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STRONGLY UNEQUAL PERMUTATIONS OF S_n AND SUDOKU

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ABSTRACT. In this paper we study the strongly unequal permutations of symmetric group S_n and its application to the counting of sudoku solutions. We present an estimate for the number of strongly unequal permutations of S_n and a recursive relation for the same. The number of different 4×4 sudoku solutions is presented using strongly unequal permutations of S_4.

Key Words Permutations, Strongly unequal Permutations, Sudoku problems, Sudoku solutions

Classification: 20B30, 03E, 05A

1. PRELIMINARY RESULTS

In this paper we study about strongly unequal permutations of S_n, which is defined as

Definition 1.1. Any two permutations σ_1, $\sigma_2 \in S_n$ are said to be strongly unequal if

$$\sigma_1(i) \neq \sigma_2(i) \quad \forall\ i \in \{1, 2, 3, \ldots, n\} \tag{1.1}$$

Let $(|U_n(\sigma)|)$ be the number of permutations of S_n which are strongly unequal with σ. One may ask two basic questions namely, (i) given a permutation σ of S_n, how many permutations are strongly unequal with σ? and (ii) whether the number of strongly unequal permutations is same for every permutation of S_n? These question leads to the study of partitioning of S_n and obtain

Theorem 1.2. *Every permutation of S_n has exactly $|U_n|$ strongly unequal permutations, where*

$$|U_n| = |U_n(\sigma)| = n! - (n-1)! - (n-1)\left(\sum_{k=1}^{n-2}(-1)^{k-1}\binom{n-2}{k}(n-(k+1))!\right) \tag{1.2}$$

This study of number of strongly unequal permutations of a given permutation of S_n leads to a recursive relation

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Theorem 1.3. *$|U_{n+1}| = n(|U_n| + |U_{n-1}|)$, for $n \geq 2$.*

In other words, the number of strongly unequal permutations of any permutation of S_{n+1} is n times the sum of the number of strongly unequal permutation(s) of any permutation of S_n and S_{n-1}. From the concept of strongly unequal permutation we prove the following lemmas

Lemma 1.4. *Any $n+1$ permutations of S_n are not strongly unequal with one another.*

Lemma 1.5. *There exist a collection of n strongly unequal permutations in S_n.*

One of the main objectives of this study is to use these as tools to study sudoku problems and solutions. Sudoku was first published anonymously by Garns (1979) for Dell Pencil Puzzles. The Sudoku puzzle received a large amount of attention in the United States and Europe in 2005 after a regular Sudoku puzzle began appearing in the London Times.

A Sudoku problem is a partially filled $m^2 \times m^2$ matrix which can be filled in exactly one way. The $m^2 \times m^2$ matrix is filled with integers from $\{1, 2, 3, \ldots, m^2\}$ such that every row, every column and every $m \times m$ submatrices as shown below have integers from $\{1, 2, 3, \ldots, m^2\}$ appearing exactly once. Such a filled matrix is called as Sudoku solution.

		3	
	2		
1			
			4

4	1	3	2
3	2	4	1
1	4	2	3
2	3	1	4

FIGURE 1. A 4×4 Sudoku problem and its unique solution (for $m = 2$)

			7					
1								
			4	3		2		
								6
			5		9			
						4	1	8
				8	1			
		2					5	
	4					3		

2	6	4	7	1	5	8	3	9
1	3	7	8	9	2	6	4	5
5	9	8	4	3	6	2	7	1
4	2	3	1	7	8	5	9	6
8	1	6	5	4	9	7	2	3
7	5	9	6	2	3	4	1	8
3	7	5	2	8	1	9	6	4
9	8	2	3	6	4	1	5	7
6	4	1	9	5	7	3	8	2

FIGURE 2. A 9×9 Sudoku problem and its unique solution (for $m = 3$)

For example in figure[1] we have listed a 4×4 Sudoku problem and its unique solution (for $m = 2$). Sometimes 4×4 Sudoku problem is referred as Shidoku problem

[10, 4]. In figure [2] we have listed a 9×9 Sudoku problem and its unique solution (for $m = 3$).

The idea here is due to the observation that all the m^2 rows (columns, sub-square) of any $m^2 \times m^2$ Sudoku solution will have the following properties that,

- every row (column, sub-square) is an element of S_{m^2}, due to the rule that no repetition of integers in every row(columns, sub-square). As said before S_{m^2} is set of all $m^2!$ permutations of integers $\{1, 2, 3, \ldots, m^2\}$ for any $m \in \mathbb{N}$.
- any two rows (columns) will be strongly unequal due to the rule that no repetition is allowed in every columns(rows).

Hence, given a Sudoku solution ($m^2 \times m^2$ matrix) all its m^2 rows (columns) will be strongly unequal with one another. In other words, the given Sudoku solution is made out of m^2 strongly unequal permutations of S_{m^2}. Lets prove by method of contradiction. Suppose not, then there exist two rows such that i^{th} entry of these two rows are equal, then it implies the i^{th} column has a repeated integer which leads to a contradiction with the Sudoku rule. Similar argument holds for columns.

Definition 1.6. A collection of k permutations of S_n are said to be strongly unequal if they are strongly unequal with one another. In other words, any two permutations in the collection will be strongly unequal. Such a collection is referred as, a collection of k strongly unequal permutations of S_n.

It is to be noted that any arrangement of m^2 strongly unequal permutations of S_{m^2} will not necessarily construct a Sudoku solution due to the rule that the same integers may not appear twice in the sub-squares. We are interested in set of m^2 permutations that are strongly unequal with one another such that they form a Sudoku solution.

The number of possible Sudoku problems and Sudoku solutions have been of interest to many [2, 4, 5, 7, 8, 9]. That is, in how many ways one can fill the $m^2 \times m^2$ matrix without repeating any integers $\{1, 2, \ldots, m^2\}$ in each row, column and sub-square.

For $m = 2$, Laura [10] proved that there are 288 different Sudoku solutions exist. She has used a similar procedure followed by Felgenhauer and Jarvis [1] who used a computer program to find the all possible solutions for $m = 3$. Felgenhauer and Jarvis [1] used a computer program to show that there are 6,670,903,752,021,072,936,960 different Sudoku solutions for $m = 3$. Russell and Jarvis[3] showed that there are 5,472,730,538 essentialy different Sudoku solutions for $m = 3$. It is also to be noted that a proof that does not use computer is not yet known[10]. The main objective of introducing the concept of strongly unequal permutations is to attempt to give an algebraic proof to the number of Sudoku solutions. In this paper we will prove the following theorem for $m = 2$.

Theorem 1.7. *There are* 288 *different* 4×4 *Sudoku Solutions.*

A few observation and challenges in higher dimension is also given.

2. Proof of Theorem 1.2

Proof. Let $\sigma = (a_1, a_2, \ldots, a_n)$ be any arbitrary permutation of S_n. The idea here is to remove all the permutations which are not strongly unequal with σ. Therefore its enough to prove that there are N (as defined below) permutations which are not strongly unequal with σ, where

$$N = (n-1)! + (n-1)\left(\sum_{k=1}^{n-2}(-1)^{k-1}\binom{n-2}{k}(n-(k+1))!\right) \tag{2.1}$$

Please refer to figure [3]. Its easy to verify that all possible $n!$ permutations can be divided into n subsets namely, $P_{a_1}, P_{a_2}, \ldots, P_{a_n}$, each having $(n-1)!$ permutations, where the subset P_{a_r} contains all the permutations with a_r arranged in the first place as indicated in the figure [3]. (Why P_{a_r} has $(n-1)!$ permutations for any r? and prove that $P_{a_i} \cap P_{a_j} = \phi$ for any $i \neq j$. We leave these as exercise.)

Its to be observed that every permutation of P_{a_1} is not strongly unequal with σ, since all the permutations of P_{a_1} starts with a_1. Hence all the $(n-1)!$ permutations of P_{a_1} are to be removed. The term $(n-1)!$ in equation [2.1] is because of this analysis.

Now the idea is to identify the permutations in the remaining $(n-1)$ subsets, namely $P_{a_2}, P_{a_3}, \ldots, P_{a_n}$ which are not strongly unequal with σ. Lets take an arbitrary subset, P_{a_r}, i.e., the set of all $(n-1)!$ permutations starting with a_r. Now we identify, $(n-2)$ subsets of P_{a_r}, namely $P_{(a_r,a_2)}$, $P_{(a_r,a_3)}, \ldots, P_{(a_r,a_{r-1})}$, $P_{(a_r,a_{r+1})}, \ldots, P_{(a_r,a_n)}$, where $P_{(a_r,a_l)}$ is set of all permutations with a_r in the first place and a_l is the l^{th} place, for any $l \in \{1, 2, \ldots, n\}/\{1, r\}$. Its easy to verify that all the $(n-2)!$ permutations of $P_{(a_r,a_l)}$ are not strongly unequal with σ (because of a_l in the l^{th} place). (Question:- Why? $P_{(a_r,a_l)}$ has $(n-2)!$ permutations? Since a_r is in the first place, we are fixing one of the remaining $n-1$ places except the r^{th} place. Hence we will get $n-2$ subsets each having $(n-2)!$ permutations). We have to remove all the permutations of $P_{(a_r,a_l)}$, for all $l \in \{2, 3, 4, \ldots, n\}/\{r\}$, therefore, we have to remove all the permutations of

$$\bigcup_{\substack{i=2 \\ i \neq r}}^{n} P_{(a_r,a_i)}$$

Its to be noted that, $|P_{(a_r,a_p)} \cap P_{(a_r,a_q)}| = (n-3)!$ for any $p \neq q$ since there are $(n-3)!$ permutations possible with a_r in the first place, a_p and a_q at the p^{th} and q^{th} places respectively (fixing 3 elements the remaining $n-3$ places can be permuted in $(n-3)!$ ways).

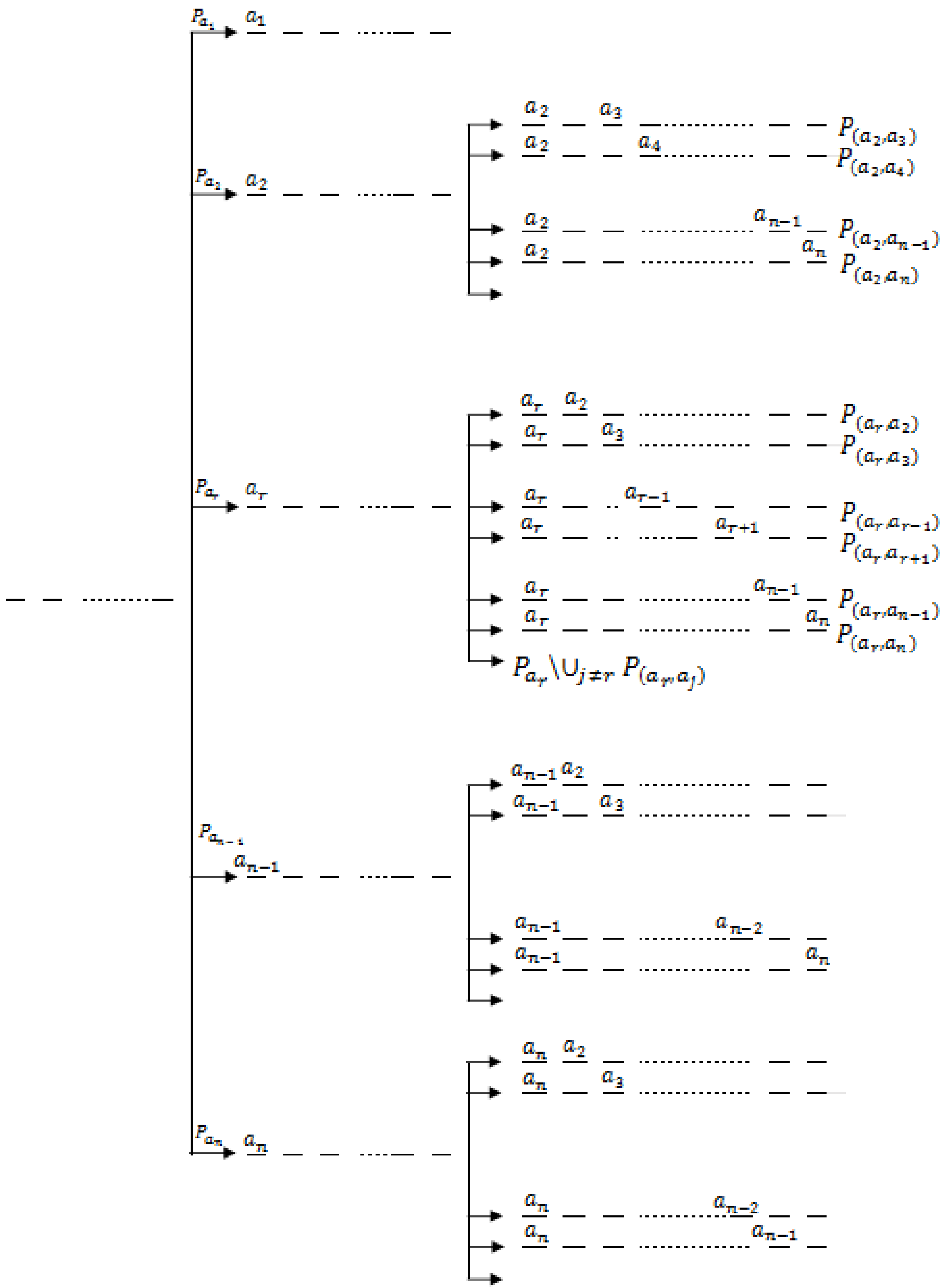

FIGURE 3. Partitioning of $n!$ permutations to identify permutations which are not strongly unequal with $(a_1, a_2, \ldots, a_n)$

By inclusion-exclusion principle [6]

$$\left|\bigcup_{\substack{i=2\\i\neq r}}^{n} P_{(a_r,a_i)}\right| = \sum_{k=1}^{n-2}\left(\sum_{\substack{\phi\neq J\subseteq\{2,3,\ldots,n\}/\{r\}\\ \&\ |J|=k}} (-1)^{k-1}\left|\bigcap_{j\in J} P_{(a_r,a_j)}\right|\right) \tag{2.2}$$

Claim:- Let J be any non-empty subset of $\{2,3,\ldots,n\}/\{r\}$, then

$$\left|\bigcap_{j\in J} P_{(a_r,a_j)}\right| = (n-(|J|+1))!$$

Its easy to verify that $\bigcap_{j\in J} P_{(a_r,a_j)}$ are set of all permutations having a_r in the first places and $|J|$ elements arranged in their respective places. Hence $|J|+1$ places are fixed out of n places. The left over $n-(|J|+1)$ places can be permuted with $n-(|J|+1)$ elements of $\{1,2,3,\ldots,n\}/J\cup\{r\}$ in $(n-(|J|+1))!$ ways and hence the claim.

From the above claim it is to be noted that for any two non-empty subsets of $\{2,3,\ldots,n\}/\{r\}$, namely, J and J' with $|J| = |J'|$

$$\left|\bigcap_{j\in J} P_{(a_r,a_j)}\right| = \left|\bigcap_{j\in J'} P_{(a_r,a_j)}\right| = (n-(|J|+1))! \tag{2.3}$$

Equation [2.2] implies

$$\begin{aligned}\left|\bigcup_{\substack{i=2\\i\neq r}}^{n} P_{(a_r,a_i)}\right| &= \sum_{k=1}^{n-2}\left(\sum_{\substack{\phi\neq J\subseteq\{2,3,\ldots,n\}/\{r\}\\ \&\ |J|=k}} (-1)^{k-1}\left|\bigcap_{j\in J} P_{(a_r,a_j)}\right|\right)\\ &= \sum_{|J|=1}^{n-2}\left((-1)^{|J|-1}\binom{n-2}{|J|}\quad (n-(|J|+1))!\right)\end{aligned}$$

Due to the fact that there are $\binom{n-2}{|J|}$ ways to select subsets having $|J|$ elements from $\{2,3,\ldots,n\}/\{r\}$. The above equation gives the numbers of permutations starting with a_r and not strongly unequal with the $(a_1,a_2,\ldots,a_n)$, where $r\in\{2,3,\ldots,n\}$. Please refer to the equation [2.1] due to the above analysis we have the following

$$(n-1)\left(\sum_{k=1}^{n-2}(-1)^{k-1}\binom{n-2}{k}(n-(k+1))!\right) \tag{2.4}$$

Hence we have proved that there are N (as given in equation [2.1]) number of permutations which are not strongly unequal with $(a_1,a_2,\ldots,a_n)$. Since there are $n!$ permutations possible, given any $(a_1,a_2,\ldots,a_n)$ there will be exactly $n!-N$ $(=|U_n(\sigma)|)$ permutations which are strongly unequal. Since σ is arbitrary $|U_n(\sigma)| = |U_n|$. Hence the proof. $\square$

3. **Proof of Theorem 1.3 and a few Observations**

A few observations: The table [1] list a few experiment on the number ($|U_n|$) of strongly unequal permutations of S_n for various n and is computed using theorem [1.2]. Its easy to observe the pattern which leads to the identity.

n	$\|U_n\|$ The number of strongly unequal permutations of any given permutation of S_n	$\|U_n\| = (n-1) \times (\|U_{n-1}\| + \|U_{n-2}\|)$ $n \geq 3$	$\frac{\|U_n\|}{\|U_{n-1}\|} \approx$ n
1	0	-	-
2	1	-	-
3	2	2	2.000000
4	9	9	4.500000
5	44	44	4.888889
6	265	265	6.022727
7	1854	1854	6.996226
8	14833	14833	8.000539
9	133496	133496	8.999933
10	1334961	1334961	10.000008
11	14684570	14684570	10.999999
12	176214841	176214841	12.000000
13	2290792932	2290792932	12.999999
14	32071101049	32071101049	14.000000
15	481066515734	481066515734	14.999999
...	...	...	...

TABLE 1. Table of number of strongly unequal permutations of any permutation of S_n ($|U_n|$) for various n

Theorem 1.3 $|U_{n+1}| = n(|U_n| + |U_{n-1}|)$, for $n \geq 2$.

In other words, the number of strongly unequal permutations of any permutation of S_{n+1} is n times the sum of the number of strongly unequal permutation(s) of any permutation of S_n and S_{n-1}.

Proof. In a much more simple words, the permutations of U_{n+1} can be constructed from the permutations of U_n and U_{n-1}.

Let $\sigma = (a_1, a_2, \ldots, a_n, a_{n+1})$ be any arbitrary permutation of S_{n+1} and let $U_{n+1} \subset S_{n+1}$ be the set of all strongly unequal permutations of σ. The idea here is to prove that $|U_{n+1}| \geq n(|U_n| + |U_{n-1}|)$ and $|U_{n+1}| \leq n(|U_n| + |U_{n-1}|)$.

From our assumption, $\sigma = (a_1, a_2, \ldots, a_n, a_{n+1}) \in S_{n+1}$

$$\implies \exists\,!\, i \in \{1, 2, \ldots, n+1\} \text{ such that } a_i = n+1 \tag{3.1}$$

$$\implies (a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n, a_{n+1}) = \sigma\,' \in S_n \tag{3.2}$$

Claim 1: $|U_{n+1}| \geq n \times |U_n|$

Let $U_n(\sigma\,') \subset S_n$ be the set of all strongly unequal permutations of σ'. Let $(b_1, b_2, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{n+1})$ be any arbitrary permutation of $U_n(\sigma')$.

$$\implies a_j \neq b_j \quad \forall j \in \{1, 2, \ldots, n+1\}/\{i\} \tag{3.3}$$

Please note that we are playing with the index set for convenience.

Now, lets construct a permutation of S_{n+1} by inserting $n+1$ in the i^{th} place of the arbitrary permutation of $U_n(\sigma')$, namely $\gamma = (b_1, b_2, \ldots, b_{i-1}, n+1, b_{i+1}, \ldots, b_{n+1}) \in S_{n+1}$. Its easy to verify that γ is not strongly unequal with σ. (Why?, because $\gamma(i) = \sigma(i) = n+1$, that is, because of the fact that the i^{th} entries of these two permutations are equal to $n+1$.) Its is to be noted that the other entries of these two permutations are unequal, that is, $\gamma(j) \neq \sigma(j)\ \forall j \in \{1, 2, \ldots, i-1, i+1, \ldots n+1\}$ by equation [3.3].

Now the idea is to swap the i^{th} entry of γ with the remaining n entries to get strongly unequal permutations of σ. Let $k \neq i$ and without loss of generality let $k < i$. Define $\sigma_1 = (b_1, b_2, \ldots, b_{k-1}, n+1, b_{k+1}, \ldots, b_{i-1}, b_k, b_{i+1}, \ldots, b_n)$. We have just swapped the k^{th} and i^{th} entries of γ.

Claim 1.1: $\sigma_1 \in U_{n+1}$, that is σ_1 is strongly unequal with σ. For $j \in \{1, 2, \ldots, n+1\}/\{k, i\}$

$$\begin{aligned} \sigma_1(j) &= b_j \quad \text{(by definition)} \\ &\neq a_j \quad \text{(equation [3.3])} \\ &= \sigma(j) \quad \text{(by definition)} \\ \implies \sigma_1(j) &\neq \sigma(j) \quad \forall j \in \{1, 2, \ldots, n+1\}/\{k, i\} \end{aligned}$$

Now, $\sigma_1(k) = n+1 \neq \sigma(k)$, since $\sigma(i) = n+1$ & $i \neq k$, hence $\sigma_1(k) \neq \sigma(k)$. And, $\sigma_1(i) = b_k$ & $\sigma(i) = a_i = n+1 \implies \sigma_1(i) \neq \sigma(i)$ since $b_k \neq n+1$.

Therefore $\sigma_1(j) \neq \sigma(j)\ \forall j \in \{1, 2, \ldots, n+1\}$ hence $\sigma_1 \in U_{n+1}$.

Since the arbitrary $k \in \{1, 2, \ldots, i-1, i+1, \ldots, n+1\}$ its clear that we have constructed n permutations of U_{n+1} from a permutation of U_n. By this idea we have generated n strongly unequal permutations of σ from an arbitrary permutations of $U_n(\sigma')$. Its easy to verify that if we apply this method for two different permutations of $U_n(\sigma')$ we get two different sets of n permutations such that they are strongly unequal with σ. Therefore, we have generated $n \times |U_n|$ permutations which are strongly unequal with σ. Hence $|U_{n+1}| \geq n \times |U_n|$. Let T_1 be the set of all permutations that are strongly unequal with σ and generated by this algorithm.

Claim 2: $|U_{n+1}| \geq n \times |U_{n-1}|$

We have $\sigma = (a_1, a_2, \ldots, a_n, a_{n+1}) \in S_{n+1}$. By equation [3.1], we have $a_i = n+1$. Let $k \in \{1, 2, \ldots, i-1, i+1, \ldots, n+1\}$ and without loss of generality let $k > i$. Now, we construct a permutation $\sigma'' = (a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{k-1}, a_{k+1}, \ldots, a_{n+1})$ by dropping i^{th} and k^{th} entries. It is to be noted that σ'' is a permutation of $n-1$ integers namely, $\{a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{k-1}, a_{k+1}, \ldots, a_{n+1}\}$. Let W be the set of all strongly unequal permutations of σ''. Since, σ'' is isomorphic to any permutation of S_{n-1}, we get $|W| = |U_{n-1}|$. Therefore, we have got $|U_{n-1}|$ permutations which are strongly unequal with σ''. Let $(c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_{k-1}, c_{k+1} \ldots, c_{n+1})$ be any arbitrary permutation of W.

$$\implies a_j \neq c_j \quad \forall j \in \{1, 2, \ldots, n+1\}/\{i, k\} \tag{3.4}$$

Please note that we are playing with the index set for our convenience. Now define,

$$\sigma_2 = (c_1, c_2, \ldots, c_{i-1}, a_k, c_{i+1}, \ldots, c_{k-1}, a_i, c_{k+1} \ldots, c_{n+1}) \in S_{n+1} \tag{3.5}$$

Claim 2.1: $\sigma_2 \in U_{n+1}$, that is σ_2 is strongly unequal with σ. For $j \in \{1, 2, \ldots, n+1\}/\{i, k\}$

$$\begin{aligned} \sigma_2(j) &= c_j \quad \text{(by definition)} \\ &\neq a_j \quad \text{(equation [3.4])} \\ &= \sigma(j) \quad \text{(by definition)} \\ \implies \sigma_2(j) &\neq \sigma(j) \quad \forall j \in \{1, 2, \ldots, n+1\}/\{i, k\} \end{aligned}$$

Now $\sigma_2(i) = a_k \neq a_i = \sigma(i)$, since $a_i = n+1$ & $i \neq k$, hence $\sigma_2(i) \neq \sigma(i)$. And, $\sigma_2(k) = a_i = n+1$ & $\sigma(k) = a_k \implies \sigma_2(k) \neq \sigma(k)$ since $a_k \neq n+1$.

Therefore $\sigma_2(j) \neq \sigma(j)$ $\forall j \in \{1, 2, \ldots, n+1\}$ hence $\sigma_2 \in U_{n+1}$.

Its clear that we have constructed $|W| = |U_{n-1}|$ permutations for a fixed k. Since $k \in \{1, 2, \ldots, i-1, i+1, \ldots, n+1\}$, there are n choices of k and hence we have generated $n \times |U_{n-1}|$ strongly unequal permutations of σ. Its easy to verify that if we apply this method for two different permutations of W we get two different sets of n permutations such that they are strongly unequal with σ. Therefore, we have generated $n \times |U_{n-1}|$ permutations which are strongly unequal with σ. Hence $|U_{n+1}| \geq n \times |U_{n-1}|$. Let T_2 be the set of all permutations that are strongly unequal with σ and generated by this algorithm.

Claim 3: Let δ be any permutation of U_{n+1} then ($\delta \in T_1$ & $\delta \notin T_2$) or ($\delta \notin T_1$ & $\delta \in T_2$). Let $\delta = (c_1, c_2, \ldots, c_{n+1}) \in U_{n+1}(\sigma)$. Then $\forall j \in \{1, 2, \ldots, n+1\}$

$$c_j \neq a_j \tag{3.6}$$

Since, $\delta \in U_{n+1} \subset S_{n+1}$, $\exists\, k \in \{1, 2, \ldots, i-1, i+1, \ldots, n+1\}$ such that $\delta(k) = c_k = n+1$. Without loss of generality lets assume that $k < i$. Now lets construct a new permutation δ' by swapping the i^{th} and k^{th} entries of δ, $\delta' = (c_1, c_2, \ldots c_{k-1}, c_i, c_{k+1}, \ldots, c_{i-1}, c_k, c_{i+1}, \ldots, c_{n+1})$

If $a_k = \delta'(k) = c_i$ then $\delta \in T_2$, this follows directly from the proof of claim 2. Similarly, If $a_k \neq \delta'(k) = c_i$ then $\delta \in T_1$, this follows directly from the proof of claim 1. From the above argument it follows directly that $\delta \in T_1$ iff $\delta \notin T_2$. Hence the claim.

Claim 4: $|U_{n+1}| \geq n \times (|U_n| + |U_{n-1}|)$

From Claims 1, 2, and 3 this follows directly.

Claim 5: $|U_{n+1}| \leq n \times (|U_n| + |U_{n-1}|)$

From Claim 3 this follows directly. Hence the theorem. □

4. **Proof of Lemma 1.4 and 1.5**

Lemma 1.4 Any $n+1$ permutations of S_n are not strongly unequal with one another.

Proof. Lets prove by the method of contradiction. Suppose there exist a collection of $n+1$ strongly unequal permutations of S_n, namely $A_1, A_2, \ldots, A_{n+1}$. Then by definition [1.6], for any $i \neq j$

$$\begin{aligned} & A_i(k) \neq A_j(k), \quad 1 \leq k \leq n \\ \Longrightarrow \quad & A_1(k), A_2(k), \ldots, A_{n+1}(k) \text{ are all unequal } \& \ A_i(k) \in \{1, 2, \ldots, n\} \ \forall\ i \end{aligned}$$

$$\Rightarrow \Leftarrow$$

Which is a contradiction since $n+1$ different $A_i(k)$s are not possible. Therefore, the existence of $n+1$ strongly unequal permutations of S_n is not possible. Hence any $n+1$ permutations of S_n are not strongly unequal with one another. □

Lemma 1.5 There exist a collection of n strongly unequal permutations in S_n.

Proof. It's enough to produce an example to prove the above statement. Let A_1 be any $(a_1, a_2, \ldots, a_n) \in S_n$. By cycle notation of A_1, we generate the next permutation as $A_2 = (a_2, a_3, \ldots, a_n, a_1)$. Similarly we can generate n permutations and let the i^{th} permutation be $A_i = (a_i, a_{i+1}, \ldots, a_n, a_1, \ldots, a_{i-1})$ for any $1 \leq i \leq n$. Now the claim is $A_1, A_2, \ldots, A_n$ are all strongly unequal with one another. Suppose not, then there exist a pair of permutations, A_i & A_j with $i \neq j$ such that they are not strongly unequal.

$$\begin{aligned} & \Longrightarrow \exists k \in \{1, 2, \ldots, n\}\ s.t.\ A_i(k) = A_j(k) \quad \text{(by definition [1.1])} \\ & \Longleftrightarrow a_{(i+k-1) \mod n} = a_{(j+k-1) \mod n} \qquad (\because n \equiv 0 \mod n, \text{we let } a_{n \mod n} = a_n) \\ & \Longleftrightarrow i + k - 1 \mod n = j + k - 1 \mod n \quad (\because a_l \neq a_m\ for\ l \neq m) \end{aligned}$$

$\iff i+k-1 \equiv j+k-1 \mod n$

$\iff i \equiv j \mod n \qquad (\because a \equiv b \mod n \iff a+k \equiv b+k \mod n$ for any fixed k)

$\iff i = j \qquad (\because i, j \leq n)$

$\Rightarrow\Leftarrow$ Contradiction to the assumption that $A_i \neq A_j$. Hence the claim. □

Observation 4.1. By combining lemmas [1.4] and [1.5] its easy to observe that the maximum possible collection of strongly unequal permutations in S_n is n. Now, given any $n-1$ strongly unequal permutations of S_n, there exist a unique permutation of S_n such that these collection of n $(=(n-1)+1)$ permutations will be strongly unequal with one another. This follows directly from the definition of n strongly unequal permutations. Hence the challenge of finding n strongly unequal permutations of S_n reduces to finding $n-1$ strongly unequal permutations of S_n.

5. A few observations and proof of Theorem 1.7

Lets start with an important observation between strongly unequal permutations of S_4 and existence of Sudoku solutions out of them. Let $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ be a collection of 4 strongly unequal permutations of S_4. We claim that its always possible to construct a Sudoku solution out of $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$. Let $\sigma_1 = (a, b, c, d)$ and let it fill the first row of the 4×4 Sudoku matrix. Now its easy to verify that one of the permutations of $\{(c, d, a, b), (c, d, b, a), (d, c, a, b), (d, c, b, a)\}$ will be σ_2 and let it fill the second row of the Sudoku matrix. Now the left over σ_3 and σ_4 can fill the third and fourth row by similar argument. Hence the claim. Its to be noted that its possible to construct at least 8 Sudoku solutions from a given collection of 4 strongly unequal permutations of S_4.

Theorem 1.7 There are 288 different 4×4 Sudoku Solutions.

Proof. Given 4 strongly unequal permutations of S_4, we can arrange them in 4! ways and its clear that not all the arrangements are going to give a Sudoku solution because repetition of numbers is not allowed in any sub-square as per the Sudoku rule.

Idea here is to look for all possible ways of identifying 4 strongly unequal permutations such that they form a Sudoku solution.

First row can be any one of the permutations of S_4, which can be selected in ${}^{4!}C_1$ $(={}^{24}\ C_1 = 24)$ ways. Let $\gamma_1 = (a, b, c, d)$ be any arbitrary permutation of S_4 to fill the first row. By theorem [1.2], there will be 9 permutations of S_4 which will be strongly unequal with γ_1. Its easy to list these nine permutations as follows, **(b,a,d,c)**, (b, c, d, a), (b, d, a, c), (c, a, d, b), (c, d, a, b), (c, d, b, a), (d, a, b, c), (d, c, a, b) and (d, c, b, a). Lets fill the third row with any one of these 9 strongly unequal permutations. Therefore, the third row can be filled in ${}^9C_1$. Among these 9 permutations the permutation (b, a, d, c) which is a special one. Upon filling the third row by any one of the remaining 8 permutations, the partially filled Sudoku becomes a Sudoku problem, that is, it will lead to a unique Sudoku solution.

Among these 9 permutation there will be a permutation (b, a, d, c) which gives 4 different Sudoku solution,the rest(24×8) are all gives unique Sudoku solution.This contributes $(24 \times 1 \times 4) + (24 \times 8 \times 1) = 288$ □

Observation 5.1. Can every collection of 9 strongly unequal permutations of S_9 be arranged to construct a 9×9 Sudoku solution?

The answer is no. Here is the counter example. $\sigma_1 = (1, 2, 3, 4, 5, 6, 7, 8, 9)$, $\sigma_2 = (4, 3, 1, 5, 6, 2, 8, 9, 7)$, $\sigma_3 = (5, 1, 2, 3, 4, 8, 9, 7, 6)$, $\sigma_4 = (2, 4, 5, 1, 9, 7, 3, 6, 8)$, $\sigma_5 = (3, 5, 9, 8, 7, 4, 6, 2, 1)$, $\sigma_6 = (6, 9, 8, 7, 3, 5, 4, 1, 2)$, $\sigma_7 = (8, 6, 7, 9, 2, 3, 1, 4, 5)$, $\sigma_8 = (9, 7, 4, 6, 8, 1, 2, 5, 3)$, $\sigma_9 = (7, 8, 6, 2, 1, 9, 5, 3, 4)$.

6. **Results and Discussions:**

In this article we have demonstrated the idea of using strongly unequal permutations of S_4 to find the number of 4×4 Sudoku solutions. We have proved a closed form representation for the number of strongly unequal permutations of any given permutation of S_n. We have proved a recursive relation involving $|U_{n-2}|$, $|U_{n-1}|$ and $|U_n|$. In the table [1] we have also observed a pattern in the ratio of U_n with U_{n-1} and is shown for various values of n. As part of computational effort we have listed all the permutations that are strongly unequal with the perumutation $324165 \in S_6$ in table [2] and have also listed all the permutations that are strongly unequal with the perumutations 6712354, $1325467 \in S_7$ in table [3].

Future work: The same idea can be extended to find the number of 9×9 Sudoku solutions. By lemma [1.5] there exist a set of 9 strongly unequal permutations of S_9. Is it possible to arrange such permutations to form a sudoku solution and from observation [5.1] its clear that there do exist collection of 9 strongly unequal permutations of S_9 which can not be arranged to form a 9×9 Sudoku solution. A few basic questions to work on are (i) given a collection of 9 strongly unequal permutations of S_9, how many 9×9 Sudoku solutions can be constructed. (ii) how many different collections of 9 strongly unequal permutations of S_9 is possible. Please note that, observation [4.1] may help us in finding a collection of 9 strongly unequal permutations of S_9 and observation [5.1] opens up lot of interesting questions in higher dimension.

Table 2. Strongly Unequal Permutations of the Permutation **3 2 4 1 6 5** #265

612354	142356	413256	613452	613524	645312	536421	562341	645231	251643
162354	146253	641352	163452	163524	643512	635421	253416	642531	251436
162534	142653	461352	136452	136524	463512	632451	253641	462531	251346
612534	142536	416352	135426	135624	436512	632541	256341	462351	251634
615234	145236	413652	135642	135246	435612	635241	265341	642351	261534
165234	145623	413526	136542	132546	431526	532416	263541	462513	216534
156234	146523	415326	163542	132654	431652	532641	236541	642513	215634
152634	615423	415632	613542	136254	431256	536241	235641	645213	215346
152346	165423	416532	615342	163254	643251	563241	235416	465213	213546
512346	156423	461532	165342	613254	463251	653241	236451	456213	213654
512634	516423	641532	156342	132456	436251	653214	263451	452613	216354
516234	561423	451632	153642	631254	432651	563214	246351	542613	261354
561234	651423	451326	153426	631524	432516	536214	243651	546213	213456
651234	541623	541326	513426	531246	435216	532614	243516	652413	
651243	541236	541632	513642	531624	435621	635214	245316	562413	
561243	451236	651432	516342	531642	436521	632514	245631	256413	
516243	451623	561432	561342	531426	463521	231456	246531	265413	
512643	641523	516432	651342	631542	643521	231654	265431	246513	
512436	461523	156432	651324	631452	645321	231546	256431	245613	
152436	416523	165432	561324	635412	465321	235614	562431	241536	
152643	415623	615432	516324	536412	456321	236514	652431	241653	
156243	415236	146532	513624	563412	453621	263514	546231	241356	
165243	412536	145632	513246	653412	453216	265314	542631	261453	
615243	412653	145326	153246	546312	543216	256314	542316	216453	
612543	416253	143526	153624	543612	543621	253614	452316	215436	
162543	461253	143652	156324	453612	546321	562314	452631	215643	
162453	641253	146352	165324	456312	653421	652314	456231	216543	
612453	412356	143256	615324	465312	563421	652341	465231	261543	

Table 3. List of strongly unequal permutations of 6712354 and 1325467, #578

7651243	4671523	7564132	3651742	3564712	7431526	3574216	7653241	2436175	7456231	5264713	2143576
7561243	4167523	7654132	3657142	3574612	4631572	3546271	3257146	2436715	4576231	5274613	2143675
5671243	4176523	5164732	7631542	3547612	7643125	3546721	3251746	2437615	4567231	5247613	2146735
5167243	4157623	5174632	3671542	3546712	7463125	3547621	3271546	2473615	4657231	5246713	2147635
5176243	4156723	5147632	3167542	3546172	4673125	3574621	3261745	7243615	4267531	5246173	2174635
7156243	4156273	5146732	3176542	3541672	4637125	3564721	3267145	2463715	4276531	5241673	2164735
7146235	4157236	5143672	3157642	3574126	4631725	3564271	3276145	2463175	4257631	5274136	2164573
7164235	7641235	5143726	3156742	3547126	4631275	3654271	3271645	2643175	4256731	5247136	2146573
7146523	7461235	5143276	3154672	3541726	7436125	3654721	7231645	2643715	4253671	5241736	2174536
7164523	4671235	7154632	3154726	3541276	7431625	7634521	7236145	2643571	4253716	2541736	2147536
7154623	4167235	7164532	3154276	7534126	7436215	3674521	2631745	2463571	4253176	2547136	2154736
7154236	4176235	7146532	3147526	7534612	4637215	3647521	2637145	2436571	4237516	2574136	2154673
5147236	4136275	7143526	3174526	5634172	4673215	3467521	2673145	7243516	4273516	7254136	2156743
5174236	4136725	7143625	3146572	5634712	7463215	3476521	7263145	2473516	4236571	2541673	2157643
5146273	4137625	7134625	3164572	5643712	7643215	3457621	7231546	2437516	4263571	2546173	2176543
5146723	4173625	7134526	3164275	5643172	7436521	3456721	7253146	2453176	4263175	2546713	2167543
5147623	4163725	7136542	3164725	5463172	4637521	3456271	2573146	2453716	4263715	2547613	2671543
5174623	4163275	7163542	3174625	5463712	4673521	3457216	2537146	2453671	4273615	2574613	7261543
5164723	4163572	7153642	3147625	7543612	7463521	7634215	2531746	2456731	4237615	7254613	2657143
5164273	4136572	5134276	3146725	5473612	7643521	3674215	5231746	2457631	4236715	2564713	2651743
7654123	4173526	5134726	3146275	5437612	4653721	3647215	5237146	7246531	4236175	2564173	2561743
7564123	4137526	5134672	3471625	5436712	4653271	3467215	5273146	2476531	4231675	2654173	2567143
5674123	4153276	5136742	3476125	5436172	4563271	3476215	5263741	2467531	4231576	2654713	2576143
5647123	4153726	5137642	3467125	5431672	4563721	3241576	5273641	2647531	4271635	7264513	7256143
5641723	4153672	5173642	3461725	7543126	7453621	3241675	5237641	2674531	4276135	2674513	7251643
5641273	4156732	5163742	3461275	5473126	4573621	3246175	5236741	7264531	4267135	2647513	2571643
5461273	4157632	5173246	3641275	5437126	4537621	3246715	5234671	2654731	4261735	2467513	5271643
5461723	4176532	5137246	3641725	5431726	4536721	3247615	5234716	2564731	4261573	2476513	5276143
5467123	4167532	7153246	3647125	5431276	4536271	3274615	5234176	7254631	4271536	7246513	5267143
5476123	4671532	7163245	3674125	4531276	7453216	3264715	2534176	2574631	4251736	2457613	5261743
7546123	7461532	7136245	7634125	4531726	4573216	3264175	2534716	2547631	4257136	2456713	2153746
7541623	7641532	3176245	3641572	4537126	4537216	3264571	2534671	2546731	4251673	2456173	2137546
5471623	4657132	3167245	3461572	4573126	5437216	3246571	2536741	2543671	4256173	2451673	2173546
7541236	4651732	3671245	3471526	7453126	5473216	3274516	2537641	2543716	4256713	2457136	2163745
5471236	4561732	7631245	3451276	4531672	7543216	3247516	2573641	2543176	4257613	2451736	2173645
4571236	4567132	3157246	3451726	4536172	5436271	3254176	7253641	5243176	4276513	2471536	2137645
7451236	4576132	3571246	3457126	4536712	5436721	3254716	2563741	5243716	4267513	7241536	2136745
4571623	7456132	7531246	3451672	4537612	5437621	3254671	2653741	5243671	4657213	2461573	2134675
7451623	7451632	7653142	3456172	4573612	5473621	3256741	7263541	5246731	4567213	2641573	2134576
7456123	4571632	7563142	3456712	7453612	7543621	3257641	2673541	5247631	4576213	7264135	
4576123	5471632	5673142	3457612	4563712	5463721	3276541	2637541	5274631	7456213	2674135	
4567123	7541632	5637142	3476512	4563172	5463271	3267541	7236541	5264731	7546213	2647135	
4561723	7546132	5631742	3467512	4653172	5643271	3657241	7234516	7654231	5476213	2641735	
4561273	5476132	7536142	3647512	4653712	5643721	3567241	2634571	7564231	5467213	2461735	
4651273	5467132	7531642	3674512	7643512	5634721	3576241	2634715	5674231	5647213	2467135	
4651723	5461732	3571642	7634512	7463512	5634271	7536241	2634175	5647231	5674213	2476135	
4657123	5641732	3576142	3654712	4673512	7534621	5637241	7234615	5467231	7564213	7246135	
7641523	5647132	3567142	3654172	4637512	7534216	5673241	2431576	5476231	7654213	7241635	
7461523	5674132	3561742	3564172	7436512	3547216	7563241	2431675	7546231	5264173	2471635	

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