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NUMERICAL ESTIMATION OF UNCERTAINTY PRINCIPLE FOR LÉVY DRIVEN ORNSTEIN UHLENBECK PROCESSES IN FINANCE

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ABSTRACT. According to uncertainty principle, the more precisely the position of some particle is determined, the less precisely its momentum can be known and vice versa. This study applies the uncertainty principle to geometric Brownian motion, mean reverting Ornstein Uhlenbeck (OU) and more general Lévy driven OU processes which are widely used in finance. It is found out that the variance of the process itself which can be regarded as the momentum and the variance of its Fourier transform, which corresponds to the position are inversely related. Various approximation and numerical techniques are applied to Lévy driven OU processes since their Fourier transforms involve non elementary integrals.

Key Words Uncertainty Principle, Mean Reverting Stochastic Processes, Fourier Transform.

1. INTRODUCTION

A moving particle in quantum physics and a financial asset typically have the same properties. Since any financial asset moves along in time, the dispersion of it can be considered as the dispersion in the momentum. Whenever the dispersion of gets closer to zero, that is, it becomes more and more stable, after a threshold the dispersion of its position immediately tends to infinity due the uncertainty principle. The uncertainty principle merely says that we cannot be perfectly clear about where something is and where it is going. Although this principle always holds it is hard to be observed in daily life. However, when particles are of atomic size then the effect of this principle is clearly visible. As a consequence of this principle there is a relationship between a randomly distributed variable and its Fourier transform which will be our main yardstick for this study. First the uncertainty principle is introduced and its linkage with Fourier transform is explained. After stating the uncertainty principle for the easiest case namely the geometric Brownian motion, we then consider the mean reverting OU processes which are widely used for modeling financial securities. The last and the main concern of the article is devoted to Lévy driven OU processes. We first start with basic definitions and important formulas. We then find the Fourier transform of a Lévy driven OU process where some major theorems are stated and

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used. Here due to the complexities only positive exponentially distributed jumps are taken into consideration. The non-elementary integrals lead us to approximate them with Taylor series conjoined with numerical integration and a root finding algorithm. We state a new approximation technique for finding the roots of the Fourier transform of the Lévy driven OU process which is a mixture of Taylor series expansion and nonlinear least squares estimation. It is found out that after a certain point the variance of the Fourier transform interestingly diverges. The paper concludes with possible related future research topics.

2. UNCERTAINTY PRINCIPLE

The uncertainty principle [7] states that the standard deviation of the position of a moving particle Δ_x and the standard deviation of its momentum Δ_p should respect the following inequality:

$$\Delta_x \cdot \Delta_p \geq \frac{h}{4\pi} \tag{2.1}$$

where h is the Planck constant (approximately $6,6 \cdot 10^{-34}$). In order to demonstrate the equivalence of uncertainty principle for random variables the following definitions and theorem are given:

Definition 2.1. Let $f(x)$ be an arbitrary probability density function. $M(t)$ is called the moment generating function of $f(x)$ defined by $M(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$.

A key problem with moment generating functions is that moments and the moment generating function may not exist, as the integrals need not converge absolutely.

Definition 2.2. Let $f(t)$ be an arbitrary function. A function $F(\omega)$ is called the Fourier transform of $f(t)$ defined by $F(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)dt$.

Contrary to the moment generating function, Fourier transform always exists. Intuitively, the Fourier transform carries a function from time domain to frequency domain.

Theorem 2.3 (Plancherel [11])**.** *Let $f(t)$ has the Fourier transform $F(\omega)$ then*

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d(\omega)$$

Theorem 2.4 (Uncertainty Principle)**.** *Suppose $f(t)$ has the Fourier transform $F(\omega)$. Let*

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d(\omega)$$

$$d^2 = \frac{1}{E} \int_{-\infty}^{\infty} t^2 |f(t)|^2 dt \quad ; \quad D^2 = \frac{1}{2\pi E} \int_{-\infty}^{\infty} \omega^2 |F(\omega)|^2 d(\omega)$$

If $\sqrt{t} f(t) \to 0$ as $|t| \to \infty$ then $D \cdot d \geq \frac{1}{2}$.

Fourier transform may not always be a probability density function at all and its variance may not always be defined. However, $D \cdot d = 1$ when we have Gaussian distribution which will be proved in the next section.

3. GEOMETRIC BROWNIAN MOTION

The driving process of a stock is usually assumed to follow a geometric Brownian motion as stated by [1]

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t) \tag{3.1}$$

Where μ is the drift, σ is the volatility parameter and $B(t)$ is the standard Brownian motion. The solution of (3.1) is quite straightforward by Ito formula [8]

$$S(t) = S(0) \cdot e^{\left(\mu - \frac{\sigma^2}{2}\right) + \sigma \dot{B}(t)} \tag{3.2}$$

and since Brownian motion is normally distributed we have

$$s(t) = ln(S(t)) \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right) \tag{3.3}$$

Let $f(x) \sim N(0,1)$ be standard normal distributed. Now consider the moment generating function of $f(x)$ which is

$$M(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{\frac{x^2}{2}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\left(\frac{-x^2}{2} + tx\right)} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} e^{-\frac{1}{2}t^2} dx = e^{\frac{1}{2}t^2}$$

since $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} dx = 1$.

Now for an arbitrary mean and variance, the moment generating function of $f(x) \sim N(\mu, \sigma^2)$ becomes $M(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$. Defining $z = \frac{x-\mu}{\sigma}$ implies $x = z\sigma + \mu$ and by change of variable technique the result is $M(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$. Since Fourier transform (characteristic function) is defined as $\Phi(t) = M(it)$ we have $\Phi(t) = e^{i\mu t - ((\sigma^2 t^2)/2)}$. Now, the Fourier transform of a normal distribution with mean zero and variance σ^2 is another normal distribution with mean zero but a variance of $\frac{1}{\sigma^2}$. Hence, the relation between the variances of a normal probability distribution function with mean zero and its Fourier transform can be expressed in uncertainty principle form as $\sigma^2 \frac{1}{\sigma^2} = 1$. When we set $t = 1$, without loss of generality, the relation for the geometric Brownian motion case easily follows.

4. UNCERTAINTY PRINCIPLE FOR OU PROCESSES

Modeling data in continuous time with uncertainty is a major issue. The corresponding driving process is usually assumed to follow a particular pattern. Mean reverting processes for term structures drew quite a lot of attention in the literature. Benchmark for these is the OU process by [10]:

$$dX(t) = -cX(t)dt + \sigma dB(t), \quad X(0) \in \mathbb{R} \tag{4.1}$$

where $\sigma \in \mathbb{R}^+, c > 0$ and $B(t)$ is the standard Brownian motion. OU process can be solved analytically by integration by parts as

$$Y(t) = X(t)e^{ct} \Rightarrow dY(t) = ce^{ct}X(t)dt + e^{ct}dX(t)$$
$$= ce^{ct}X(t)dt + e^{ct}[-cX(t)dt + \sigma dB(t)] \Rightarrow dY(t) = \sigma e^{ct}dB(t)$$
$$Y(0) = x \Rightarrow Y(t) = x + \sigma \int_0^t e^{cs}dB(s) \Rightarrow X(t)e^{ct} = x + \sigma \int_0^t e^{cs}dB(s)$$
$$\Rightarrow X(t) = xe^{-ct} + \sigma \int_0^t e^{c(s-t)}dB(s)$$

Mean and variance of 4.1 are

$$\mathbb{E}(X(t)) = \mathbb{E}(xe^{-ct}) + \sigma\mathbb{E}\left[\int_0^t e^{c(s-t)}dB(s)\right] = xe^{-ct}. \tag{4.2}$$

$$Var(X(t)) = \mathbb{E}(X(t) - \mathbb{E}(X(t))^2 = \mathbb{E}\left(\sigma e^{-ct}\int_0^t e^{c(s-t)}dB(s)\right)^2$$

(4.3) and from Ito isometry we have

$$\sigma^2 e^{-2ct}\mathbb{E}\left[\int_0^t e^{2cs}ds\right] = \sigma^2 e^{-2ct}\frac{e^{2cs}}{2c}\Big|_0^t = \sigma^2 e^{-2ct}\frac{e^{2ct}-1}{2t} = \frac{\sigma^2}{2c}(1 - e^{-2ct}).$$

If we consider the OU process stated in (4.1) and follow the derivation of the characteristic function of the normal distribution the Fourier transform of (4.1) becomes

$$e^{-itX(t)} = e^{\left(-iuX(0)e^{-2ct} + \frac{u^2}{2}\frac{\sigma^2}{2c}(1-e^{-2ct})\right)} \tag{4.4}$$

Since our main concern is the comparison of variances we can set $X(0) = 0$. Moreover, we can also write (4.4) as

$$e^{-itX(t)} = e^{\left(\frac{u^2}{\frac{1}{\sigma^2}}\frac{1}{2c}(1-e^{-2ct})\right)}$$

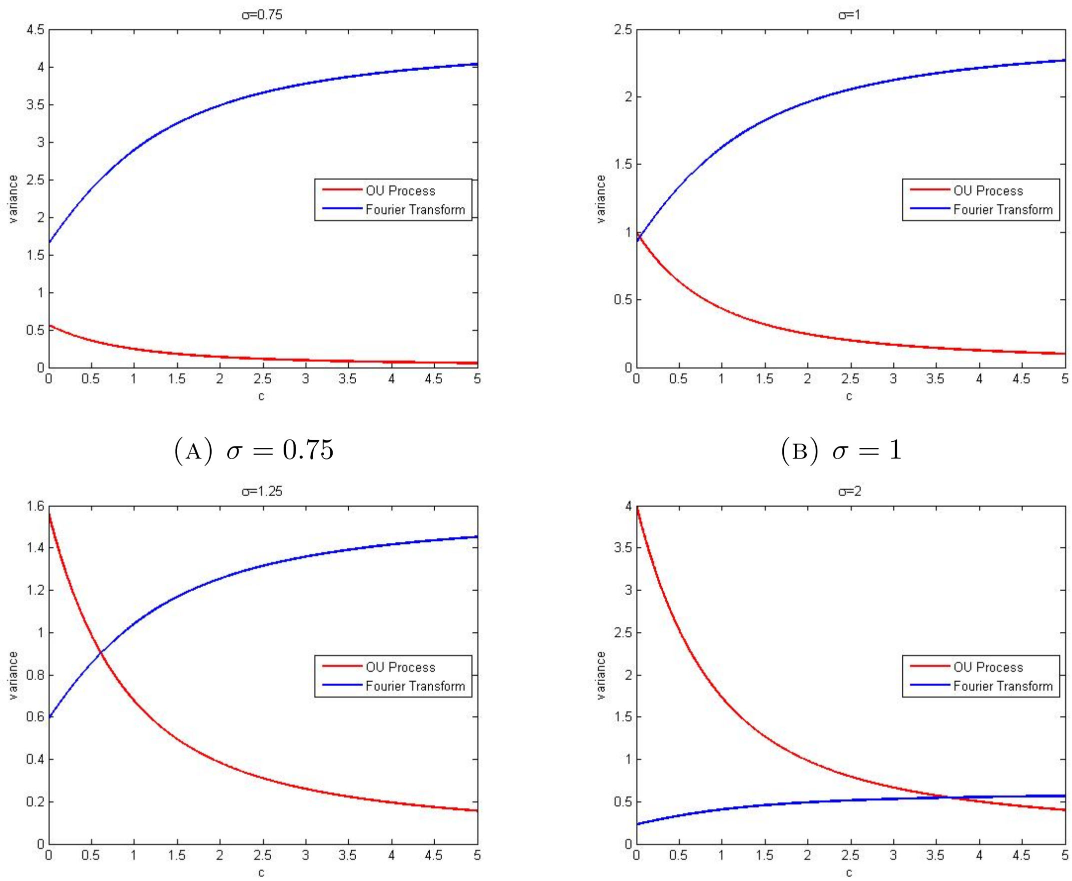
Following the assertion of Plancharel's Theorem we have the following:

$$\frac{\sigma^2}{2c}(1 - e^{-2ct})\frac{1}{2\pi}\frac{1}{\sigma^2}e^{\left(\frac{1}{2c}(1-e^{-2ct})\right)} = 1$$

In order for this equality to hold, when $\sigma \to 0$ then $\frac{1}{\sigma} \to \infty$. Hence it is apparent that a decrease in the variance of the OU Process in question yields bigger uncertainty. The variances of the OU Process and its Fourier transform are compared in Figure 1.

Whenever the variance of the OU Process gets smaller the variance of its Fourier transform widens. The situation is reversed when the variance of the OU process increases. When $\sigma = 1$ the variances are again inversely related according to the mean reversion rate c. Hence if the momentum is certain the position becomes uncertain according to c and vice versa if the uncertainty effect from σ is nil. However, none of the variances immediately diverge. The uncertainty principle holds but is hard to observe when σ is not infinitesimally close to zero.

FIGURE 1. Comparison of the variance functions of OU Process and its Fourier transform for different values of σ when $t = 1$.



(A) $\sigma = 0.75$

(B) $\sigma = 1$

(C) $\sigma = 1.25$

(D) $\sigma = 2$

5. UNCERTAINTY PRINCIPLE FOR LÉVY DRIVEN OU PROCESSES

OU processes are generalized by adding a random jump component. This is usually done with a Poisson process conjoined with a Poisson random measure. Formal definitions are stated below (for detailed explanation see [3]).

Definition 5.1 (Poisson Process)**.** Poisson processes are extensively used for modeling breaks in financial data. The homogeneous Poisson process counts events that occur at a fixed rate λ called the intensity. The process is characterized by

$$\mathbb{P}[N(t+\tau) - N(t) = k] = \frac{e^{-\lambda\tau}(\lambda\tau)^k}{k!}, \quad k = 0, 1, \ldots \tag{5.1}$$

where $N(t+\tau) - N(t) = k$ is the number of events in time interval $(t, t+\tau]$ and λ being the expected number of jumps that occur per unit time. The mean and variance of the Poisson process given in (5.1) is $\mathbb{E}(x) = Var(x) = \lambda t$.

Definition 5.2 (Compound Poisson Process)**.** If jumps are Y_i with arbitrary sizes and $\mathbb{E}(Y_i) = \beta$ then $Q(t) = \sum_{i=1}^{N(t)} Y_i$ is called a compound Poisson process where $\mathbb{E}(x) = Var(x) = \beta\lambda t$.

Definition 5.3 (Compensated Compound Poisson Process)**.** The process $M(t) = Q(t) - \beta\lambda t$ is called the compensated compound Poisson process. Its expectation is 0.

Definition 5.4 (Poisson Random Measure)**.** Let $(\mathbb{S}, S, m)$ be a sigma finite measure space with $m(s) > 0$ and $(\Omega, F, \mathbb{P})$ the underlying probability space. A Poisson random measure π with intensity m is a collection of random variables $\pi(A)_{A\in S}$ with the following properties:

(i) $\pi(A) = \text{Poisson}(m(A)), \quad \forall A \in S$
(ii) If $A_1, A_2, \ldots, A_k \in S$ are disjoint then, $\pi(A_1), \pi(A_2), \ldots, \pi(A_k)$ are independent.

The Poisson random measure counts the expected number of jumps for a given interval.

Definition 5.5 (Lévy Process)**.** A stochastic process is said to be a Lévy process if it has independent and stationary increments and is continuous in probability.

Definition 5.6 (Lévy Ito Decomposition)**.** A Lévy process has three independent components (γ, σ^2, ν) called the Lévy triplet where γ is a linear drift, σ^2 is a diffusion process captured by a Brownian motion and ν is a Poisson random measure for Poisson processes with different jump sizes.

Definition 5.7 (Lévy Khintchine Formula)**.** The distribution of a Lévy process is characterized by

$$\mathbb{E}(e^{iux}) = e^{iu\gamma} e^{-\frac{u^2\sigma^2}{2}} e^{\int_{-\infty}^{\infty}[e^{iux}-1-iuxI_{|x|<1}]\nu(dx)}$$

where I is the indicator function and ν is a finite measure satisfying $\int_{-\infty}^{\infty}(1\Lambda x^2)\nu(dx) < \infty$.

Consider the following OU Process

$$dX(t) = -cX(t)dt + dL(t) \tag{5.2}$$

where $L(t)$ is a Lévy process with the triplet (γ, σ^2, ν) and ν is a Poisson random measure defined by $\nu(dx) = \lambda F(dx) = \lambda\alpha e^{-\alpha x}dx, \quad x > 0$. Although could be generalized, for computational and accuracy purposes (which will be clearer) only exponentially distributed positive jumps and a maturity of unity is considered. The solution of (5.2) straightforward as:

$$X(t) = X(0)e^{-ct} + \int_0^t e^{-c(s-t)}dL(s)$$

which can be partitioned by Lévy Ito decomposition as

$$X(t) = X(0)e^{-ct} + \gamma \int_0^t e^{-c(s-t)} ds + \sigma \int_0^t e^{-c(s-t)} dB(s)$$
$$+ \int_0^t \int_{|x|=1}^{\infty} e^{-c(s-t)} xM(dsdx) + \int_0^t \int_{|x|=\epsilon}^{1} e^{-c(s-t)} x\widetilde{M}(dsdx)$$

where M denotes the compound Poisson and $\widetilde{M}$ denotes the compensated compound Poisson process. The expectation of $X(t)$ is

$$\begin{aligned}
\mathbb{E}(X(t)) &= X(0)e^{-ct} + \gamma \int_0^t e^{-c(s-t)} ds + \int_0^t \int_1^{\infty} e^{-c(s-t)} x\nu(dx)ds \\
&= X(0)e^{-ct} + \gamma \frac{1-e^{-ct}}{c} + \frac{1-e^{-ct}}{c} \int_1^{\infty} \lambda\alpha e^{-\alpha x} dx \qquad (5.3) \\
&= X(0)e^{-ct} + \gamma \frac{1-e^{-ct}}{c} + \frac{1-e^{-ct}}{c} \frac{\lambda}{\alpha} e^{-\alpha}(1+\alpha) \\
&= X(0)e^{-ct} + \frac{\gamma}{c}(1-e^{-ct}) + \frac{\lambda(1+\alpha)e^{-\alpha}}{\alpha c}(1-e^{-ct})
\end{aligned}$$

and the variance can be computed as

$$\begin{aligned}
Var(X(t)) &= \sigma^2 \int_0^t e^{2c(s-t)} ds + \int_0^t \int_{-\infty}^{\infty} e^{2c(s-t)} x^2 \nu(dx)ds \\
&= \frac{\sigma^2}{2c}(1-e^{-2ct}) + \frac{\lambda}{\sigma^2 c}(1-e^{-2ct}) \qquad (5.4) \\
&= \frac{(1-e^{-2ct})}{c}\left(\frac{\sigma^2}{2} + \frac{\lambda}{\alpha^2}\right)
\end{aligned}$$

In order to compute the characteristic function of $X(t)$ we need the following lemma:

Lemma 5.8. *Let $f : [0,T] \to \mathbb{R}$ be left continuous and $Z(t)$ be a Lévy process. Then*

$$\mathbb{E}\left(e^{-i\int_0^t f(t)dZ(t)}\right) = e^{\left(\int_0^T \Psi(f(t))dt\right)}$$

where $\Psi(u)$ is the characteristic exponent of Z.

Now let us compute the characteristic function of $X(t)$

$$\mathbb{E}\left(e^{iuX(t)}\right) = e^{iuX(0)e^{-ct}} \mathbb{E}\left(e^{iu\int_0^t e^{-c(s-t)} dL(s)}\right) = e^{iuX(0)e^{-ct}} e^{\int_0^T \Psi(ue^{c(s-t)})}$$

Now consider $L(t)$ defined by

$$L(t) = \gamma t \sigma B(t) + \int_0^t \int_{|x|=1}^{\infty} xM(dsdx) + \int_0^t \int_{|x|=\epsilon}^{1} x\widetilde{M}(dsdx)$$

From Lévy Khintchine Formula we have

$$\mathbb{E}(e^{iuL_1}) = e^{iu\gamma} e^{-\frac{u^2\sigma^2}{2}} e^{\int_{-\infty}^{\infty}[e^{iux}-1-iuxI_{|x|<1}]\nu(dx)}$$

where I is the indicator function. Then the characteristic component of L is

$$\Phi(u) = (iu\gamma)\left(-\frac{u^2\sigma^2}{2}\right)\left(\int_{-\infty}^{\infty}[e^{iux} - 1 - iuxI_{|x|<1}]\nu(dx)\right)$$

and

$$\Phi(ue^{c(s-t)}) =$$
$$\left((iu\gamma)e^{-c(s-t)}\right)\left(-\frac{u^2\sigma^2}{2}e^{2c(s-t)}\right)\left(\int_{-\infty}^{\infty}[e^{iue^{c(s-t)}} - 1 - iue^{-c(s-t)}I_{|x|<1}]\nu(dx)\right)$$

then

$$\int_0^t \Phi(ue^{c(s-t)})ds = \left(iu\gamma\int_0^t e^{-c(s-t)}ds\right)\left(-\frac{u^2\sigma^2}{2}\int_0^t e^{2c(s-t)}ds\right)$$
$$+\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[e^{iue^{c(s-t)x}} - 1 - iue^{c(s-t)}xI_{|x|<1}]ds\nu(dx)$$

We finally have

$$\mathbb{E}\left(e^{iuX(t)}\right) = e^{\left(iuX(0)e^{-ct}\right)}e^{\left(iu\gamma\left(\frac{1-e^{-ct}}{c}\right)-\frac{\sigma^2u^2}{2}\left(\frac{1-e^{-2ct}}{2c}\right)\right)}$$
$$e^{\lambda t}e^{\left(iu\frac{\lambda}{c}\left(\frac{1-e^{-\alpha}(1+\alpha)}{\alpha}\right)\right)}e^{\left(\int_0^t\int_{-\infty}^{\infty} e^{iuxe^{c(s-t)}}ds\lambda x\alpha e^{-\alpha x}dx\right)} \tag{5.5}$$

Since our main concern is the comparison of variances we can set $X(0) = 0$ and $\gamma = 0$. However, we cannot say that the variance is $\frac{1}{\sigma^2}$ multiplied by a constant as in the OU process since u's also appear elsewhere Thus we have

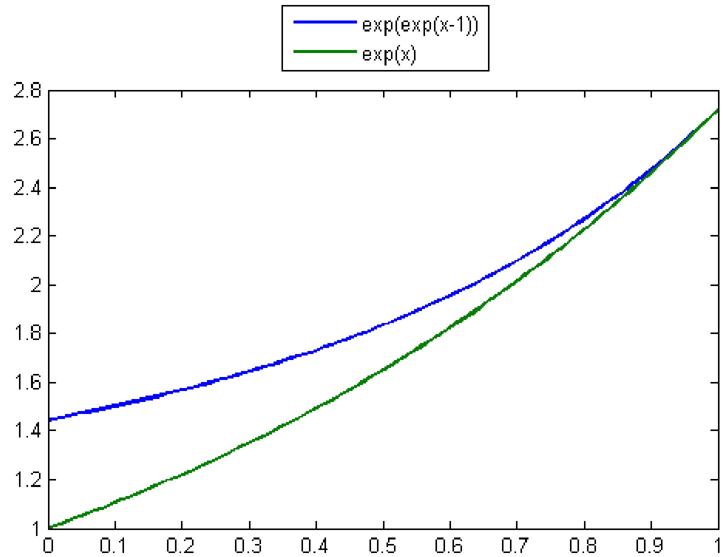
$$\mathbb{E}\left(e^{iuX(t)}\right) = e^{-\frac{\sigma^2u^2}{2}\left(\frac{1-e^{-2ct}}{2c}\right)}e^{\lambda t}e^{\left(iu\frac{\lambda}{c}\left(\frac{1-e^{-\alpha}(1+\alpha)}{\alpha}\right)\right)}e^{\left(\int_0^t\int_{-\infty}^{\infty} e^{iuxe^{c(s-t)}}ds\lambda x\alpha e^{-\alpha x}dx\right)}$$

Now the mean of $\mathbb{E}e^{iuX(t)}$ is

$$\mathbb{E}\left(\mathbb{E}\left(e^{iuX(t)}\right)\right) =$$
$$\int_{-\infty}^{\infty} u\left[e^{-\frac{\sigma^2u^2}{2}\left(\frac{1-e^{-2ct}}{2c}\right)}e^{\lambda t}e^{\left(iu\frac{\lambda}{c}\left(\frac{1-e^{-\alpha}(1+\alpha)}{\alpha}\right)\right)}e^{\left(\int_0^t\int_0^{\infty} e^{iuxe^{c(s-t)}}ds\lambda x\alpha e^{-\alpha x}dx\right)}\right]du$$

Let us consider the integrand with respect to s and x in the exponent of last term which is $\lambda\int_0^t\int_0^{\infty} e^{iuxe^{c(s-t)}}dsx\alpha e^{-\alpha x}dx$. Notice that lower limit of the final integral is changed from $-\infty$ to 0 since we only have positive exponentially distributed jumps. This double integral has no analytic solution therefore we approximate the term $e^{c(s-t)}$ with the first two Taylor series term at the expansion point $t = 1$ as $1+c(s-t)$. Here adding more terms again gives non elementary results therefore only the first two terms are taken. Although it seems a naive approximation, our choice of $t = 1$ and positive exponentially distributed jumps makes it quite accurate as seen in Figure 2.

FIGURE 1. Approximation of $e^{e^{c(s-t)}}$ when $c = t = 1$.



The reason of our particular choices are now clearer. Now we have

$$\lambda\int_0^t\int_0^\infty e^{iux(1+c(s-t))}dsx\alpha e^{-\alpha x}dx = \int_0^\infty \lambda\frac{e^{iux(1+c(s-t))}}{iuxc}\Big|_0^t x\alpha e^{-\alpha x}dx$$
$$= \int_0^\infty \lambda\frac{e^{iux}e^{iuxc}}{iuxc}\Big|_0^t x\alpha e^{-\alpha x}dx$$

when we set $c = \alpha = 1$ the integral becomes

$$\int_0^\infty \lambda\frac{2e^{iux}}{iux}x\alpha e^{-x}dx = \int_0^\infty -\lambda\frac{2e^{iux-x}}{iu}dx = \lambda\frac{2ie^{iux-1}x}{u(iu-1)}\Big|_0^\infty = \frac{2i\lambda}{u(iu-1)}.$$

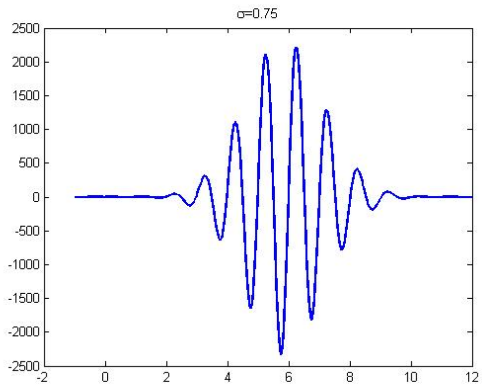
For other values of c solution of the integral is harder and results are too long. Our aim here is to see the impact of variance and jump components and since the effect of mean reversion rate is analyzed in detail for sole OU processes we will not take into account mean reversion rates other than unity. We also set $\alpha = 1$ but for other values of α the integral can easily be computed. Now the mean becomes

$$\mathbb{E}\left(\mathbb{E}\left(e^{iuX(t)}\right)\right) = e^{\frac{1-e^{-2}}{2}+2\lambda(2-e^{-1})}\int_{-\infty}^{\infty} ue^{-\frac{u^2\sigma^2}{2}+iu+e^{\frac{2i}{u(iu-1)}}} \tag{5.6}$$

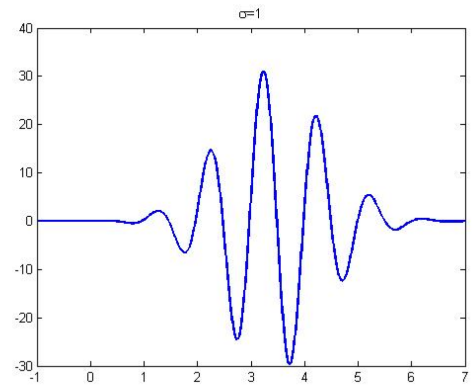
The last term inside the integral makes it non elementary therefore we again approximate it with the first two Taylor series terms around 1, we then have

$$\begin{aligned}&\mathbb{E}\left(\mathbb{E}\left(e^{iuX(t)}\right)\right)\\ &= e^{\frac{1-e^{-2}}{2}+2\lambda(2-e^{-1})}\int_{-\infty}^{\infty} ue^{-\frac{u^2\sigma^2}{2}+iu+e^{(1-i)}+(-1+2i)e^{(1-i)}(u-1)}du\end{aligned} \tag{5.7}$$

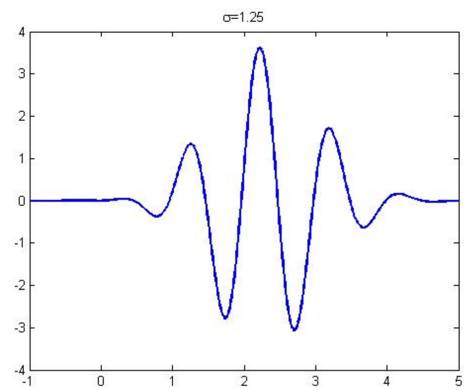
FIGURE 2. Graphs of the mean estimating function for the Fourier transform of Lévy driven OU process for different values of σ when $t = 1$.



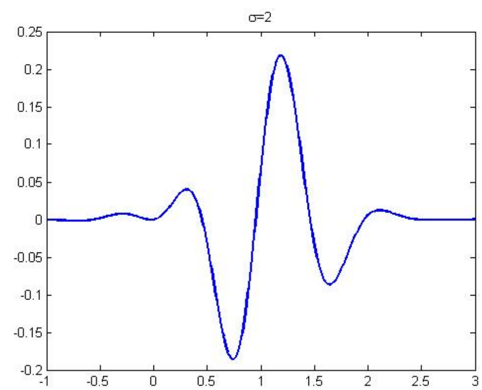
(A) $\sigma = 0.75$



(B) $\sigma = 1$



(C) $\sigma = 1.25$



(D) $\sigma = 2$

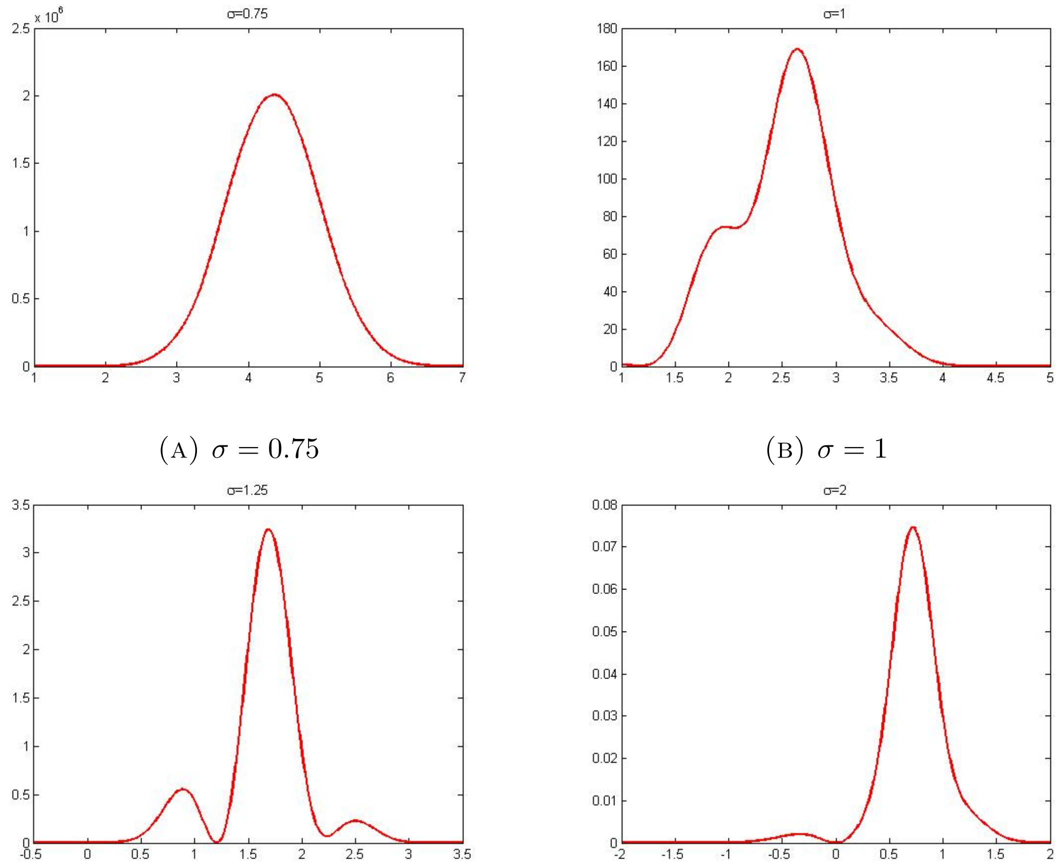
The oscillations and their wave lengths increase when the σ decreases. Now let us denote the mean by μ. Then the variance becomes

$$\begin{aligned}&\mathbb{E}(u-\mu)^2 e^{iuX(t)} = \\ &e^{\frac{1-e^{-2}}{2}+2\lambda(2-e^{-1})}\int_{-\infty}^{\infty}(u-\mu)^2 e^{-\frac{u^2\sigma^2}{2}+iu+e^{(1-i)}+(-1+2i)e^{(1-i)}(u-1)}du\end{aligned} \tag{5.8}$$

Since $e^{\frac{1-e^{-2}}{2}+2\lambda(2-e^{-1})}$ are just constants we try to understand the shape of the variance estimating function. Notice that it involves complex values therefore we only sketch the graph of its real part since it corresponds to the usual variance. Moreover, the variance function had some negativities, hence their absolute value is taken into consideration.

Notice how the variance estimator function shoots up when σ decreases from 1.25 to 1. It almost tends to infinity when σ is lowered to 0.75.

FIGURE 3. Graphs of the variance estimating function for the Fourier transform of Lévy driven OU process for different values of σ when $t = 1$.



(A) $\sigma = 0.75$ (B) $\sigma = 1$

(C) $\sigma = 1.25$ (D) $\sigma = 2$

6. SERIES EXPANSION AND ROOT FINDING ALGORITHM FOR LÉVY DRIVEN OU PROCESSES

In order to determine the position of roots for estimating the variance with Simpson's rule the roots of the integral stated in equation (5.8), we first compute its Taylor Series Expansion which is

$$\sum_{k=0}^{\infty} \frac{u^2 \left(-\frac{u^2\sigma^2}{2} + iu + e^{(1-i)} + (-1+2i)e^{(1-i)}(u-1)\right)^k}{k!}$$

Therefore, we follow polynomial root finding algorithm due to Cayley Hamilton Theorem.

Theorem 6.1 (Cayley Hamilton)**.** *For a given n by n square matrix the characteristic polynomial of A defined by $p(\lambda) = det(\lambda I_n - A)$ where I_n is the identity matrix and det stands for the determinant. Then we have $p(A) = 0$.*

Definition 6.2 (Companion Matrix)**.** The companion matrix of the polynomial $p(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1} + c_n t^n$ is defined by

$$C(p) = \begin{bmatrix} 0 & 0 & 0 & \dots & -c_0 \\ 1 & 0 & 0 & \dots & -c_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -c_{n-1} \end{bmatrix}$$

(Detailed proofs are available in [2, 5, 6].)

Following Cayley Hamilton Theorem, the eigenvalues of the companion matrix which in fact is the roots of the polynomial in question. Here we compute eigenvalues via QR algorithm [4, 9] for a desired level of Taylor Expansion order. The real parts of the roots are ordered and their minimum and maximum values are picked. Having seen the tendency of real roots next step is to calculate the corresponding variance value for the selected σ's. In order to compute them we consider the integral on the neighborhood of real roots and apply Simpson's rule. Our algorithm stops when 0,01 error is reached. It is also worthwhile mentioning that the proposed root finding method with Taylor expansion and Cayley Hamilton Theorem becomes significantly slower when the Taylor series order is more than 15. The reason for this is the due to the saddle shape of the Fourier transform. Our proposed algorithm is too fast for smoother distributions like the OU Process without jumps. We therefore apply a new innovative technique for estimation of the roots by using logarithmic least squares estimation after certain number of roots are computed. The findings are listed in Tables 1–4:

Table 1. Estimation of the variance of Fourier transform of the Lévy driven OU Process for $\sigma = 2$

	$\sigma = 2$	
Order	Minimum	Maximum
1	1,0537	1,0537
2	0.9157	1,2100
3	0,8390	1,1929
4	0,7920	1,1759
5	0,7616	1,2350
6	0,7413	1,2594
7	0,7275	1,2545
8	0,7176	1,2622
9	0,7101	1,2852
10	0,7040	1,2990
11	0,6900	1,3066
12	0,6768	1,3125
13	0,6655	1,3321
14	0,6557	1,3474
Trend	y=-0,14ln(x)+1,0132	y=-0,0943ln(x)+1,0821
Number Of Iterations	760	487
Value	0,0845	1,6657
Variance	0,0622	

TABLE 2. Estimation of the variance of Fourier transform of the Lévy driven OU Process for $\sigma = 1,25$

	$\sigma = 1,25$	
Order	Minimum	Maximum
1	1,4684	1,4684
2	1,4204	1,5330
3	1,3946	1,5935
4	1,3760	1,6431
5	1,3600	1,6832
6	1,3447	1,7160
7	1,3295	1,7436
8	1,3141	1,7476
9	1,2988	1,7890
10	1,2839	1,8085
11	1,2689	1,8266
12	1,2505	1,8436
13	1,2315	1,8598
14	1,2125	1,8752
Trend	y=-0,093ln(x)+1,4926	y=0,1612ln(x)+1,4335
Number Of Iterations	481	894
Value	0,9182	2,5290
Variance	2,1919	

TABLE 3. Estimation of the variance of Fourier transform of the Lévy driven OU Process for $\sigma = 1$

	$\sigma = 1$	
Order	Minimum	Maximum
1	2,4883	2,4883
2	2,4372	2,5428
3	2,4006	2,5901
4	2,3709	2,6312
5	2,3453	2,6673
6	2,3224	2,6995
7	2,3018	2,7285
8	2,2829	2,7551
9	2,2656	2,7797
10	2,2498	2,8026
11	2,2342	2,8242
12	2,2130	2,8446
13	2,1872	2,8646
14	2,1647	2,8830
Trend	y=-0,105ln(x)+2,4723	y=0,1569ln(x)+2,4391
Number Of Iterations	558	833
Value	1,8082	3,4943
Variance	31,11	

As can be observed from the table at $\sigma = 1$ the variance of the Fourier transform of Lévy Process is approximately 31. However, when $\sigma = 0,75$ which is only a small decrease the variance shoots up to 16346! We can conclude that at some point the inequality is broken from the Levy driven OU process in favor of momentum, therefore in order to satisfy the inequality the variance of the Fourier transform which corresponds to discrepancy in position tends to infinity. In that sense one should be very cautious when a certain variance parameter to a Lévy driven OU process is

TABLE 4. Estimation of the variance of Fourier transform of the Lévy driven OU Process for $\sigma = 1$

	$\sigma = 0,75$	
Order	Minimum	Maximum
1	3,9916	3,9916
2	3,9395	4,0438
3	3,9004	4,0882
4	3,8684	4,1276
5	3,8413	4,1635
6	3,8176	4,1967
7	3,7966	4,2278
8	3,7778	4,2569
9	3,7607	4,2845
10	3,7451	4,3110
11	3,7270	4,3343
12	3,5979	4,4404
13	3,5962	4,4506
14	3,4825	4,6001
Trend	y=-0,162ln(x)+4,0656	y=0,1984ln(x)+3,8942
Number Of Iterations	833	1060
Value	2,9761	5,2763
Variance	**16346**	

assigned since an incremental decrease may instantly end up with a huge discrepancy in its position.

7. CONCLUSION

This study applies the Heisenberg's uncertainty principle to three major processes which are widely used in finance. It is observed that for geometric Brownian motion and OU processes without jumps, where both are normally distributed, the impact of the principle is not easily observable unless the variance parameter tends to zero. However, the effect of the uncertainty principle is clearly visible when the process in question is a Lévy driven OU process. After a certain point, which is quite far away from zero compared to the other mentioned processes the variance of the Fourier transform tends to infinity. Due to the complex structure of the Fourier transform of the Lévy driven process, we used Taylor series expansion in order to simplify the non-elementary integrals. Moreover, we used only exponentially distributed positive jumps. More generalized jump size distributions can be a future research topic. Another obstacle we faced was the computation of the variance of the Fourier transform. Here we applied the Taylor expansion to find the roots of the variance estimating function. Although our proposed algorithm works quite fast for smooth distributions, due to the saddle like structure, it becomes too slow after an order of 15. We therefore applied an innovative technique by using a logarithmic least squares estimation of the roots. After a desired tolerance level, we stop and use the roots as lower and upper limit for the integration of the variance estimator function. Here Simpson's rule is used for approximation of the variance.

REFERENCES

[1] F. Black and M. Scholes, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* 81 (3): 637–654, 1973.

[2] A. Cayley, A Memoir on the Theory of Matrices. *Phil.Trans.* 148, 1858.

[3] R. Cont and P. Tankov, *Financial Modeling with Jump Processes*, Chapman Hall/CRC Financial Mathematics Series, 2004.

[4] J. G. F. Francis, The QR Transformation, *The Computer Journal*, 4 (3): 265–271, 1961.

[5] F. G. Frobenius, Ueber lineare Substutionen und bilineare Formen. *J. reine angew Math.(Crelle J.)* 84: 1–63, 1878.

[6] W. R. Hamilton, On a New and General Method of Inverting a Linear and Quaternion Function of a Quaternion. *Proceedings of the Royal Irish Academy (Royal Irish Academy)* 8: 182–183, 1864.

[7] W. Heisenberg, Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik, *Zeitschrift für Physik* 43 (3–4): 172–198, 1927.

[8] K. Ito, Stochastic Integral *Proc. Imperial Acad. Tokyo*, 20: 519–524, 1944.

[9] V. Kublanovskaya, On some algorithms for the solution of the complete eigenvalue problem, *USSR Computational Mathematics and Mathematical Physics* 1 (3): 637–657, 1963

[10] L. Ornstein and G. E. Uhlenbeck, On the Theory of the Brownian Motion, *Physical Review* 36: 823–841, 1930.

[11] M. Plancherel, Contribution a l'etude de la representation d'une fonction arbitraire par les integrales dfinies. *Rendiconti del Circolo Matematico di Palermo*, 30: 298–335, 1910.