}.$$

Thus we have

$$\begin{aligned}
II &= E\left[\int_{T-s}^{T}\alpha\tilde{u}(Y(\tau),\tau)e^{-\gamma(\tau-T+s)}d\tau \mid Y(T-s)=x\right]\\
&\le \alpha\int_{T-s}^{T}\sup_{x\in I}\overline{u}\cdot e^{-(T-\tau)am}e^{-\gamma(\tau-T+s)}d\tau\\
&= \alpha\sup_{x\in I}\overline{u}\cdot e^{-Tam+\gamma T-\gamma s}\int_{T-s}^{T}e^{(am-\gamma)\tau}d\tau\\
&= \alpha\sup_{x\in I}\overline{u}\cdot e^{-Tam+\gamma T-\gamma s}\frac{1}{am-\gamma}\{e^{(am-\gamma)T}-e^{(am-\gamma)(T-s)}\}\\
&= \alpha\sup_{x\in I}\overline{u}\cdot e^{-Tam+\gamma T-\gamma s}\frac{1}{am-\gamma}e^{(am-\gamma)T}(1-e^{(am-\gamma)(-s)})\\
&= \alpha\sup_{x\in I}\overline{u}\cdot e^{-\gamma s}\frac{1}{am-\gamma}(1-e^{(am-\gamma)(-s)})\\
&= \alpha\sup_{x\in I}\overline{u}\cdot\frac{e^{-ams}-e^{-\gamma s}}{\gamma-am}(\gamma-am>0).
\end{aligned}$$

Similarly for the estimate of lower bound, we have

$$II \ge \begin{cases}\alpha\inf_{x\in I}\overline{u}\cdot\dfrac{e^{-aMs}-e^{-\gamma s}}{\gamma-aM} & (\gamma-aM\neq 0),\\ \alpha\inf_{x\in I}\overline{u}\cdot se^{\gamma s} & (\gamma-aM=0).\end{cases} \tag{4.12}$$

Refer to Appendix 5.3 for the proof.

As a consequence, since $u(x,t)=\tilde{u}(x,T-t)$, $v(x,t)=\tilde{v}(x,T-t)$, we obtain the main theorem which gives bounds for the solution of (KS) as follows:

Theorem 4.2. *Suppose that the conditions given in Proposition* 4.1 *are satisfied. Let* (u,v) *be the classical solution of* (KS) *defined through the solution* $(\tilde{u},\tilde{v})$ *of* $(\mathrm{KS})^*$ *by the change of variable* t *by* $T-t$. *Then, for any* s $(0<s<T)$, *the following inequalities hold:*

$$\inf_{x\in I}\overline{u}(x)\cdot e^{-saM} \le u(x,s) \le \sup_{x\in I}\overline{u}(x)\cdot e^{-sam}, \tag{4.13}$$

$$K_M(s) \le v(x,s)-\Phi(\overline{v})(x,s) \le K_m(s), \tag{4.14}$$

where $\Phi(\overline{v})(x,s) := e^{-\gamma s}\left(\frac{A_0}{2}+\sum_{n=1}^{\infty}A_n e^{-(\frac{n\pi}{L_2-L_1})^2 s}\cos\frac{n\pi}{L_2-L_1}(x-L_1)\right)$,

$$A_n := \frac{2}{L_2-L_1}\int_{L_1}^{L_2}\overline{v}(x)\cos\frac{n\pi}{L_2-L_1}(x-L_1)dx \quad (n=0,1,2,\dots),$$

$$K_m(s) := \alpha \sup_{x\in I} \overline{u} \cdot \frac{e^{-ams} - e^{-\gamma s}}{\gamma - am},$$

$$K_M(s) := \begin{cases} \alpha \inf_{x\in I} \overline{u} \cdot \dfrac{e^{-aMs} - e^{-\gamma s}}{\gamma - aM} & (\gamma - aM \neq 0), \\ \alpha \inf_{x\in I} \overline{u} \cdot s e^{\gamma s} & (\gamma - aM = 0) \end{cases}.$$

Remark 4.3. Theorem 4.2 would be an application or modification of maximal principle in usual analysis, can be proved easily through the stochastic analytic methods, as we have seen above.

Remark 4.4. The above inequalities (4.13) and (4.14) give the good estimates of $u(x,t)$ and $v(x,t)$ for small $t > 0$, respectively.

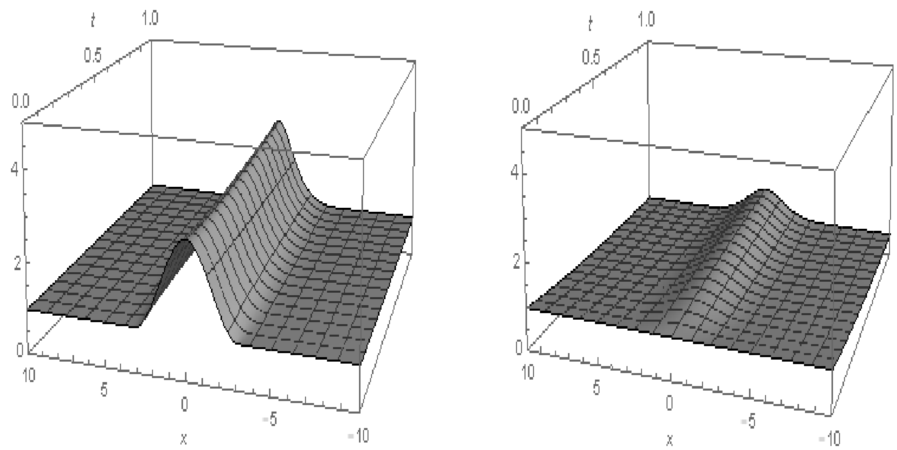

Figure 4.5. Result of the nummerical computation ($T = 1$)

We confirm the result of **Theorem 4.2** by the following example. Let parameters $a, \alpha, \gamma, L_1, L_2$ and initial functions $\overline{u}, \overline{v}$ in (KS) and the time T in (KS)* be as follows: $a = 2,\ \alpha = 1,\ \gamma = 2,\ L_1 = -10,\ L_2 = 10,\ T = 1,$

$$\overline{u}(x) = \begin{cases} 2 + \cos x(-\pi \leq x \leq \pi), \\ 1(\ -10 < x < -\pi,\ \pi < x < 10), \end{cases} \qquad \overline{v}(x) \equiv 1.$$

Then we have the above graphs of $u(x,t)$ and $v(x,t)$ by a direct numerical computation. Left figure is the graph of $u = u(x,t)$ and right one is the graph of $v = v(x,t)$. In case where T is not so large, we see that for $0 \leq t \leq T$ the solution (u, v) is so similar to $(\overline{u}, \overline{v})$.

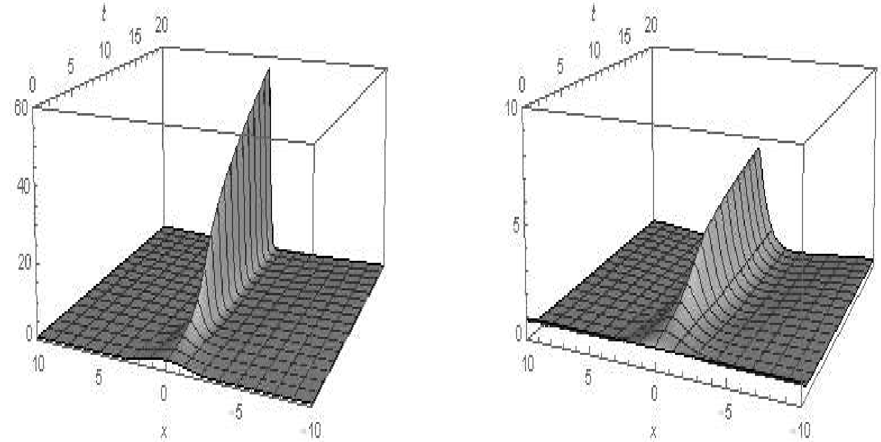

Figure 4.6. Result of the nummerical computation $(T = 20)$

Let parameters $a, \alpha, \gamma, L_1, L_2$ and initial functions $\overline{u}, \overline{v}$ in (KS) be the same as the previous ones, but in the present case take $T = 20$. We have the above graphs of $u(x,t)$ and $v(x,t)$ by a direct numerical computation. Left figure is the graph of $u = u(x,t)$ and right one is the graph of $v = v(x,t)$. In case where T is large, we see that for $0 \le t \le T$ the solution (u, v) becomes not similar to $(\overline{u}, \overline{v})$.

5. **Appendix**

5.1. **Proof of** (4.8)**.** In this subsection, we show the detailed proof of (4.8).

$$
\begin{aligned}
& d\left(\tilde{u}(X(s),s)e^{-\int_t^s a\tilde{v}_{xx}(X(\tau),\tau)d\tau}\right) \\
&= d\left(\tilde{u}(X(s),s)\right)\cdot e^{-\int_t^s a\tilde{v}_{xx}(X(\tau),\tau)d\tau} \\
&\quad + \tilde{u}(X(s),s)\cdot d\left(e^{-\int_t^s a\tilde{v}_{xx}(X(\tau),\tau)\,d\tau}\right) \\
&= \Big[(\tilde{u}_x(X(s),s)\cdot(-a\tilde{v}_x(X(s),s))\,ds + \sqrt{2}\,\tilde{u}_x(X(s),s)dB_s \\
&\quad + \tilde{u}_{xx}(X(s),s)ds + \tilde{u}_s(X(s),s)\,ds\Big]\cdot e^{-\int_t^s a\tilde{v}_{xx}(X(\tau),\tau)d\tau} \\
&\quad + \tilde{u}(X(s),s)\cdot(-a\tilde{v}_{xx}(X(s),s))\cdot e^{-\int_t^s a\tilde{v}_{xx}(X(\tau),\tau)d\tau}ds \\
&= \Big[(\tilde{u}_x(X(s),s)\cdot(-a\tilde{v}_x(X(s),s))\,ds + \sqrt{2}\tilde{u}_x(X(s),s)dB_s \\
&\quad + \tilde{u}_{xx}(X(s),s)ds - \tilde{u}_{xx}(X(s),s)ds + a\tilde{u}_x(X(s),s)\cdot\tilde{v}_x(X(s),s)ds \\
&\quad + a\tilde{u}(X(s),s)\tilde{v}_{xx}(X_t(s),s)ds\Big]\cdot e^{-\int_t^s a\tilde{v}_{xx}(X(\tau),\tau)d\tau} \\
&\quad + \tilde{u}(X(s),s)\cdot(-a\tilde{v}_{xx}(X(s),s))\cdot e^{-\int_t^s a\tilde{v}_{xx}(X(\tau),\tau)\,d\tau}ds \\
&= \sqrt{2}\tilde{u}_x(X(s),s)dB_s\cdot e^{-\int_t^s a\tilde{v}_{xx}(X(\tau),\tau)d\tau},
\end{aligned}
$$

where we have used the differential calculus of a composite function, (4.4) and the generalized Itô's formula (cf. Prop 3.2) for the second equation, and (4.1) for the third equation. Also we have used the fact that the term of local time ϕ_1 in (4.4) vanishes for the functional $\tilde{u}$ satisfying $\tilde{u}_x(L_1,t) = \tilde{u}_x(L_2,t) = 0$ (cf. section IV-7 of [2], and [7] and references therein). □

5.2. **Proof of (4.7).** As was done for the proof of (4.8), we can prove (4.7). Let's consider a functional of $\tilde{v}(Y(s),s)e^{-\gamma\int_t^s d\tau}$. By using (4.2) and (4.5) instead of (4.1) and (4.4) in 5.1, we have the following relation:

$$\begin{aligned}
&d\left(\tilde{v}(Y(s),s)e^{-\gamma\int_t^s d\tau}\right)\\
&\quad= d\left(\tilde{v}(Y(s),s)\right)\cdot e^{-\gamma\int_t^s d\tau} + \tilde{v}(Y(s),s)\cdot d\left(e^{-\gamma\int_t^s\ d\tau}\right)\\
&\quad= \Bigg[\tilde{v}_x(Y(s),s)\cdot 0ds + \sqrt{2}\tilde{v}_x(Y(s),s)dB_s + \tilde{v}_{xx}(Y(s),s)ds\\
&\qquad + \tilde{v}_s(Y(s),s)ds\Bigg]\cdot e^{-\gamma\int_t^s\ d\tau} + \tilde{v}(Y(s),s)\cdot(-\gamma)\cdot e^{-\gamma\int_t^s d\tau}ds\\
&\quad= \Bigg[\sqrt{2}\tilde{v}_x(Y(s),s)dB_s + \tilde{v}_{xx}(Y(s),s)ds - \tilde{v}_{xx}(Y(s),s)ds + \gamma\tilde{v}(Y(s),s)ds\\
&\qquad - \alpha\tilde{u}(Y(s),s)ds\Bigg]\cdot e^{-\gamma\int_t^s d\tau} + \tilde{v}(Y(s),s)\cdot(-\gamma)\cdot e^{-\gamma\int_t^s\ d\tau}ds\\
&\quad= \left(\sqrt{2}\tilde{v}_x(Y(s),s)dB_s - \alpha\tilde{u}(Y(s),s)ds\right)e^{-\gamma\int_t^s d\tau}.
\end{aligned}$$

By integrating both sides from t to T, we have

$$\begin{aligned}
&\int_t^T d\left(\tilde{v}(Y(s),s)\right)e^{-\gamma\int_t^s d\tau}ds\\
&\qquad\qquad= \int_t^T \left(\sqrt{2}\tilde{v}_x(Y(s),s)dB_s - \alpha\tilde{u}(Y(s),s)ds\right)e^{-\gamma\int_t^s d\tau}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\tilde{v}(Y(T),T)e^{-\gamma\int_t^T d\tau} - \tilde{v}(Y(t),t)e^{-\gamma\displaystyle\int_t^t d\tau}\\
&\qquad= \sqrt{2}\int_t^T e^{-\gamma\int_t^s d\tau}\tilde{v}_x(Y(s),s)dB_s - \int_t^T \alpha\tilde{u}(Y(s),s)e^{-\gamma\int_t^s d\tau}ds.
\end{aligned}$$

After putting in order, we take the expectation E in probability measure P, then we have

$$\begin{aligned}
&E[\tilde{v}(Y(T),T)e^{-\gamma(T-t)} - \tilde{v}(Y(t),t) \mid Y(t) = x]\\
&\qquad= E\left[\sqrt{2}\int_t^T e^{-\gamma(s-t)}\tilde{v}_x(Y(s),s)dB_s \mid Y(t) = x\right]\\
&\qquad\quad - E\left[\int_t^T \alpha\tilde{u}(Y(s),s)e^{-\gamma(s-t)}ds \mid Y(t) = x\right].
\end{aligned}$$

Since the first term of the right side equals to zero, and $v(x,T)$ equals to $\overline{v}(x)$, we have (4.7). □

5.3. **Proof of (4.12).** Finally we shall prove the inequality (4.12). By (4.11),

$$\tilde{u}(Y(\tau),\tau) = \tilde{u}(Y(\tau), T-(T-\tau)) \geq \inf_{x\in I} \overline{u} \cdot e^{-(T-\tau)aM}$$

hold. Thus for $\gamma \neq aM$, we have

$$\begin{aligned}
II &= E\left[\int_{T-s}^{T} \alpha\tilde{u}(Y(\tau),\tau)e^{-\gamma(\tau-T+s)}d\tau \mid Y(T-s)=x\right] \\
&\geq \alpha \int_{T-s}^{T} \inf_{x\in I} \overline{u} \cdot e^{-(T-\tau)aM} e^{-\gamma(\tau-T+s)} d\tau \\
&= \alpha \inf_{x\in I} \overline{u} \cdot e^{-TaM+\gamma T-\gamma s} \int_{T-s}^{T} e^{(aM-\gamma)\tau} d\tau \\
&= \alpha \inf_{x\in I} \overline{u} \cdot e^{-TaM+\gamma T-\gamma s} \frac{1}{aM-\gamma}\{e^{(aM-\gamma)T} - e^{(aM-\gamma)(T-s)}\} \\
&= \alpha \inf_{x\in I} \overline{u} \cdot e^{-TaM+\gamma T-\gamma s} \frac{1}{aM-\gamma} e^{(aM-\gamma)T}(1-e^{(aM-\gamma)(-s)}) \\
&= \alpha \inf_{x\in I} \overline{u} \cdot e^{-\gamma s} \frac{1}{aM-\gamma}((1-e^{(aM-\gamma)(-s)}) \\
&= \alpha \inf_{x\in I} \overline{u} \cdot \frac{e^{-aMs}-e^{-\gamma s}}{\gamma - aM}.
\end{aligned}$$

Also for $\gamma = aM$, we have

$$\begin{aligned}
II &= E\left[\int_{T-s}^{T} \alpha\tilde{u}(Y(\tau),\tau)e^{-\gamma(\tau-T+s)}d\tau \mid Y(T-s)=x\right] \\
&\geq \alpha \int_{T-s}^{T} \inf_{x\in I} \overline{u} \cdot e^{-(T-\tau)aM} e^{-\gamma(\tau-T+s)} d\tau \\
&= \alpha \inf_{x\in I} \overline{u} \cdot e^{-\gamma s} \int_{T-s}^{T} d\tau \\
&= \alpha \inf_{x\in I} \overline{u} \cdot s e^{-\gamma s}.
\end{aligned}$$

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