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STABILITY SWITCHES AND BIFURCATION ANALYSIS OF A FOOD CHAIN STAGE-STRUCTURED MODEL WITH IMPULSIVE PEST MANAGEMENT STRATEGY

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ABSTRACT. It is a natural fact that the insect population have two major life stages, immature and mature. Therefore in this paper, a stage-structured three species crop-pest-natural enemy food chain model concerning impulsive technique is used to control the pest population at a desired level. The model is analyzed into two parts. In first part, the model is studied without impulsive effect and a set of sufficient conditions for local asymptotic stability of all the feasible equilibria is obtained. Moreover, using approach as in [6], the possibility of the existence of a Hopf bifurcation for the interior equilibrium with respect to maturation delay is explored. Also obtain some threshold values of maturation delay for the stability-switching of the particular system. In succession, using the normal form theory and center manifold argument, we derive the explicit formulas which determine the stability and direction of bifurcating periodic solutions. In second part, the model is studied with impulsive effects and some sufficient conditions are obtained for extinction of pest populations. Finally, numerical simulations in supporting of theoretical findings are explored.

Key Words: Food chain, maturation delay, stability-switch, Hopf bifurcation, chaos, impulse

MSC: 92Bxx, 92F05

1. Introduction

We know that pest is a harmful insect and its outbreak often cause serious ecological and economic problems for the society [18, 26]. However, many pest control methods are available, such as biological, cultural, physical and chemical [11, 24, 25], but the farmers mostly use pesticides to control pests because of its efficiency and convenience. The pesticide kills not only pests but also their natural enemies. The common practice proves that long-term adopting chemical control may give rise to disastrous results, for example: environmental contamination, toxicosis of the man and animals and so on [8, 10]. On the other hand, the biological control is harmless to

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the human, animals and environment. It is generally used to control a particular pest using a chosen living organism; this chosen organism might be a predator, parasite or disease which attacks on the harmful insect pest.

In the last few years the researchers have increased their interests in the study of biological pests control using prey-predator interaction because the well transformation of these problems into mathematical form by impulsive differential equations [16, 20, 17]. Moreover, the effective use of biological control is often require a good understanding of the biology of natural enemies life-stages, which are particularly easy to recognize, being separated by short events such as eggs, moult or pupation. Many researchers studied stage-structured models before 1990 [4, 5, 27], but the real interest comes into picture on the stage structured models after the work of Aiello and Freedman [1]. They proposed a single species model with stage structure assuming an average age to maturity (i.e., as a constant time delay) which reflecting a delayed birth of immature and a reduced survival of immature to their maturity. The model is as follows:

$$\begin{aligned} \frac{dx(t)}{dt} &= \beta y(t) - rx(t) - \beta e^{-r\tau} y(t-\tau), \\ \frac{dy(t)}{dt} &= \beta e^{-r\tau} y(t-\tau) - \eta y^2(t), \end{aligned} \tag{1.1}$$

where $x(t)$ and $y(t)$ represent the immature and mature populations densities, respectively. Here, it is assumed that at any time $t > 0$, growth rate of immature population is proportional to the existing mature population with proportionality constant β; the death rate of immature population is r; the death rate of mature population is proportional to the square of the population with the proportionality constant η. The term $\beta e^{-r\tau} y(t-\tau)$ represents the immature who were born at time $t-\tau$ and survive at time t, therefore it represents the transformation of immature to mature, where τ represent a constant time to maturity. All the parameters τ, β, r and η are positive constants.

Further, the single species model (1.1) is extended by many researchers into different kinds of stage-structured models and obtained significant results [2, 21]. Recently, many authors studied different kinds of predator-prey system with division of the predators into immature and mature class and a good number of research has been carried out [28, 12, 15, 23]. One interested model is suggested by Satio and Takeuchi [23]. They considered two life stages for predator and proposed the following predator-prey model:

$$\begin{aligned} \dot{x}(t) &= x(t)\left(r_1 - a_{11}x(t) - a_{13}y(t)\right), \\ \dot{Y}(t) &= -r_2 Y(t) + a_{31}x(t)y(t) - a_{31}e^{-r_2\tau}x(t-\tau)y(t-\tau), \\ \dot{y}(t) &= -r_3 y^2(t) + a_{31}e^{-r_2\tau}x(t-\tau)y(t-\tau), \end{aligned} \tag{1.2}$$

where $x(t)$ is population density of prey, $Y(t)$ and $y(t)$ denote the densities of immature and mature predator population, respectively; τ represent a constant time to maturity for predator; a_{13} is the per capita rate of predation; a_{31} is the conversion rate and all other parameters have the similar meaning as in (1.1).

Furthermore, three species food chain models are investigated by many researchers, but they ignored the stage structure phenomena of species and impulsive releasing of natural enemy population, which is the new interesting idea for effectively pest control. Therefore, the aim of this paper is to study a crop-pest-natural enemy model with two life stages of natural enemy concerning the biological control method with impulsive strategy. The paper is organized as follows: in section 2, models development are discussed. The preliminaries and boundedness are given in 3. The analysis of model without impulsive effect is carried out in section 4 and all the feasible equilibria with their local stability behavior are studied in 4.1. The stability and direction of Hopf bifurcation is analyzed in 4.2. Further, in section 5, the impulsive model is analyzed. Finally, a set of numerical simulations is given to verify all the major analytical findings in 6 and then conclusions for this paper are given in the last section.

2. **Proposed Mathematical Model**

In this section, our main aim is to propose a mathematical model for the interaction of plant-pest-natural enemy. Since plant-hoppers are serious pests for the rice crops and these are suppressed by Lycosa tarantula and other spiders, it is well documented in a report of the Indian Council of Agricultural Research (ICAR) [7]. The tarantula species has two major life stages, namely, immature and mature; only mature population can harvest the pest and reproduce a new offspring. In modelling process, we assume that $x(t)$, $y(t)$, $z_1(t)$ and $z_2(t)$ are densities of crop, pest, immature and mature natural enemy at time t, respectively. The parameters a_1, b_1 are respectively the intrinsic growth rate and overcrowding rate of crops; c_1, α_1 are per capita predation rate of crop by the pest and the corresponding growth rate of pest, respectively. The parameters c_2 is per capita predation rate of pest by the natural enemy and α_2 is the corresponding growth rate of mature natural enemy. Here, d_1, d_2, d_3 are the natural death rates of pests, immature and mature natural enemies, respectively. Further, τ is the maturation delay from immature to mature natural enemies, the term $\alpha_2 e^{-d_2\tau} y(t-\tau) z_2(t-\tau)$ represents the transformation of immature to mature population. Keeping this biological situation in mind and motivated from the modelling ideas of [1, 23], in this paper, we propose following two types of three species stage structured crop-pest-natural enemy models:

2.1. **Model without impulsive effect.**

$$\begin{aligned}\frac{dx(t)}{dt} &= x(t)\left(a_1 - b_1 x(t) - c_1 y(t)\right),\\ \frac{dy(t)}{dt} &= y(t)\left(\alpha_1 x(t) - d_1 - c_2 z_2(t)\right),\\ \frac{dz_1(t)}{dt} &= \alpha_2 y(t) z_2(t) - d_2 z_1(t) - \alpha_2 e^{-d_2\tau} y(t-\tau) z_2(t-\tau),\\ \frac{dz_2(t)}{dt} &= \alpha_2 e^{-d_2\tau} y(t-\tau) z_2(t-\tau) - d_3 z_2(t),\end{aligned} \tag{2.1}$$

The model completes with the following set of initial conditions:

$$\begin{aligned}&x(\theta) = \phi_1(\theta),\ y(\theta) = \phi_2(\theta),\ z_1(\theta) = \psi_1(\theta),\\ &z_2(\theta) = \psi_2(\theta),\ \phi_i(0) > 0,\ \psi_i(0) > 0,\quad \theta \in [-\tau, 0],\quad i = 1, 2,\end{aligned} \tag{2.2}$$

where $(\phi_1, \phi_2, \psi_1, \psi_2) \in C\left([-\tau, 0], R_+^4\right)$, the Banach space of continuous functions mapping on the interval $[-\tau, 0]$ into R_+^4. For continuity of the initial conditions, we further require

$$\psi_1(0) = \int_{-\tau}^{0} \alpha_2 \phi_2(s) \psi_2(s) e^{d_2 s} ds, \tag{2.3}$$

where $\psi_1(0)$ represents the accumulated survivors of those natural enemy members who were born between $-\tau$ and 0.

2.2. **Model with impulsive effect.**

$$\begin{cases}\left.\begin{aligned}\frac{dx(t)}{dt} &= x(t)\left(a_1 - b_1 x(t) - c_1 y(t)\right),\\ \frac{dy(t)}{dt} &= y(t)\left(\alpha_1 x(t) - d_1 - c_2 z_2(t)\right),\\ \frac{dz_1(t)}{dt} &= \alpha_2 y(t) z_2(t) - d_2 z_1(t) - \alpha_2 e^{-d_2\tau} y(t-\tau) z_2(t-\tau),\\ \frac{dz_2(t)}{dt} &= \alpha_2 e^{-d_2\tau} y(t-\tau) z_2(t-\tau) - d_3 z_2(t),\end{aligned}\right\} & t \neq nT,\\ \left.\begin{aligned}&\Delta x(t) = 0,\\ &\Delta y(t) = 0,\\ &\Delta z_1(t) = p_r,\\ &\Delta z_2(t) = 0,\end{aligned}\right\} \quad t = nT,\ n = 1, 2, \ldots. \end{cases} \tag{2.4}$$

Here T is period of pulsing and p_r represents the pulse releasing amount of immature natural enemies.

3. Preliminaries and Boundedness

Assume $X(t) = (x(t), y_1(t), y_2(t), z(t))^T$ is the solution of (2.4), which is a piecewise continuous function $X : R_+ \to R_+^4$. Again, $X(t)$ is continuous in the interval $(nT, (n+1)T]$, $n \in Z_+$ and $X(nT^+) = \lim\limits_{t\to nT^+} X(t)$ exists. The global existence and uniqueness of solution of system (2.4) are guaranteed by the smoothness properties similar as in [3, 22].

Now we will reproduce and establish some lemmas before going to the main results.

Lemma 3.1 ([19]). *Consider the following delay equation*

$$\frac{dx(t)}{dt} = a_1 x(t-\tau) - a_2 x(t),$$

where $a_1,\ a_2, \tau > 0; x(t) > 0$ *for* $-\tau \le t \le 0$. *If* $a_1 < a_2$, *we have* $\lim_{t\to\infty} x(t) = 0$.

Lemma 3.2 ([22]). *Let the function* $m \in PC'[R^+, R]$ *and* $m(t)$ *be left-continuous at* $t_k,\ k = 1, 2, \dots,$ *satisfies the inequalities*

$$\begin{cases} m'(t) \le p(t)m(t) + q(t),\ t \ge t_0, \quad t \ne t_k, \\ m(t_k^+) \le d_k m(t_k) + b_k,\ t = t_k, \quad k = 1, 2, \dots, \end{cases} \tag{3.1}$$

where $p, q \in PC[R^+, R]$ *and* $d_k \ge 0$, b_k *are constants, then*

$$\begin{aligned} m(t) \ \le\ & m(t_0) \prod_{t_0<t_k<t} d_k \exp\left(\int_{t_0}^t p(s)ds\right) \\ & + \sum_{t_0<t_k<t} \left(\prod_{t_k<t_j<t} d_j \exp\left(\int_{t_k}^t p(s)ds\right)\right) b_k \\ & + \int_{t_0}^t \prod_{s<t_k<t} d_k \exp\left(\int_s^t p(\sigma)d\sigma\right) q(s)ds,\ t \ge t_0. \end{aligned} \tag{3.2}$$

If all the directions of the inequalities in (3.1) *are reversed, then* (3.2) *holds true for the reversed inequality.*

Since boundedness implies a natural restriction, therefore, we can state and prove the following lemma:

Lemma 3.3. *The solutions of the system* (2.1) *and* (2.4) *with initial conditions* (2.2) *and* (2.3) *are bounded.*

Proof. Let $V(t) = \alpha_1\alpha_2 x(t) + c_1\alpha_2 y(t) + c_1c_2 z_1(t) + c_1c_2 z_2(t)$ and $p = \min\{a_1, d_1, d_2, d_3\}$, we have

$$\dot V(t) = a_1\alpha_1\alpha_2 x(t) - b_1\alpha_1\alpha_2 x^2(t) - c_1 d_1\alpha_2 y(t) - c_1c_2d_2 z_1(t) - c_1c_3d_3 z_2(t).$$

$$\dot{V}(t) + pV(t) \leq 2a_1\alpha_1\alpha_2 x(t) - b_1\alpha_1\alpha_2 x^2(t) \leq K.$$

Where $K = a_1^2\alpha_1\alpha_2/b_1$. Thus $V(t) \leq K/p + (V(0) - K/p)\, e^{-pt} \to K/p$ as $t \to \infty$ and hence the solution of the system (2.1) is bounded.

Similarly, we obtain that $D^+V(t) + pV(t) \leq K$, for $t \in (nT, (n+1)T]$. When $t = nT$, $V(t^+) = V(t) + p_r$. By Lemma 3.2 for $t \in (nT, (n+1)T]$, we have

$$\begin{aligned} V(t) &\leq V(0)e^{-pt} + \int_0^t Ke^{-p(t-s)}ds + \sum_{0<nT<t} p_r e^{-p(t-nT)} \\ &= V(0)e^{-pt} + \frac{K}{p}(1-e^{-pt}) + p_r\frac{e^{-p(t-T)} - e^{-p(t-(n+1)T)}}{1-e^{pT}} \\ &< V(0)e^{-pt} + \frac{K}{p}(1-e^{-pt}) + \frac{p_r e^{-p(t-T)}}{1-e^{pT}} + \frac{p_r e^{pT}}{e^{pT}-1} \\ &\to \frac{K}{p} + \frac{p_r e^{pT}}{e^{pT}-1}, \text{ as } t \to \infty. \end{aligned}$$

Thus $V(t)$ is uniformly ultimately bounded. This completes the proof. □

4. **Analysis of Model without impulsive effect**

In the next section, we will investigate the feasible equilibrium of the system (2.1) and study their stability behaviors.

4.1. **Nonnegative equilibria and their local stability.** In this section, our main objective is to investigate the local behavior of all feasible equilibria and existence of a Hopf bifurcation at interior equilibrium. The equation for the variable z_1 in the system (2.1) can be rewritten as

$$\begin{aligned} \frac{dz_1(t)}{dt} &= \alpha_2 y(t)z_2(t) - d_2z_1(t) - \alpha_2 e^{-d_2\tau}y(t-\tau)z_2(t-\tau) \\ &:= -d_2z_1(t) + f\left(y(t), z_2(t), y(t-\tau), z_2(t-\tau)\right), \end{aligned}$$

if $y(t)$, $z_2(t)$ are bounded and $y(t) \to y^*$, $z_2 \to z_2^*$ as $t \to \infty$, then $z_1(t) \to f\left(y^*, z_2^*, y^*, z_2^*\right)/d_2$ as $t \to \infty$, i.e., the asymptotic behavior of z_1 is completely dependent on $y(t)$ and $z_2(t)$. Hence, the asymptotic behavior of our proposed model will remain the same with the following reduced system:

$$\begin{aligned} \frac{dx(t)}{dt} &= x(t)\left(a_1 - b_1x(t) - c_1y(t)\right), \\ \frac{dy(t)}{dt} &= y(t)\left(\alpha_1x(t) - d_1 - c_2z_2(t)\right), \\ \frac{dz_2(t)}{dt} &= \alpha_2 e^{-d_2\tau}y(t-\tau)z_2(t-\tau) - d_3z_2(t), \end{aligned} \tag{4.1}$$

Using simple algebraic manipulations, we get four feasible equilibria for the system (4.1), namely,

(a) trivial equilibrium $E_0(0,0,0)$;
(b) boundary equilibrium $E_1\left(a_1/b_1,0,0\right)$;
(c) planner equilibrium $E_2(\bar{x},\bar{y},0)$ exists only when (**H1**) $a_1\alpha_1 > b_1d_1$;
(d) interior equilibrium $E_3(x^*,y^*,z_2^*)$ exists if (**H2**) $a_1\alpha_1\alpha_2 > \Delta$.

Where

$$\bar{x}=\frac{d_1}{\alpha_1},\ \bar{y}=\frac{a_1\alpha_1-b_1d_1}{c_1\alpha_1},\ x^*=\frac{a_1\alpha_2-c_1d_3e^{d_2\tau}}{b_1\alpha_2},\ y^*=\frac{d_3e^{d_2\tau}}{\alpha_2},$$

$$z_2^*=\frac{a_1\alpha_1\alpha_2-\Delta}{b_1c_2\alpha_2},\ \Delta=c_1d_3\alpha_1e^{d_2\tau}+b_1d_1\alpha_2.$$

Further, (**H2**) implies that

$$\tau<\frac{1}{d_2}\log\left(\frac{a_1\alpha_1\alpha_2-b_1d_1\alpha_2}{c_1d_3\alpha_1}\right):=\bar{\tau}.$$

The characteristic equation for trivial equilibrium $E_0(0,0,0)$ is given by

$$(\lambda-a_1)(\lambda+d_1)(\lambda-d_3)=0. \tag{4.2}$$

The characteristic equation (4.2) has one positive and two negative roots, hence equilibrium E_0 is a unstable saddle point.

Similarly, the characteristic equation for boundary equilibrium E_1 is as follows:

$$(\lambda+d_3)(\lambda+a_1)\left(\lambda+d_1-\frac{a_1\alpha_1}{b_1}\right)=0. \tag{4.3}$$

Clearly, all the eigenvalues are negative only when $a_1\alpha_1 < b_1d_1$, which stabilize E_1, otherwise it is unstable. Again, the characteristic equation for planner equilibrium $E_2(\bar{x},\bar{y},0)$ becomes

$$\left(\lambda+d_3-\alpha_2\bar{y}e^{-d_2\tau}\right)\left(\lambda^2+b_1\bar{x}\lambda+c_1\alpha_1\bar{x}\bar{y}\right)=0. \tag{4.4}$$

Since, both roots of the quadratic equation $\lambda^2+b_1\bar{x}\lambda+c_1\alpha_1\bar{x}\bar{y}=0$ have negative real parts, hence the equilibrium E_2 is locally asymptotically stable if $\alpha_2\bar{y}e^{-d_2\tau} < d_3$ for all $\tau > (1/d_2)\log\alpha_2\bar{y}/d_3 := \tau_{cr}$.

Finally, the characteristic equation for interior equilibrium $E_3(x^*,y^*,z_2^*)$ is given as:

$$\lambda^3+A_1(\tau)\lambda^2+A_2(\tau)\lambda+A_3(\tau)+\left(B_1(\tau)\lambda^2+B_2(\tau)\lambda+B_3\right)e^{-\lambda\tau}=0, \tag{4.5}$$

where

$$A_1(\tau)=b_1x^*(\tau)+d_3,\ A_2(\tau)=b_1d_3x^*(\tau)+c_1\alpha_1x^*(\tau)y^*(\tau),$$

$$A_3(\tau)=c_1d_3\alpha_1x^*(\tau)y^*(\tau),\ B_1=-\alpha_2y^*(\tau)e^{-d_2\tau},$$

$$B_2(\tau)=\alpha_2y^*(\tau)\left(c_2z_2^*(\tau)-b_1x^*(\tau)\right)e^{-d_2\tau},$$

$$B_3(\tau)=\alpha_2x^*(\tau)y^*(\tau)\left(b_1c_2z_2^*(\tau)-c_1\alpha_1y^*(\tau)\right)e^{-d_2\tau}.$$

We write A_i, B_i in place of $A_i(\tau)$, $B_i(\tau)$ for $i = 1, 2, 3$ in the rest of the analysis. The characteristic equation (4.5) can be rewritten as:

$$P(\lambda,\tau) + Q(\lambda,\tau)e^{-\lambda\tau} = 0, \tag{4.6}$$

where

$$P(\lambda,\tau) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3,\ \ Q(\lambda,\tau) = B_1\lambda^2 + B_2\lambda + B_3. \tag{4.7}$$

When $\tau = 0$, the characteristic equation (4.5) becomes

$$\lambda^3 + b_1x^*\lambda^2 + (c_1\alpha_1x^*y^* + c_2\alpha_2y^*z_2^*)\lambda + b_1c_2\alpha_2x^*y^*z_2^* = 0. \tag{4.8}$$

Since $(A_1(0)+B_1(0))(A_2(0)+B_2(0)) - (A_3(0)+B_3(0)) = b_1c_1\alpha_1x^{*2}y^* > 0$, therefore, using Routh-Hurwitz criterion, all the solutions of the characteristic equation (4.8) have negative real parts. Thus the interior equilibrium E_3 is locally asymptotically stable for $\tau = 0$ if it exists.

In the following, we investigate the existence of purely imaginary roots $\lambda = i\omega(\omega > 0)$ of characteristic equation (4.6). We apply Beretta and Kuang [6] geometric criterion which gives the existence of purely imaginary roots of a characteristic equation with delay dependent coefficients.

Lemma 4.1. *If (**H2**) holds, then the following are satisfies:*

1. $P(0,\tau) + Q(0,\tau) \neq 0$,
2. $P(i\omega,\tau) + Q(i\omega,\tau) \neq 0$ *for all* $\omega \in R$,
3. $\limsup\{|Q(\lambda,\tau)/P(\lambda,\tau)| \ :\ |\lambda| \to \infty,\ Re\lambda \geq 0\} < 1$,
4. $F(\omega,\tau) = |P(i\omega,\tau)|^2 - |Q(i\omega,\tau)|^2$ *for each* τ *has at most a finite number of real zeros,*
5. *each positive root* $\omega(\tau)$ *of* $F(\omega,\tau) = 0$ *is continuous and differentiable in* τ *whenever it exists.*

Proof. 1. For $\tau \in [0,\bar{\tau})$,

$$P(0,\tau) + Q(0,\tau) = b_1c_2\alpha_2x^*y^*z_2^*e^{-d_2\tau} \neq 0.$$

2. $P(i\omega,\tau) + Q(i\omega,\tau) = -(A_1+B_1)\omega^2 + A_3 + B_3 + i[-\omega^3 + (A_2+B_2)\omega] \neq 0$.
3. Since $P(\lambda,\tau)$ is a third degree polynomial in λ and $Q(\lambda,\tau)$ second degree, hence, lim $\sup\{|Q(\lambda,\tau)/P(\lambda,\tau)| \ :\ |\lambda| \to \infty,\ Re\lambda \geq 0\} = 0 < 1$.
4. Let F be defined as

$$F(\omega,\tau) = |P(i\omega,\tau)|^2 - |Q(i\omega,\tau)|^2.$$

From

$$|P(i\omega,\tau)|^2 = \omega^6 + \left(A_1^2 - 2A_2\right)\omega^4 + \left(A_2^2 - 2A_1A_3\right)\omega^2 + A_3^2$$

and

$$|Q(i\omega,\tau)|^2 = B_1^2\omega^4 + \left(B_2^2 - 2B_1B_3\right)\omega^2 + B_3^2,$$

we have

$$F(\omega,\tau) = \omega^6 + p(\tau)\omega^4 + q(\tau)\omega^2 + r(\tau),$$

where

$$p(\tau) = \frac{(a_1\alpha_2 - c_1 d_3 e^{d_2\tau})(a_1 b_1 \alpha_2 - (b_1 + 2\alpha_1)c_1 d_3 e^{d_2\tau})}{b_1\alpha_2^2},$$

$$q(\tau) = c_1^2\alpha_1^2 x^{*2} y^{*2} + 2b_1 c_2 \alpha_2 x^* y^* z_2^* e^{-d_2\tau} - c_2\alpha_2^2 y^{*2} z_2^* e^{-2d_2\tau(2b_1 x^* + c_2 z_2^*)},$$

$$r(\tau) = -3c_1 d_3 \alpha_1 + (a_1\alpha_1 - b_1 d_1)\alpha_2 e^{-d_2\tau}.$$

It is obvious that property (iv) is satisfied.

5. Since $F(\omega,\tau)$ is continuous in ω and τ and it is differentiable with respect to ω. Therefore, from Implicit Function Theorem each root of $F(\omega,\tau) = 0$ is continuous and differentiable in τ.

Hence all the conditions of the Lemma are satisfied, which ensure the existence of purely imaginary roots for the characteristic equation (4.5). □

Now let $\lambda = i\omega$ ($\omega > 0$) be a root of (4.5). Substituting it into (4.5) and separating the real and imaginary parts, we get

$$B_2\omega \sin\omega\tau + \big(B_3 - B_1\omega^2\big)\cos\omega\tau = A_1\omega^2 - A_3,$$

$$\big(B_1\omega^2 - B_3\big)\sin\omega\tau + B_2\omega\cos\omega\tau = \omega^3 - A_2\omega, \tag{4.9}$$

which gives

$$\sin\omega\tau = \frac{B_1\omega^5 - (A_1B_2 + A_2B_1 + B_3)\,\omega^3 + (A_2B_3 + A_3B_2)\,\omega}{B_1^2\omega^4 + (B_2^2 - 2B_1B_3)\omega^2 + B_3^2},$$

$$\cos\omega\tau = \frac{(B_2 - A_1B_1)\,\omega^4 + (A_1B_3 + A_3B_1 - A_2B_2)\,\omega^2 - A_3B_3}{B_1^2\omega^4 + (B_2^2 - 2B_1B_3)\omega^2 + B_3^2}. \tag{4.10}$$

We can define the angle $\theta(\tau) \in [0, 2\pi]$, $\forall\, \tau \geq 0$ as the solution of (4.10):

$$\sin\theta(\tau) = \frac{B_1\omega^5 - [A_1B_2 + A_2B_1 + B_3]\,\omega^3 + [A_2B_3 + A_3B_2]\,\omega}{B_1^2\omega^4 + [B_2^2 - 2B_1B_3]\omega^2 + B_3^2},$$

$$\cos\theta(\tau) = \frac{[B_2 - A_1B_1]\,\omega^4 + [A_1B_3 + A_3B_1 - A_2B_2]\,\omega^2 - A_3B_3}{B_1^2\omega^4 + [B_2^2 - 2B_1B_3]\omega^2 + B_3^2}, \tag{4.11}$$

where $\omega = \omega(\tau)$ and such $\theta(\tau)$ is uniquely well defined for all τ, so that $F(\omega(\tau),\tau) = 0$. Hence

$$\omega^6 + p(\tau)\omega^4 + q(\tau)\omega^2 + r(\tau) = 0. \tag{4.12}$$

Again the polynomial function F can be written as

$$F(\omega,\tau) = h(\omega^2,\tau),$$

where h is a cubic polynomial, defined by

$$h(z,\tau) := z^3 + p(\tau)z^2 + q(\tau)z + r(\tau).$$

Applying the Descartes' rule of signs for the number of positive roots of $h(z,\tau)=0$, we get the following four cases:

Case I: Let

$$I_{11}=\{\tau\geq 0 \mid p(\tau)>0 \;\; q(\tau)>0 \text{ and } r(\tau)>0\},$$

$$I_{12}=\{\tau\geq 0 \mid r(\tau)>0, \text{ at least one } p(\tau)<0 \;\; q(\tau)<0\},$$

$$I_1=I_{11}\cup I_{12}.$$

In the interval I_{12}, $h(z,\tau)=0$ either has 0 or 2 positive roots. When the polynomial $h(z,\tau)$ has no positive zero in I_{12}, then it also has no positive zero in the interval I_1. Thus in this case, purely imaginary root of (4.5) never exists.

Case II: Let

$$I_{21}=\{\tau\geq 0 \mid p(\tau)>0 \text{ and } r(\tau)<0\},$$

$$I_{22}=\{\tau\geq 0 \mid p(\tau)<0, \; q(\tau)<0 \text{ and } r(\tau)<0\},$$

$$I_{23}=\{\tau\geq 0 \mid P(\tau)<0, \; q(\tau)>0 \text{ and } r(\tau)<0\}.$$

In the region I_{23}, either one or three positive zeros of $h(z,\tau)$ exist. Suppose only one positive zero is feasible in I_{23}, then in the interval $I_2=I_{21}\cup I_{22}\cup I_{23}$, $h(z,\tau)$ has only one positive zero. Therefore, $i\omega^*$ with $\omega^*=\omega(\tau^*)>0$ is a purely imaginary root of (4.5) if and only if τ^* is a zero of the S_n, where

$$S_n(\tau)=\tau-\frac{\theta(\tau)+2n\pi}{\omega(\tau)}, \;\; \tau\in I_2, \text{ with } n\in N_0.$$

Now we will verify the following lemma:

Lemma 4.2 (Beretta and Kuang [6]). *Assume that $\omega(\tau)$ is a positive real root of (4.5) defined for $\tau\in I$, $I\subseteq R_{+0}$, and at some $\tau^*\in I$,*

$$S_n(\tau^*)=0, \quad \textit{for some } n\in N_0. \tag{4.13}$$

Then a pair of simple conjugate pure imaginary roots $\lambda_+(\tau^*)=i\omega(\tau^*)$, $\lambda_-(\tau^*)=-i\omega(\tau^*)$ of (4.5) exists at $\tau=\tau^*$ which crosses the imaginary axis from left to right if $\delta(\tau^*)>0$ and crosses the imaginary axis from right to left if $\delta(\tau^*)<0$, where

(4.14)
$$\delta(\tau^*)=Sign\left\{\frac{dRe(\lambda)}{d\tau}\bigg|_{\lambda=i\omega(\tau^*)}\right\}=Sign\left\{\frac{\partial F}{\partial\omega}(\omega(\tau^*),\tau^*)\right\}Sign\left\{\frac{dS_n(\tau)}{d\tau}\bigg|_{\tau=\tau^*}\right\}.$$

Since $\partial F(\omega,\tau)/\partial\omega|_{\omega=\omega(\tau^*)}=[6\omega^5+4p(\tau)\omega^3+2q(\tau)\omega]_{\omega=\omega(\tau^*)}>0$, therefore, from (4.14), we get

$$\delta(\tau^*)=Sign\left\{\frac{dRe(\lambda)}{d\tau}\bigg|_{\lambda=i\omega(\tau^*)}\right\}=Sign\left\{\frac{dS_n(\tau)}{d\tau}\bigg|_{\tau=\tau^*}\right\}.$$

Here, we can easily observe that $S_n(0)<0$ and $S_n(\tau)>S_{n+1}(\tau)\;\forall\;\tau\in I_2$, $n\in N_0$. Thus, if S_0 has no zero in I_2, then the function S_n also have no zero in I_2 and if the function S_n has positive zeros, denoted by τ_n^j for some $\tau \in I_2,\ n \in N_0$, then without loss of generality, we may assume that

$$\frac{dS_n(\tau_n^j)}{d\tau} \neq 0 \text{ with } S_n(\tau_n^j) = 0.$$

Applying similar logic as in [6], stability switches occur at the zeros of $S_0(\tau)$, denoted by τ_0^j, if (**H2**) holds. Let us assume that

$$\tau^* = \min\{\tau \in I_2 \mid S_0(\tau) = 0\} \ \text{ and } \ \tau^{**} = \max\{\tau \in I_2 \mid S_0(\tau) = 0\}.$$

Using the Hopf bifurcation theorem for functional differential equation [13], we can conclude the existence of Hopf bifurcation in the following theorem:

Theorem 4.3. *Let (**H2**) hold. The local behavior of the system* (2.1) *at interior equilibrium is described as:*

1. *If the function $S_0(\tau)$ has no positive zero in I_2, then the interior equilibrium $E_3(x^*, y^*, z_2^*)$ is locally asymptotically stable for all $\tau \geq 0$.*
2. *If the function $S_n(\tau)$ has at least positive zero in I_2 for some $n \in N_0$, then E_3 is locally asymptotically stable for $\tau \in [0, \tau^*) \cup (\tau^{**}, \bar{\tau}]$ and unstable and a Hopf bifurcation occurs for $\tau \in (\tau^*, \tau^{**})$, i.e., stability switches of stability-instability-stability occur.*

Case III: If in the interval $I_3 = I_{12}$, $h(z, \tau) = 0$ has two positive roots, denoted by ω_1 and ω_2, we get following two sequences of functions on I_3:

$$S_n^{(1)}(\tau) = \tau - \frac{\theta_1(\tau) + 2n\pi}{\omega_1(\tau)}\ ;\ S_n^{(2)}(\tau) = \tau - \frac{\theta_2(\tau) + 2n\pi}{\omega_2(\tau)}, \quad n \in N_0,$$

where $\theta_1(\tau)$ and $\theta_2(\tau)$ are the solutions of (4.11) when $\omega = \omega_1, \omega_2$ respectively. Similarly, we can also obtain for $\tau \in I_3$ that $S_n^{(k)}(0) \leq 0$ and $S_n^{(k)}(\tau) > S_{n+1}^{(k)}(\tau)$ with $n \in N_0$, $k = 1, 2$. Thus if $S_0^{(1)}(\tau) > S_0^{(2)}(\tau)$, then $S_n^{(1)}(\tau) > S_n^{(2)}(\tau)$ and hence stability switch depends on all real roots of $S_n^{(1)}(\tau) = 0$, otherwise the stability switches depend on roots of both $S_n^{(1)}(\tau) = 0$ and $S_n^{(2)}(\tau) = 0$. Furthermore, we can also obtain the similar results stated in Theorem 4.3.

Case IV: If in the interval $I_4 = I_{23}$, $h(z, \tau)$ has three positive zeros, we can obtain the parallel results as in case III.

4.2. **Direction and stability of Hopf bifurcation.** In the previous section, we obtained the conditions, under which system (4.1) undergoes Hopf bifurcation, taking maturation delay (τ) as the critical parameter. Using the normal form theory and center manifold reduction as described in Hassard et al. [14], we can investigate the direction of Hopf bifurcation and the properties of these bifurcating periodic solutions. Hence, we always assume that system (4.1) undergoes Hopf bifurcations at the critical value τ^* of τ and there exists a pair of pure imaginary roots, i.e., $\pm i\omega(\tau^*)$ of the characteristic equation (4.5).

Using the Appendix A, we can compute the following values:

$$
\begin{aligned}
C_1(0) &= \frac{i}{2\omega^*\tau^*}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2},\\
\mu_2 &= \frac{Re\{C_1(0)\}}{Re\{\lambda'(\tau^*)\}},\\
\beta_2 &= 2Re\{C_1(0)\},\\
T_2 &= \frac{I_m\{C_1(0)\} + \mu_2 I_m\{\lambda'(\tau^*)\}}{\omega^*\tau^*},
\end{aligned}
\tag{4.15}
$$

which determine the behavior of bifurcating periodic solution in the center manifold at the critical value τ^*, i.e., μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solution exists for $\tau > \tau^*$ ($\tau < \tau^*$); β_2 determines the stability of the bifurcating periodic solution: the bifurcating periodic solution is stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$) and T_2 determines the period of the bifurcating periodic solution: the period increases (decreases) if $T_2 > 0$ ($T_2 < 0$).

In the next section, the effect of impulsive harvesting of pest on the system (2.4) is studied.

5. **Analysis of model with impulsive effect**

When pests eradicate completely, then mature natural enemies are also extinct and immature natural enemies survive due to impulsive releasing. Therefore, the system(2.4) converts into following form:

$$
\begin{cases}
\left.\begin{aligned}
\frac{dx(t)}{dt} &= x(t)\left(a_1 - b_1 x(t)\right),\\
\frac{dz_1(t)}{dt} &= -d_2 z_1(t),
\end{aligned}\right\} & t \neq nT,\\
\left.\begin{aligned}
\Delta x(t) &= 0,\\
\Delta z_1(t) &= p_r,
\end{aligned}\right\} & t = nT,\ n = 1, 2, \ldots.
\end{cases}
\tag{5.1}
$$

The first equation of model (5.1) represents logistic growth, therefore, $x(t) \to a_1/b_1$ as $t \to \infty$ and it is globally asymptotically stable.

Integrating and solving the second equation of the model (5.1) between pulses, we obtain that

$$
z_1(t) = z_1(nT)e^{-d_2(t-nT)}, \quad t \in (nT, (n+1)T], \quad n \in Z_+,
\tag{5.2}
$$

where $z_1(nT)$ is the initial value at nT. Using the fourth equation of (5.1), we deduce stroboscopic map such that

$$
z_1((n+1)T) = z_1(nT) + p_r \triangleq f(z_1).
$$

It is easy to know that the above equation has a unique positive equilibrium $z_1^* = \frac{p_r}{1-e^{-d_2T}}$, which satisfies $z_1 < f(z_1) < z_1^*$ if $0 < z_1 < z_1^*$ and $z_1 < f(z_1) > z_1^*$ if $z_1 > z_1^*$. From Cull [9], it is derive that z_1^* is globally asymptotically stable. Thus from (5.2), we obtain the following periodic solution

$$\widetilde{z_1}(t) = z_1^* e^{-d_2(t-nT)}, \quad t \in (nT, (n+1)T], \quad n \in Z_+,$$

which is globally asymptotically stable. Hence we get the result:

Lemma 5.1. *Consider the system*

$$\begin{cases} \dfrac{dz_1(t)}{dt} = -d_2 z_1(t), & t \neq nT, \\ \Delta z_1(t) = p_r, & t = nT, \ n = 1, 2, \ldots. \end{cases} \tag{5.3}$$

The system (5.3) *has a periodic solution*

$$\widetilde{z_1}(t) = z_1^* e^{-d_2(t-nT)}, \quad t \in (nT, (n+1)T], \quad n \in Z_+,$$

which is globally asymptotically stable, where $z_1^* = \frac{p_r}{1-e^{-d_2T}}$.

5.1. **Extinction of pest population.** Using the Lemma 5.1, it is obtaining that the system (5.3) has a pest and mature natural enemies extinction positive periodic solution $(a_1/b_1, 0, \widetilde{z_1}(t), 0)$, which is globally asymptotically stable. In the next, we will discuss the stability of the pest and mature natural enemies extinction periodic solution $(a_1/b_1, 0, \widetilde{z_1}(t), 0)$ of the system (2.4).

Theorem 5.2. *The pest and mature natural enemies extinction periodic solution* $(a_1/b_1, 0, \widetilde{z_1}(t), 0)$ *of the system* (2.4) *is globally asymptotically stable provided that* $\alpha_1 b_1 > a_1 d_1$.

Proof. Let $(x(t), y(t), z_1(t), z_2(t))$ be any solution of (2.4). From the first equation of the system (2.4), we have

$$\frac{dx(t)}{dt} \leq x(t)\left(a_1 - b_1 x(t)\right),$$

Consider the comparison system

$$\frac{du(t)}{dt} = u(t)\left(a_1 - b_1 u(t)\right), \tag{5.4}$$

Since (5.4) is a logistic equation, therefore using comparison theorem for differential equation it is obtain that $x(t) \leq u(t) \to a_1/b_1$ as $t \to \infty$. Thus there exists a positive constant $\epsilon_1 > 0$ (small enough) such that

$$x(t) < \frac{a_1}{b_1} + \epsilon_1. \tag{5.5}$$

From second equation of (2.4), we have

$$\frac{dy(t)}{dt} \leq y(t)\left(\alpha_1\left(\frac{a_1}{b_1} + \epsilon_1\right) - d_1\right). \tag{5.6}$$

Since $d_1/\alpha_1 > a_1/b_1$ for existence of planner equilibrium of the model (2.1) and $\epsilon_1 \to 0$ as $t \to \infty$, therefore, from (5.6) it is obtain that $y(t) \to 0$ as $t \to \infty$. Without loss of generality we may assume that there exists a positive constant $\epsilon_2 > 0$ (small enough) such that $y(t) < \epsilon_2$.

Again from the first equation of (2.4), we have

$$\frac{dx(t)}{dt} > x(t)\left((a_1 - c_1\epsilon_2) - b_1 x(t)\right). \tag{5.7}$$

Similarly it is clear that there exists $\epsilon_1 > 0$ such that

$$x(t) > \frac{a_1 - c_1\epsilon_2}{b_1} - \epsilon_1. \tag{5.8}$$

Since here it is assumed that ϵ_1 and ϵ_2 are small enough, therefore $\epsilon_1,\ \epsilon_2 \to 0$ as $t \to \infty$ and hence from (5.5) and (5.8) it is concluded that $x(t) \to a_1/b_1$ as $t \to \infty$.

Now from fourth equation of the system (2.4), we have

$$\frac{dz_2(t)}{dt} \le \alpha_2\epsilon_2 e^{-d_2\tau} z_2(t-\tau) - d_3 z_2(t). \tag{5.9}$$

Using Lemma 3.1, it is clear that $z_2(t) \to 0$ as $t \to \infty$ provided that $\alpha_2\epsilon_2 e^{-d_2\tau} < d_3$. Therefore, there exists a positive constant $\epsilon_3 > 0$ (small enough) such that $z_2(t) < \epsilon_3$.

From system (2.4), we may get the following subsystem:

$$\begin{cases} \dfrac{dz_1(t)}{dt} \le \alpha_2\epsilon_2\epsilon_3 - d_2 z_1(t), \quad t \ne nT, \\ \Delta z_1(t) = p_r, \quad t = nT, \quad n = 0, 1, 2, \ldots. \end{cases} \tag{5.10}$$

Consider the following comparison system:

$$\begin{cases} \dfrac{dv(t)}{dt} = \alpha_2\epsilon_2\epsilon_3 - d_2 v(t),\ t \ne nT, \\ v(t^+) = v(t) + p_r,\ t = nT. \end{cases} \tag{5.11}$$

Using the Lemma 5.1, we obtain that the system (5.11) has a periodic solution

$$\widetilde{v}(t) = \frac{\alpha_2\epsilon_2\epsilon_3}{d_2} + \frac{p_r e^{-d_2(t-nT)}}{1 - e^{-d_2 T}}, \quad t \in (nT, (n+1)T], \quad n \in Z_+,$$

which is globally asymptotically stable. In view of comparison theorem of impulsive equation [3], we have $z_1(t) \le v(t)$ and $v(t) \to \widetilde{z}_1(t)$ as $t \to \infty$ since ϵ_2 and $\epsilon_3 \to 0$ as $t \to \infty$. Then there exists a positive constant ϵ_4 (small enough) such that

$$z_1(t) \le v(t) < \widetilde{z}_1(t) + \epsilon_4. \tag{5.12}$$

Similarly, we can get the following subsystem of system (2.4):

$$\begin{cases} \dfrac{dz_1(t)}{dt} \ge -\alpha_2\epsilon_2\epsilon_3 e^{-d_2 T} - d_2 z_1(t), \quad t \ne nT, \\ \Delta z_1(t) = p_r, \quad t = nT, \quad n = 0, 1, 2, \ldots. \end{cases} \tag{5.13}$$

Consider the following comparison system of (5.13):

$$\begin{cases} \dfrac{dw(t)}{dt} = -\alpha_2\epsilon_2\epsilon_3 e^{-d_2T} - d_2w(t), \; t \neq nT, \\ w(t^+) = w(t) + p_r, \; t = nT. \end{cases} \tag{5.14}$$

Again using the Lemma 5.1 and comparison theorem of impulsive equation, in the similar manner we obtain that the system (5.14) has a periodic solution

$$\widetilde{w}(t) = \frac{-\alpha_2\epsilon_2\epsilon_3 e^{d_2T}}{d_2} + \frac{p_r e^{-d_2(t-nT)}}{1-e^{-d_2T}}, \quad t \in (nT,(n+1)T], \quad n \in Z_+,$$

which is also globally asymptotically stable and $z_1(t) \geq v(t)$ and $v(t) \to \widetilde{z_1}(t)$ as $t \to \infty$. Then there exists a positive constant ϵ_5 (small enough) such that

$$z_1(t) \geq v(t) > \widetilde{z_1}(t) - \epsilon_5. \tag{5.15}$$

Thus from (5.12) and (5.15), it is concluded that $x(t) \to a_1/b_1$, $y(t) \to 0$, $z_1(t) \to \widetilde{z_1}(t)$ and $z_2(t) \to 0$ as $t \to \infty$. Hence the proof. □

6. **Numerical simulation**

To verify the previously established results, consider a three species crop-pest-natural enemy stage structured food chain model with the following parameter values:

$$\begin{aligned} \frac{dx(t)}{dt} &= x(t)\left(a_1 - x(t) - y(t)\right), \\ \frac{dy(t)}{dt} &= y(t)\left(\alpha_1 x(t) - d_1 - 0.6z_2(t)\right), \\ \frac{dz_2(t)}{dt} &= 1.3e^{-0.4\tau}y(t-\tau)z_2(t-\tau) - 0.3z_2(t). \end{aligned} \tag{6.1}$$

If we choose $a_1 = 1$, $\alpha_1 = 0.1$ and $d_1 = 0.5$, then we obtain that for the trivial equilibrium $E_0(0,0,0)$, the characteristic equation has three eigenvalues, $\lambda_1 = 1$, $\lambda_2 = -0.5$ and $\lambda_3 = 0.3$. This verifies that trivial equilibrium of the system (6.1) is a unstable saddle point. Similarly, the characteristic equation for the boundary equilibrium $E_1(1,0,0)$ has $\lambda_1 = -1, \lambda_2 = -0.3$ and $\lambda_3 = -0.2$ eigenvalues and hence the boundary equilibrium is locally asymptotically stable (see Figure 1(a)).

Now let $a_1 = 2$, $d_1 = 0.05$ and $\alpha_1 = 1.2$. Then the condition (**H1**) for the positivity of equilibrium E_2 is satisfy and the characteristic equation for the equilibrium $E_2(0.0416667, 1.95833, 0)$ of the system (6.1) has three eigenvalues, namely, $\lambda_1 = -0.0208333 - 0.312222i$, $\lambda_2 = -0.0208333 + 0.312222i$ and $\lambda_3 = -0.3 + 2.54583e^{-0.4\tau}$ eigenvalues. Here, λ_3 is negative only when $\tau > \bar{\tau} = 5.34608$, in this case, the planner equilibrium E_2 is locally asymptotically stable for $\tau > 5.34608$ (see Figure 1(b)). We, mainly focus on the dynamics of the interior equilibrium. To study the local behavior of interior equilibrium $E_3(x^*(\tau), y^*(\tau), z_2^*(\tau))$ of the system (6.1), we take the same

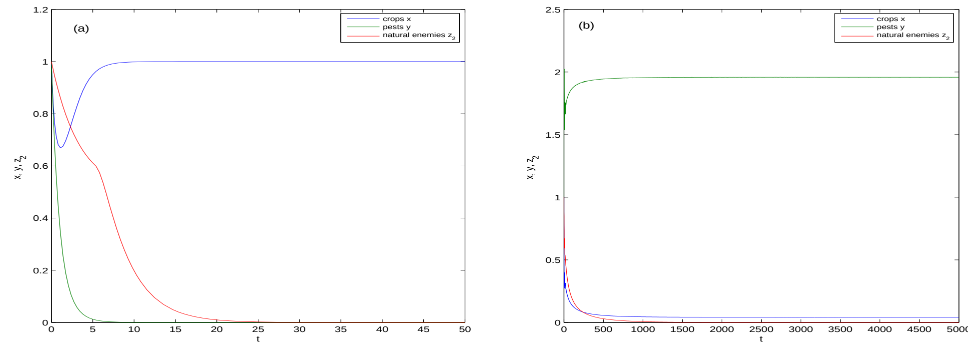

FIGURE 1. Nonnegative equilibrium points are stable (a) boundary equilibrium E_1 is stable for $a_1 = 1,\ d_1 = 0.5,\ \alpha_1 = 0.1$; (b) planner equilibrium E_2 is stable for $a_1 = 2,\ d_1 = 0.05,\ \alpha_1 = 1.2$ and $\tau = 5.4 > \tau_{cr} = 5.34608$.

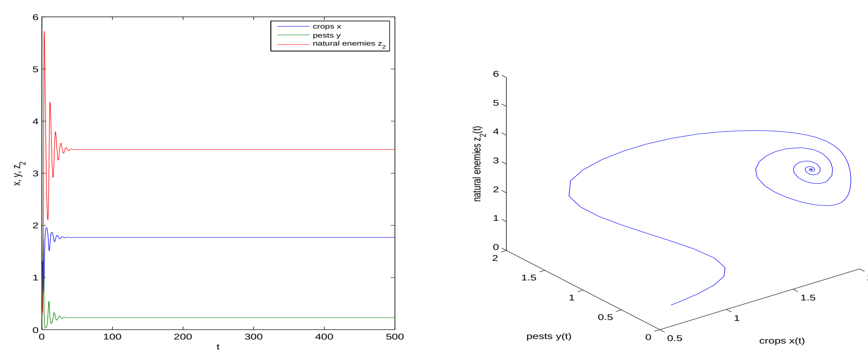

FIGURE 2. The interior equilibrium E_3 is locally asymptotically stable for $\tau = 0$.

parameters values for $a_1,\ d_1$ and α_1. We obtain that the equilibrium is positive if $\tau < \tau_1 = 5.34608$ (see Figure 3).

To apply the Descartes' rule of signs, we plot the coefficients $p(\tau)$, $q(\tau)$ and $r(\tau)$ of the function $h(z, \tau)$ with the maturation delay τ (see Figure 4). We can easily check that in the intervals I_{12}, I_{23}, the function $h(z, \tau) = 0$ has 0 and 1 positive roots respectively. Thus exactly one zero of $h(z, \tau)$ is feasible in only the interval $I_2 = [0, 2.59955)$ (see Figure 4(d)). Further, all the conditions of the lemma 4.1 are satisfied in the interval I_2. Now taking S_0 on one axis and τ on another axis, we obtain that the function $S_0(\tau) = \tau - \theta(\tau)/\omega(\tau)$ has two zeros $\tau^* \approx 0.743$ and $\tau^{**} \approx 1.568$ in the interval I_2, i.e., there are two critical values of the maturation delay of natural enemies at which the stability switching occurs (see Figure 5).

Using the same set of parametric values, it is obtained that the interior equilibrium $E_3(x^*, y^*, z_2^*)$ is locally asymptotically stable if $\tau < \tau^* = 0.743$ and a Hopf bifurcation occurs if $\tau \geq 0.743$, see Figures 6(a) and 6(a). The equilibrium again

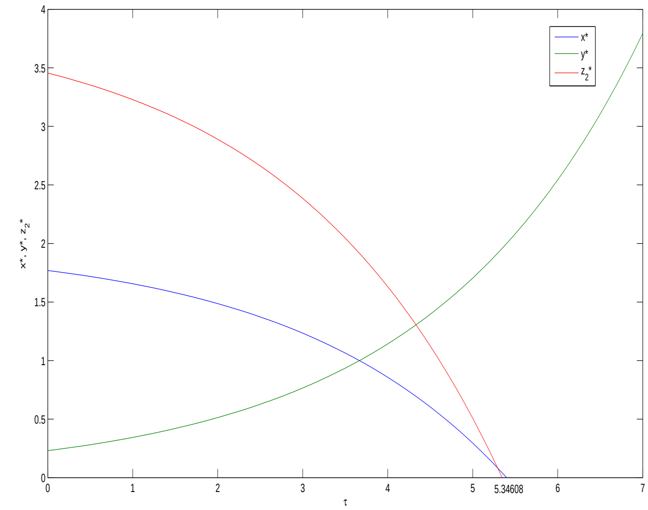

FIGURE 3. The equilibrium $E_3(x^*, y^*, z_2^*)$ is positive for $\tau < \bar{\tau} = 5.34608$.

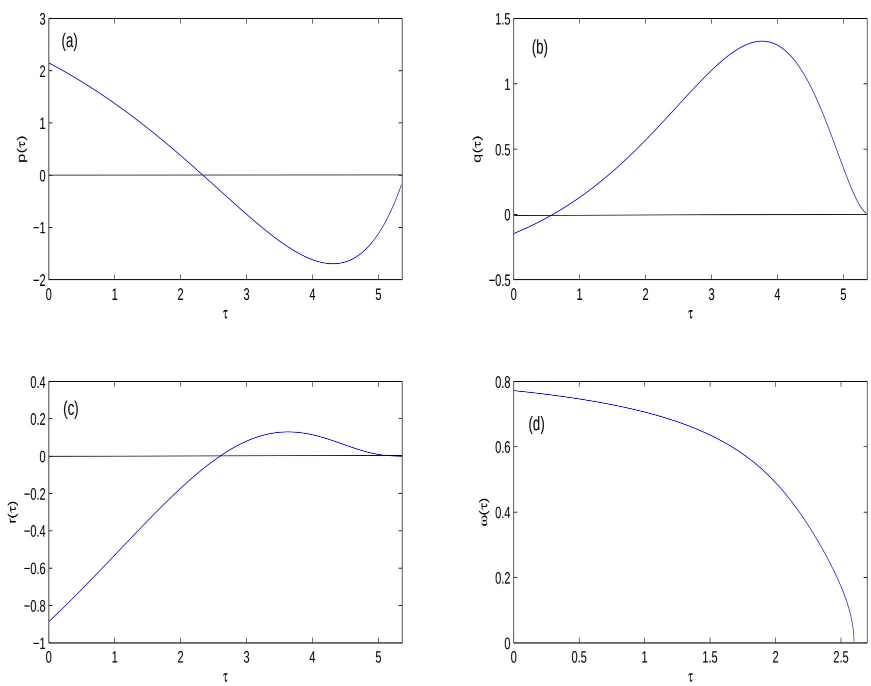

FIGURE 4. (a) $p(\tau)$; (b) $q(\tau)$ and (c) $r(\tau)$ in the interval $[0, 5.34608)$; (d) curve of $\omega(\tau)$ on $\tau \in [0, 2.59955)$.

becomes locally asymptotically when $\tau > \tau^{**} = 1.568$, see Figure 7. Thus the interior equilibrium of the system (6.1) is locally asymptotically stable for $\tau \in [0, 0.743) \cup (1.568, 5.34608]$ and is unstable for $\tau \in (\tau^*, \tau^{**})$. Hence the stability

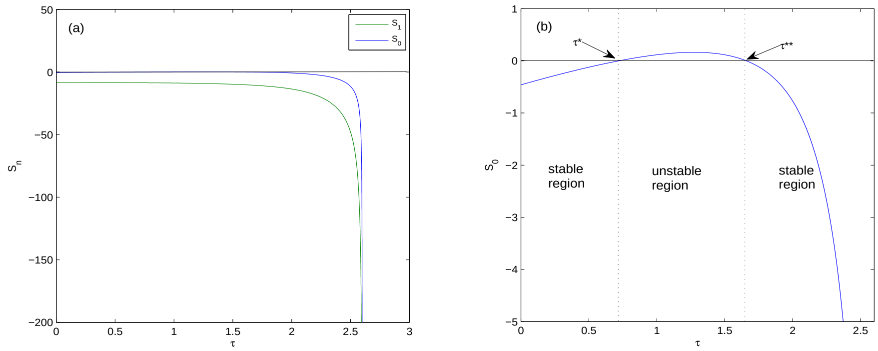

FIGURE 5. (a) Graphs of functions S_0 and S_1 for $\tau \in [0, 2.59955)$; (b) curve of S_0 for the same value of τ.

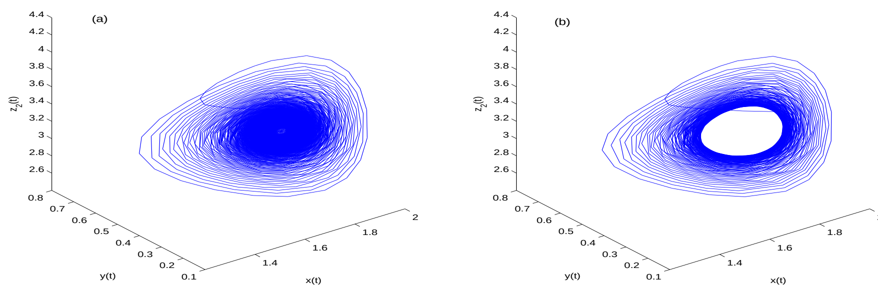

FIGURE 6. The dynamics of the interior equilibrium for first critical value of maturation delay τ: (a) E_3 is locally asymptotically stable for $\tau = 0.742 < \tau^* = 0.743$; (b) occurrence of Hopf bifurcation at E_3 for $\tau = 0.75 > \tau^* = 0.743$.

switches from *stability-instability-stability* occurs. This is the verification of the Theorem 4.3.

Furthermore, for the system (6.1), it is clear that $Re(c_1(0))|_{\tau=\tau^*} = -3.9481 < 0$ and $Re(c_1(0))|_{\tau=\tau^{**}} = 9.3706 > 0$, according to the formula given in section 4.2. Therefore, Hopf bifurcation for the interior equilibrium at τ^* (resp. τ^{**}) is forward (resp. backward) and the bifurcating periodic solution on the center manifold are orbitally asymptotically stable (see Figure 8). Finally, we observe with the following set of parameters: $a_1 = 7,\ b_1 = 1,\ c_1 = 1,\ c_2 = 0.5,\ d_1 = 0.05,\ d_2 = 0.6,\ d_3 = 1.2,\ \alpha_1 = 1.5,\ \alpha_2 = 2$, then system (2.1) has a complex dynamics of multiple bifurcation (i.e., chaos) for the maturation delay $\tau = 1.5$ (see Figure 9).

7. Conclusion

In this paper, we have proposed two models with and without impulsive effect for three species crop-pest-natural enemy food chain system with stage structure

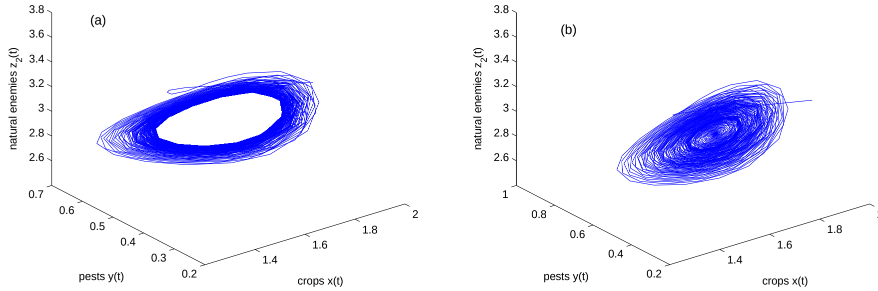

FIGURE 7. The dynamics of the interior equilibrium for second critical value of maturation delay τ: (a) occurrence of Hopf bifurcation at E_3 for $\tau = 1.56 < \tau^{**} = 1.568$; (b) E_3 is locally asymptotically stable for $\tau = 1.57 > \tau^{**} = 1.568$.

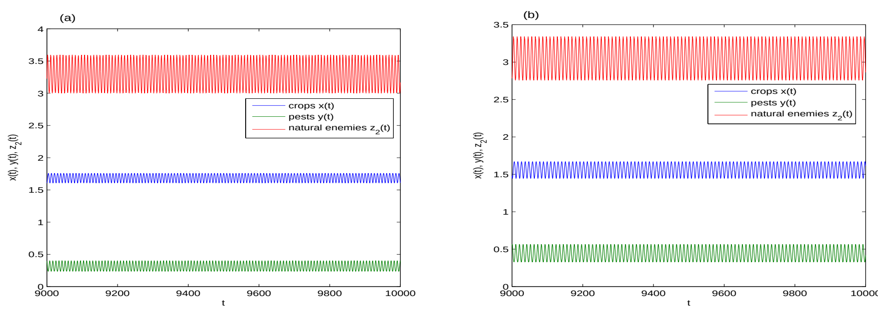

FIGURE 8. (a) The bifurcated periodic solution for the interior equilibrium is locally asymptotically stable at $\tau^* = 0.743$; (b) The bifurcated periodic solution for the interior equilibrium is locally asymptotically stable at $\tau^{**} = 1.568$.

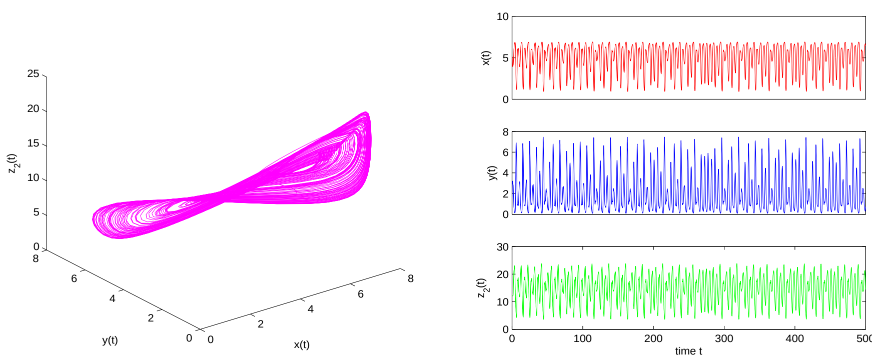

FIGURE 9. The chaotic behavior of the system at the interior equilibrium for parameter set: $a_1 = 7,\ b_1 = 1,\ c_1 = 1,\ c_2 = 0.5,\ d_1 = 0.05,\ d_2 = 0.6,\ d_3 = 1.2,\ \alpha_1 = 1.5,\ \alpha_2 = 2$ and $\tau = 1.5$.

and maturation delay for the natural enemy. In the first part of the analysis, a model without impulsive effect has studied and discussed the local stability of four nonnegative equilibria of the system (2.1). Here it is found that the trivial equilibrium $E_0(0,0,0)$ is always unstable; the boundary equilibrium $E_1(a_1/b_1,0,0)$ is locally asymptotically stable if $a_1\alpha_1 < b_1 d_1$. Further, the planner equilibrium $E_2(\bar{x},\bar{y},0)$ is locally asymptotically stable if $a_1\alpha_1 > b_1 d_1$ and $\tau > (1/d_2)\log(\alpha_2\bar{y}/d_3)$, otherwise, it is unstable. The interior equilibrium is locally asymptotically stable if $\tau < \tau^*$, a Hopf bifurcation occurs in the interval $\tau^* < \tau < \tau^{**}$ and if the maturation delay crossed the second critical τ^{**}, then the interior equilibrium becomes again stable. Thus the maturation delay plays an important role in switching of stability from *stability-instability-stability.*

In the second part of analysis, a model with impulsive effect is studied and obtained sufficient condition for the global stability of the pest and mature natural enemy free periodic solution. Furthermore, using a numerical simulation, we obtained that the existence of bifurcation and also observed the chaotic behavior of the system for a particular range of the maturation delay. In particular, it is observed that the larger maturation delay may lead to extinction of the natural enemy, i.e., natural enemies may extinct due to its stage structure. This shows that the maturation delay of natural enemy is the controlling parameter for the pest population. Thus stage structure and impulse have a great importance in the dynamics of the three species crop-pest-natural enemy food chain.

Appendix A.

Let, $x_1 = x - x^*$, $x_2 = y - y^*$, $x_3 = z_2 - z_2^*$, $\bar{x}_i(t) = x_i(\tau t)$, $\tau = \tau^* + \mu$ and dropping the bars for simplification of notations, system (4.1) is transformed into an FDE in $C = C([-1,0],R^3)$ as

$$\dot{x}(t) = L_\mu(x_t) + F(\mu, x_t), \tag{A.1}$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^T \in R^3$ and $L_\mu : C \to R$, $F : C \times R \to R$ are given, respectively, by

$$\begin{aligned} L_\mu(\phi) &= (\tau^*+\mu)\begin{pmatrix} -b_1x^* & -c_1x^* & 0 \\ \alpha_1 y^* & 0 & -c_2y^* \\ 0 & 0 & -d_3 \end{pmatrix}\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{pmatrix} \\ &\quad +(\tau^*+\mu)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \alpha_2 z_2^* e^{-d_2(\tau^*+\mu)} & \alpha_2 y^* e^{-d_2(\tau^*+\mu)} \end{pmatrix}\begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \end{pmatrix}, \end{aligned}$$

and

$$F(\mu,\phi) = (\tau^*+\mu)\begin{pmatrix} -b_1\phi_1^2(0) - c_1\phi_1(0)\phi_2(0) \\ \alpha_1\phi_1(0)\phi_2(0) - c_2\phi_2(0)\phi_3(0) \\ \alpha_2 e^{-d_2(\tau^*+\mu)}\phi_2(-1)\phi_3(-1) \end{pmatrix}, \tag{A.2}$$

where $\phi = (\phi_1, \phi_2, \phi_3)^T \in C$. By the Riesz representation theorem, there exists a function $\eta(\theta,\mu)$ of bounded variation for $\theta \in [-1, 0]$ such that

$$L_\mu\phi = \int_{-1}^{0} \eta(\theta,\mu)\phi(\theta) \quad \text{for } \theta \in C. \tag{A.3}$$

In fact, we can choose

$$\begin{aligned} \eta(\theta,\mu) &= (\tau^*+\mu)\begin{pmatrix} -b_1x^* & -c_1x^* & 0 \\ \alpha_1y^* & 0 & -c_2y^* \\ 0 & 0 & -d_3 \end{pmatrix}\delta(\theta) \\ &\quad -(\tau^*+\mu)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \alpha_2z_2^*e^{-d_2(\tau^*+\mu)} & \alpha_2y^*e^{-d_2(\tau^*+\mu)} \end{pmatrix}\delta(\theta+1), \end{aligned} \tag{A.4}$$

where δ is Dirac delta function. For $\phi \in C^1([-1, 0], R^3)$, define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \int_{-1}^{0} d\eta(s,\mu)\phi(s), & \theta = 0, \end{cases} \tag{A.5}$$

and

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-1, 0), \\ F(\mu,\phi), & \theta = 0. \end{cases} \tag{A.6}$$

Then the system (A.1) is equivalent to

$$\dot{x}_t = A(\mu)x_t + R(\mu)x_t, \tag{A.7}$$

where $x_t(\theta) = x(t+\theta)$ for $\theta \in [-1, 0]$. For $\psi \in C^1([0, 1], (R^3)^*)$, define

$$A^*\psi(s) = \begin{cases} \frac{-d\psi(s)}{ds}, & s \in (0, 1], \\ \int_{-1}^{0} d\eta^T(s, 0)\psi(s), & s = 0, \end{cases} \tag{A.8}$$

and a bilinear inner product

$$\langle\psi(s),\phi(\theta)\rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi, \tag{A.9}$$

where $\eta(\theta) = \eta(\theta, 0)$. Then $A(0)$ and A^* are adjoint operators. By the discussion in section 4.1, we know that $\pm i\omega^*\tau^*$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^*. We need to compute the eigenvector of $A(0)$ and A^* corresponding to $i\omega^*\tau^*$ and $-i\omega^*\tau^*$, respectively.

Suppose that $q(\theta) = (1, \alpha, \beta)^T e^{i\theta\omega^*\tau^*}$ is the eigenvector of $A(0)$ corresponding to $i\omega^*\tau^*$, then $A(0)q(\theta) = i\omega^*\tau^* q(\theta)$. It follows from the definition of $A(0)$ and $\eta(\theta, \mu)$ that

$$\tau^* \begin{pmatrix} i\omega^* + b_1 x^* & c_1 x^* & 0 \\ -\alpha_1 y^* & i\omega^* & c_2 y^* \\ 0 & -\alpha_2 z_2^* e^{-d_2\tau^*} e{-i\omega^*\tau^*} & i\omega^* - \alpha_2 y^* e^{-d_2\tau^*} e{-i\omega^*\tau^*} \end{pmatrix} q(0) = 0.$$

Then, we can easily obtain

$$q(0) = (1, \alpha, \beta)^T,$$

where $\alpha = -\frac{i\omega^* + b_1 x^*}{c_1 x^*}$ and $\beta = \frac{\alpha\alpha_2 z_2^* e^{-d_2\tau^*} e^{-i\omega^*\tau^*}}{i\omega^* + d_3 - \alpha_2 y^* e^{-d_2\tau^*} e^{-i\omega^*\tau^*}}$.

Similarly, let $q^*(\theta) = D(1, \alpha^*, \beta^*)e^{-i\theta\omega^*\tau^*}$ be the eigenvector of A^* corresponding to $-i\omega^*\tau^*$, then similarly we can obtain

$$\alpha^* = \frac{b_1 x^* - i\omega^*}{\alpha_1 y^*}, \quad \beta^* = \frac{c_2 y^*(-i\omega^* + b_1 x^*)}{\alpha_1 y^*(i\omega^* - d_3 + \alpha_2 y^* e^{-d_2\tau^*} e^{i\omega^*\tau^*})}.$$

By (A.9) we get

$$\begin{aligned} \langle q^*(s), q(\theta)\rangle &= \bar{D}(1, \bar{\alpha}^*, \bar{\beta}^*)(1, \alpha, \beta)^T \\ &\quad - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{D}(1, \bar{\alpha}^*, \bar{\beta}^*)e^{-\omega^*\tau^*(\xi-\theta)} d\eta(\theta)(1, \alpha, \beta)^T e^{i\omega^*\tau^*\xi} d\xi \\ &= \bar{D}\left[1 + \alpha\bar{\alpha}^* + \beta\bar{\beta}^* - (1, \bar{\alpha}^*, \bar{\beta}^*)\int_{-1}^{0} \phi(\theta) d\eta(\theta)(1, \alpha, \beta)^T\right] \\ &= \bar{D}\left[1 + \alpha\bar{\alpha}^* + \beta\bar{\beta}^* + \tau^*\left(\alpha z_2^* + \beta y^*\right)\alpha_2\bar{\beta}^* e^{-d_2\tau^*} e^{-i\omega^*\tau^*}\right]. \end{aligned}$$

Then we choose

$$\bar{D} = \frac{1}{1 + \alpha\bar{\alpha}^* + \beta\bar{\beta}^* + \tau^*\left(\alpha z_2^* + \beta y^*\right)\alpha_2\bar{\beta}^* e^{-d_2\tau^*} e^{-i\omega^*\tau^*}},$$

such that $\langle q^*(s), q(\theta)\rangle = 1$ and $\langle q^*(s), \bar{q}(\theta)\rangle = 0$.

In the following, we use the ideas in Hassard et al. [14] to compute the coordinates describing center manifold C_0 at $\mu = 0$. Define

$$z(t) = \langle q^*, x_t\rangle, \ W(t, \theta) = x_t(0) - 2Re[z(t)q(\theta)], \tag{A.10}$$

On the center manifold C_0, we have

$$W(t, \theta) = W(z(t), \bar{z}(t), \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots, \tag{A.11}$$

where z and $\bar{z}$ are local coordinates for C_0 in C in the direction of q^* and $\bar{q}^*$. Note that W is real if x_t is real. We deal only with the real solution. For solution $x_t \in C_0$ of (A.7), since $\mu = 0$, we have

$$\begin{aligned} \dot{z}(t) &= i\omega^*\tau^* z + \bar{q}^*(0)F(0, W(z, \bar{z}, 0) + 2Re\{zq(0)\}) \\ &\stackrel{def}{=} i\omega^*\tau^* + \bar{q}^*(0)F_0(z, \bar{z}) = i\omega^*\tau^* + g(z, \bar{z}), \end{aligned}$$

where

$$g(z,\bar{z}) = \bar{q}^*(0)F_0(z,\bar{z}) = g_{20}(\theta)\frac{z^2}{2} + g_{11}(\theta)z\bar{z} + g_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots . \tag{A.12}$$

From (A.10) and (A.11), we have

$$x_t(\theta) = (x_{1t}(\theta), x_{2t}(\theta), x_{3t}(\theta)) = W(t,\theta) + zq(\theta) + \overline{zq(\theta)},$$

and

$$q(\theta) = (1,\alpha,\beta)^T e^{i\theta\omega^*\tau^*}.$$

Thus, we can easily obtain that

$$\begin{aligned}
x_{1t}(0) &= W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + z + \bar{z} + O\left(|(z,\bar{z})|^3\right),\\
x_{2t}(0) &= W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \alpha z + \overline{\alpha z} + O\left(|(z,\bar{z})|^3\right),\\
x_{3t}(0) &= W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \beta z + \overline{\beta z} + O\left(|(z,\bar{z})|^3\right),
\end{aligned}$$

$$\begin{aligned}
x_{2t}(-1) = W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} &+ W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + \alpha z e^{-i\omega^*\tau^*}\\
&+ \overline{\alpha z}e^{i\omega^*\tau^*} + O\left(|(z,\bar{z})|^3\right),\\
x_{3t}(-1) = W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} &+ W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + \beta z e^{-i\omega^*\tau^*}\\
&+ \overline{\beta z}e^{i\omega^*\tau^*} + O\left(|(z,\bar{z})|^3\right).
\end{aligned}$$

From the definition of $F(\mu, x_t)$, we have

$$\begin{aligned}
g(z,\bar{z}) &= \tau^*\bar{D}(1,\bar{\alpha}^*,\bar{\beta}^*)\begin{pmatrix} -b_1x_{1t}^2(0) - c_1x_{1t}(0)x_{2t}(0)\\ \alpha_1x_{1t}(0)x_{2t}(0) - c_2x_2t(0)x_{3t}(0)\\ \alpha_2e^{-d_2\tau^*}x_{2t}(-1)x_{3t}(-1)\end{pmatrix}\\
&= \tau^*\bar{D}\left\{z^2\left[-b_1 - \alpha(c_1-\alpha_1\bar{\alpha}^*) - c_2\alpha\beta\bar{\alpha}^* + \alpha\alpha_2\beta\bar{\beta}^*e^{-d_2\tau^*}e^{-2i\omega^*\tau^*}\right]\right.\\
&\quad +2z\bar{z}\left[-b_1 - (c_1-\alpha_1\bar{\alpha}^*)Re\{\alpha\} - c_2Re\{\alpha\bar{\beta}\} + \alpha_2\bar{\beta}^*e^{-d_2\tau^*}Re\{\alpha\bar{\beta}\}\right]\\
&\quad +\bar{z}^2\left[-b_1 - (c_1-\alpha_1\bar{\alpha}^*)\bar{\alpha} - c_2\bar{\alpha}^*\bar{\alpha}\bar{\beta} + \alpha_2\bar{\beta}^*\bar{\alpha}\bar{\beta}e^{-d_2\tau^*}e^{2i\omega^*\tau^*}\right]\\
&\quad +\frac{1}{2}z^2\bar{z}\left[-2b_1W_{20}^{(1)}(0) - 4b_1W_{11}^{(1)}(0)\right.\\
&\quad -(c_1-\alpha_1\bar{\alpha}^*)\left(\bar{\alpha}W_{20}^{(1)}(0) + 2\alpha W_{11}^{(1)}(0) + 2W_{11}^{(2)}(0) + W_{20}^{(2)}(0)\right)\\
&\quad -c_2\bar{\alpha}^*\left(\bar{\beta}W_{20}^{(2)}(0) + 2\beta W_{11}^{(2)}(0) + 2\alpha W_{11}^{(3)}(0) + \bar{\alpha}W_{20}^{(3)}(0)\right)\\
&\quad +\alpha_2\bar{\beta}^*e^{-d_2\tau^*}\left(\bar{\beta}e^{i\omega^*\tau^*}W_{20}^{(2)}(-1) + 2\beta e^{-i\omega^*\tau^*}W_{11}^{(2)}(-1)\right.\\
&\quad \left.\left.\left. + 2\alpha e^{-i\omega^*\tau^*}W_{11}^{(3)}(-1) + \bar{\alpha}e^{i\omega^*\tau^*}W_{20}^{(3)}(-1)\right)\right]\right\}.
\end{aligned}$$

Comparing the coefficients with (A.12), we obtain

$$g_{20} = 2\tau^*\bar{D}[-b_1 - (c_1-\alpha_1\bar{\alpha}^*)\alpha - c_2\alpha\beta\bar{\alpha}^* + \alpha\alpha_2\beta\bar{\beta}^*e^{-d_2\tau^*}e^{-2i\omega^*\tau^*}],$$

$$g_{11} = 2\tau^*\bar{D}[-b_1 - (c_1 - \alpha_1\bar{\alpha}^*)Re\{\alpha\} - c_2 Re\{\alpha\bar{\beta}\} + \alpha_2\bar{\beta}^* e^{-d_2\tau^*} Re\{\alpha\bar{\beta}\}],$$

$$g_{02} = 2\tau^*\bar{D}[-b_1 - (c_1 - \alpha_1\bar{\alpha}^*)\bar{\alpha} - c_2\bar{\alpha}^*\bar{\alpha}\bar{\beta} + \alpha_2\bar{\beta}^*\bar{\alpha}\bar{\beta}e^{-d_2\tau^*}e^{2i\omega^*\tau^*}],$$

$$\begin{aligned} g_{21} \;=\; & \tau^*\bar{D}\Big[-b_1\left(2W_{20}^{(1)}(0) + 4W_{11}^{(1)}(0)\right) \\ & -(c_1 - \alpha_1\bar{\alpha}^*)\left(\bar{\alpha}W_{20}^{(1)}(0) + 2\alpha W_{11}^{(1)}(0) + 2W_{11}^{(2)}(0) + W_{20}^{(2)}(0)\right) \\ & -c_2\bar{\alpha}^*\left(\bar{\beta}W_{20}^{(2)}(0) + 2\beta W_{11}^{(2)}(0) + 2\alpha W_{11}^{(3)}(0) + \bar{\alpha}W_{20}^{(3)}(0)\right) \\ & +\alpha_2\bar{\beta}^* e^{-d_2\tau^*}\Big(\bar{\beta}e^{i\omega^*\tau^*}W_{20}^{(2)}(-1) + 2\beta e^{-i\omega^*\tau^*}W_{11}^{(2)}(-1) \\ & + 2\alpha e^{-i\omega^*\tau^*}W_{11}^{(3)}(-1) + \bar{\alpha}e^{i\omega^*\tau^*}W_{20}^{(3)}(-1)\Big)\Big]. \end{aligned}$$

In order to determine g_{21}, we need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$. From (A.7) and (A.10), we have

$$\dot{W} \;=\; \dot{x}_t - zq - \dot{\overline{zq}} = \begin{cases} AW - 2Re[\bar{q}^*(0)F_0 q(\theta)], & \theta \in [-1, 0), \\ AW - 2Re[\bar{q}^*(0)F_0 q(\theta)] + F_0, & \theta = 0, \end{cases}$$

$$\text{(A.13)} \qquad \overset{def}{=} \; AW + H(z, \bar{z}, \theta),$$

where

$$\text{(A.14)} \qquad H(z, \bar{z}, \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots.$$

Note that on center manifold C_0 near the origin

$$\dot{W} = W_z\dot{z} + W_{\bar{z}}\dot{\bar{z}}.$$

Thus, we obtain

$$\text{(A.15)} \qquad (A - 2i\omega^*\tau^*)W_{20}(\theta) = -H_{20}(\theta), \;\; AW_{11}(\theta) = -H_{11}(\theta).$$

Comparing the coefficient with (A.14) gives that

$$\text{(A.16)} \qquad H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \;\; H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta).$$

From (A.15), (A.16) and the definition of A, we have

$$\dot{W}_{20}(\theta) = 2i\omega^*\tau^* W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta).$$

Noting $q(\theta) = q(0)e^{i\omega^*\tau^*\theta}$, hence

$$\text{(A.17)} \qquad W_{20}(\theta) = \frac{ig_{20}}{\omega^*\tau^*}q(0)e^{i\omega^*\tau^*\theta} + \frac{i\bar{g}_{02}}{3\omega^*\tau^*}\bar{q}(0)e^{-2i\omega^*\tau^*\theta} + E_1 e^{2i\omega^*\tau^*\theta},$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)}) \in R^3$ is a constant vector. Similarly, from (A.15) and (A.16), we obtain

$$\text{(A.18)} \qquad W_{11}(\theta) = -\frac{ig_{11}}{\omega^*\tau^*}q(0)e^{i\omega^*\tau^*\theta} + \frac{i\bar{g}_{11}}{\omega^*\tau^*}\bar{q}(0)e^{-i\omega^*\tau^*\theta} + E_2,$$

where $E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)}) \in R^3$ is also a constant vector.

In the following we shall find out E_1 and E_2. From the definition of A and (A.15), we can obtain

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega^*\tau^*W_{20}(\theta) - H_{20}(0), \tag{A.19}$$

and

$$\int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0), \tag{A.20}$$

where $\eta(\theta) = \eta(0,\theta)$. From (A.13) and (A.14), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\tau^* \begin{pmatrix} -b_1 - c_1\alpha \\ \alpha\alpha_1 - c_2\alpha\beta \\ \alpha_2\alpha\beta e^{-d_2\tau^*}e^{-2i\omega^*\tau^*} \end{pmatrix}, \tag{A.21}$$

and

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + 2\tau^* \begin{pmatrix} -b_1 - c_1Re\{\alpha\} \\ \alpha_1Re\{\alpha\} - c_2Re\{\alpha\bar{\beta}\} \\ \alpha_2e^{-d_2\tau^*}Re\{\alpha\bar{\beta} \end{pmatrix}. \tag{A.22}$$

Substituting (A.17) and (A.18) into (A.19) and noticing that

$$\left(i\omega^*\tau^*I - \int_{-1}^{0} e^{i\omega^*\tau^*\theta}d\eta(\theta)\right) q(0) = 0,$$

and

$$\left(-i\omega^*\tau^*I - \int_{-1}^{0} e^{-i\omega^*\tau^*\theta}d\eta(\theta)\right) \bar{q}(0) = 0,$$

we obtain

$$\left(2i\omega^*\tau^*I - \int_{-1}^{0} e^{2i\omega^*\tau^*\theta}d\eta(\theta)\right) E_1 = 2\tau^* \begin{pmatrix} -b_1 - c_1\alpha \\ \alpha\alpha_1 - c_2\alpha\beta \\ \alpha_2\alpha\beta e^{-d_2\tau^*}e^{-2i\omega^*\tau^*} \end{pmatrix},$$

which leads to

$$\begin{pmatrix} 2i\omega^* + b_1x^* & c_1x^* & 0 \\ -\alpha_1y^* & 2i\omega^* & c_2y^* \\ 0 & -\alpha_2z_2^*e^{-d_2\tau^*}e^{-2i\omega^*\tau^*} & \alpha_2y^*e^{-d_2\tau^*}e^{-2i\omega^*\tau^*} \end{pmatrix} E_1$$

$$= 2\begin{pmatrix} -b_1 - c_1\alpha \\ \alpha\alpha_1 - c_2\alpha\beta \\ \alpha_2\alpha\beta e^{-d_2\tau^*}e^{-2i\omega^*\tau^*} \end{pmatrix}.$$

It follows that

$$E_1 = 2\begin{pmatrix} 2i\omega^* + b_1x^* & c_1x^* & 0 \\ -\alpha_1 y^* & 2i\omega^* & c_2y^* \\ 0 & -\alpha_2 z_2^* e^{-d_2\tau^*}e^{-2i\omega^*\tau^*} & \beta_2 - \alpha_2 y^* e^{-d_2\tau^*}e^{-2i\omega^*\tau^*} \end{pmatrix}^{-1} \times \begin{pmatrix} -b_1 - c_1\alpha \\ \alpha\alpha_1 - c_2\alpha\beta \\ \alpha_2\alpha\beta e^{-d_2\tau^*}e^{-2i\omega^*\tau^*} \end{pmatrix}.$$

Similarly, substituting (A.18) and (A.22) into (A.20), we can get
(A.23)

$$E_2 = 2\begin{pmatrix} b_1x^* & c_1x^* & 0 \\ -\alpha_1 y^* & 0 & c_2y^* \\ 0 & -\alpha_2 z_2^* e^{-d_2\tau^*} & \beta_2 - \alpha_2 y^* e^{-d_2\tau^*} \end{pmatrix}^{-1} \times \begin{pmatrix} -b_1 - c_1 Re\{\alpha\} \\ \alpha_1 Re\{\alpha\} - c_2 Re\{\alpha\bar{\beta}\} \\ \alpha_2 e^{-d_2\tau^*} Re\{\alpha\bar{\beta}\} \end{pmatrix}.$$

Thus, we can determine W_{20} and W_{11} from (A.17) and (A.18).

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