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A SEMI-ANALYTICAL APPROACH TO DETERMINE THE VELOCITY POTENTIAL AROUND TWO SPHERES IN ARBITRARY MOTION THROUGH AN IDEAL FLUID

ABHINANDAN CHOWDHURY AND MARK DELCAMBRE

Department of Mathematics, Savannah State University,
Savannah, Georgia 31404, USA
CGI Federal, Lafayette, Louisiana 70506, USA

ABSTRACT. Potential flow around two non-overlapping spheres in arbitrary motion through an unbounded inviscid liquid is considered. Bi-spherical coordinates are introduced to transform the Laplace equation as well as the boundary conditions to these coordinates. The fact that one of the coordinate lines is going through the spherical surfaces enables us to seek the solution in Legendre series with respect to one of the bi-spherical coordinates. The Legendre spectral method is shown to have an exponential convergence which is confirmed by the computations. The efficiency is so high that even for the hard cases of two almost touching spheres, an accuracy of 10^{-10} is achieved with as few as 20 terms in the expansion. Stream functions instead of velocity potentials are used for better demonstration of the flow direction of the inviscid liquid, and the contour plots of the streamlines are presented graphically.

Key Words. Velocity Potential, Two-Sphere Problem, Bi-spherical Coordinates, Legendre Spectral Method.

AMS Subject Classification. 35Q35, 42C10, 74S25, 76D99.

1. INTRODUCTION

The mechanics of two-sphere interaction (two-sphere problem) in fluid media have been attracting attention since the early 19th century [1–3]. These interactions usually occur in the study of suspension, two-phase flow, heat and mass transfer, and combustion of droplets etc. Among these phenomena, suspension is of particular interest which is basically the particulate materials in which the second (particulate) phase comprised by spherical particles (the *filler*) are randomly dispersed throughout the continuous phase (the *matrix*). Probably the most significant aspect of the two-sphere problem is to find a general solution of Laplace equation obtained in a form which is appropriate for cases in which boundary conditions are given over any two spherical surfaces. Poisson [1] was the first person who successfully solved such boundary value problem albeit for the case of electrostatic problem.

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It was argued that a successful numerical (e.g., spectral) solution is contingent on finding the appropriate curvilinear coordinates in which the boundaries of the domain of the solution are coordinate lines. It led to a new approach where solutions were sought in some curvilinear coordinate systems (such as bipolar coordinate) after transforming the Laplace equation to the curvilinear coordinates. Among others, the method of bi-spherical coordinates turned out to be a method of great generality. The fact that the bi-spherical coordinates are the best suited tool for solving a two-sphere problem was first emphasized by Lord Kelvin. He was apparently the first to introduce the bi-spherical coordinates in 1846 in a letter to Liouville (see [4, §211–212]). The first detailed application of the bi-spherical coordinates for solving the Laplace equation was given by G. B. Jeffery [5] for the potential flow around two spheres. Among different two-sphere problems, the case of temperature field in suspension has remained a widely popular one as it is somewhat simpler in implementation than viscosity or elasticity. The application of the bi-spherical coordinates to the heat conduction problem around two spheres with constant gradient at infinity was first sketched in [6]. The proposed method was numerically implemented strictly for arbitrary separation between the centers of the spheres [7, 8], where it was demonstrated successfully that the solution can be obtained in closed form albeit in infinite series with respect to Legendre polynomials.

After Jeffery [5], further study of potential flow distribution around two spheres by employing bi-spherical coordinate method was somewhat neglected until the work of Weihs and Small [9]. In the meantime, few other methods gained popularity for solving the problem of potential flow field around two spheres - but none of them involve the use of bi-spherical coordinates. In one such method proposed by Mitra [10], two sets of spherical polar coordinates systems were introduced in order to express the potential field in terms of infinite series whose coefficients satisfy an infinite set of linear equations. Few years later, Sneddon and Fulton [11] investigated the problem of determining the potential function for the irrotational flow of an ideal fluid past two spheres whose centers are fixed in space. They were even able to develop an expression for the force on one sphere due to the presence of the another – furthermore they also investigated the effect of the distance between the centers on this force.

When the method of bi-spherical coordinates was reintroduced in [9], it was for finding the exact solution for the incompressible potential flow around two adjacent spheres keeping one of the spheres very close to a wall. Moreover, as demonstrated in [12], another method based on tangent-sphere coordinate system was preferred over the method of bi-spherical coordinates in order to obtain the exact solution for the case when two spheres are actually touching each other. Overall, the true potential of this method is yet to be realized for the case of potential flow. While considering the technical aspects of the numerical scheme based on bi-spherical coordinates among other curvilinear coordinates, one property stands out is that the boundaries of the domain of the solution are coordinate lines. This property is of great advantage especially while considering the arbitrary distance between the two non-overlapping spheres as demonstrated in [7,8] for the case of temperature distribution around two spheres. The objective of this paper is to obtain a simple closed expression for the solution of simplest boundary value problem involving two spheres. The problem considered is that of determining the potential function for the irrotational flow of an incompressible, inviscid fluid past two non-overlapping spheres moving arbitrarily by using the method of bi-spherical coordinates.

2. POSING THE PROBLEM

In fluid dynamics, potential flow refers to the flow outside the boundary layer that conforms with the laws of flow in electric and magnetic fields. This is known as the inviscid flow region implying that transverse velocity gradients (i.e., shear) are minimal and viscous effects are next to none. By contrast, the boundary layer, at least in laminar flow, is dominated by viscous effects and does not follow the rules of potential flow. The potential flow of an ideal (inviscid, incompressible) liquid is governed by the Laplace equations for the velocity potential ϕ

$$\Delta\phi = 0,$$

and the velocity vector is related to the potential as

$$\boldsymbol{v} = \nabla\phi.$$

The boundary conditions for the potential stem from the condition that the liquid cannot penetrate the surface, hence the normal component of the velocity is zero:

$$\frac{\partial\phi}{\partial n} \equiv \boldsymbol{n}\cdot\nabla\phi = \boldsymbol{v}\cdot\boldsymbol{n} = \boldsymbol{V}\cdot\boldsymbol{n}, \quad \text{for} \quad \boldsymbol{x} \in \partial D, \tag{2.1}$$

where ∂D is the boundary of the region D occupied by the body, $\boldsymbol{n}$ is the outside normal to the surface. Here $\boldsymbol{V}$ is the given velocity of the center of the body (a point inside the region D). $\boldsymbol{V}$ can be a function of time, but is a constant with respect to the spatial coordinates $\boldsymbol{x}$.

3. COORDINATE TRANSFORMATION

For the problem under consideration, the boundary ∂D consists of the two spheres, i.e., $\{\partial D : x \in (|\boldsymbol{x} - \boldsymbol{z}_a| = a) \cup (|\boldsymbol{x} - \boldsymbol{z}_b| = b\}$, where $\boldsymbol{z}_a$ and $\boldsymbol{z}_b$ are the position vectors of the two spheres of radii a and b, respectively. The appropriate coordinate system for which both boundaries $|\boldsymbol{x} - \boldsymbol{z}_a| = a$ and $|\boldsymbol{x} - \boldsymbol{z}_b| = b$ are coordinate surfaces is the bi-spherical one. In order to introduce the bi-spherical coordinates, we first change to another set of Cartesian coordinates centered at a point, P, that lies on the segment connecting the two spheres. In Fig. 1 are plotted the new coordinate system $Py_1y_2y_3$ which is obtained from the original system $Ox_1x_2x_3$ after rotating about axis Ox_1 by an angle $(\frac{\pi}{2} - \varphi)$ in negative direction and consequent rotation about axis Ox_2 by an angle $(\frac{\pi}{2} - \theta)$ in negative direction.

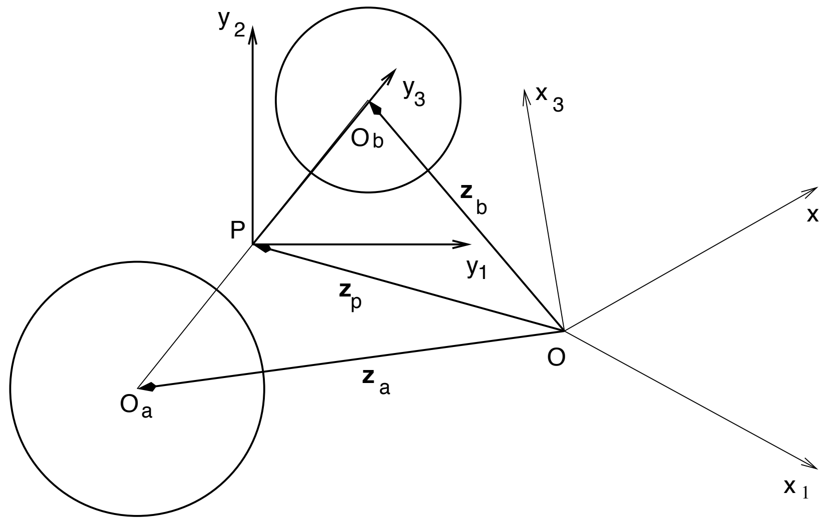


FIGURE 1. The geometry of the region.

The most important is the vector $\boldsymbol{z} = \boldsymbol{z}_b - \boldsymbol{z}_a$ which will define the axis Oy_3 (see the figure). Denote $z := |\boldsymbol{z}|$. Then Oy_2 and Oy_1 can be seen to be two arbitrary axes that are perpendicular to the axis Oy_3. The center P of the new coordinate system is chosen in a manner that both spheres are coordinate surfaces of the bi-spherical coordinate system centered at point P. The connection between the Cartesian coordinates $Py_1y_2y_3$ and the bi-spherical ones η, ξ, ζ, where $-\infty < \eta < +\infty$, $0 \le \xi < \pi$, $0 \le \zeta < 2\pi$, may be expressed as follows :

$$y_1 = c\frac{\sin\xi}{\cosh\eta - \cos\xi}\cos\zeta, \quad y_2 = c\frac{\sin\xi}{\cosh\eta - \cos\xi}\sin\zeta, \quad y_3 = c\frac{\sinh\eta}{\cosh\eta - \cos\xi}, \tag{3.1}$$

where c is called the focal distance.

The metric coefficients of the bi-spherical coordinate system are given by

$$h_\xi = h_\eta = \frac{c}{\cosh\eta - \cos\xi}, \quad h_\zeta = \frac{c\sin\xi}{\cosh\eta - \cos\xi}. \tag{3.2}$$

Let us denote by η_a and η_b the coordinate lines which represent the spheres, are functions of z and sphere radii a and b. Values η_a and η_b are of different signs if the spheres do not intersect each other and of same sign if one sphere encircles the other. We consider here the former case and, therefore, we must choose one of these numbers negative. Without loosing the generality, we select $\eta_a < 0$. Then

$$c = \frac{\sqrt{z^4 + b^4 + a^4 - 2z^2b^2 - 2a^2b^2 - 2z^2a^2}}{2z} = \sqrt{\alpha^2 z^2 - a^2}, \tag{3.3}$$

$$\eta_a = -\ln\left|\frac{c}{a} + \sqrt{1 + \frac{c^2}{a^2}}\right| \equiv -\operatorname{arcsinh}\frac{c}{a}, \tag{3.4}$$

$$\eta_b = \ln\left|\frac{c}{b} + \sqrt{1 + \frac{c^2}{b^2}}\right| \equiv \operatorname{arcsinh}\frac{c}{b}, \tag{3.5}$$

where

$$\alpha = \frac{z^2 - b^2 + a^2}{2z^2}. \tag{3.6}$$

The last quantity is positive since $z > a + b$, which is due to the fact that spheres do not intersect each other.

For the relative distances of the sphere's centers from the point P we get

$$d_a = \frac{1}{z}\sqrt{c^2 + a^2}, \qquad d_b = \frac{1}{z}\sqrt{c^2 + b^2}, \qquad d_a + d_b = 1.$$

Making use of the last formula we are able to complete the connection of systems $Py_1y_2y_3$ and $Ox_1x_2x_3$ by specifying the 'offset vector'

$$\boldsymbol{z}_p = \boldsymbol{z}_a + d_a\boldsymbol{z} = (1 - d_a)\boldsymbol{z}_a + d_a\boldsymbol{z}_b = d_b\boldsymbol{z}_a + (1 - d_b)\boldsymbol{z}_b$$

where $\boldsymbol{z}_a$ and $\boldsymbol{z}_b$ represent the locations of their respective sphere's center from the original system $Ox_1x_2x_3$. Their magnitudes can be found by computing

$$z_a = c\coth\eta_a, \quad z_b = c\coth\eta_b. \tag{3.7}$$

In order to express the result in Cartesian coordinates for possible further manipulation and presentation, we need also the inverse transformation, namely we express ξ, η and ζ in terms of y_1, y_2 and y_3. In doing so, one has to be very careful with the regions where the bi-spherical coordinates may be multi-valued. After some algebra, we obtain

$$\eta = \operatorname{arccoth}\left(\frac{y_1^2 + y_2^2 + y_3^2 + c^2}{2cy_3}\right),$$

$$\xi = \arcsin\left[\sqrt{\frac{4c^2(y_1^2 + y_2^2)}{(y_1^2+y_2^2+(y_3+c)^2)(y_1^2+y_2^2+(y_3-c)^2)}}\right],$$

$$\zeta = \begin{cases} \arcsin(y_2/\sqrt{y_1^2 + y_2^2}), & y_1 > 0, \\ \pi - \arcsin(y_2/\sqrt{y_1^2 + y_2^2}), & y_1 < 0, \\ \frac{1}{2}\pi, & y_1 = 0. \end{cases}$$

The new coordinate system $Py_1y_2y_3$ is obtained by $Ox_1x_2x_3$ after a translation on distance $OO_1 = \alpha z$ along the positive direction of the axis Ox_3. Here $z = |\boldsymbol{z}|$ and θ, φ are its directory angles measured from the original coordinate system, namely

$$z_1 = z\cos\theta, \quad z_2 = z\sin\theta\cos\varphi, \quad z_3 = z\sin\theta\sin\varphi.$$

Respectively, the relation between the old and new Cartesian systems, whose formal expression is $x = Ay + \alpha z$, adopts the form

$$\begin{aligned} x_1 &= y_1\sin\theta + (y_3 + \alpha z)\cos\theta, \\ x_2 &= -y_1\cos\theta\cos\varphi + y_2\sin\varphi + (y_3+\alpha z)\sin\theta\cos\varphi, \\ x_3 &= -y_1\cos\theta\sin\varphi - y_2\cos\varphi + (y_3+\alpha z)\sin\theta\sin\varphi, \end{aligned} \tag{3.9}$$

where the value of α is specified in (3.6). Introducing the new variables means that we are considering a new dependent function

$$\Phi(\xi,\eta,\zeta;t) := \phi[y_1(\xi,\eta,\zeta;t), y_2(\xi,\eta,\zeta;t), y_3(\xi,\eta,\zeta;t);t]. \tag{3.10}$$

To this end we need to find the time derivative and the gradient of the potential function $\phi(x,y,z,t)$ via the respective derivatives of the function $\Phi(\xi,\eta,\zeta;t)$.

First we observe that

$$\begin{pmatrix} \dfrac{\partial\Phi}{\partial\xi} \\ \dfrac{\partial\Phi}{\partial\eta} \\ \dfrac{\partial\Phi}{\partial\zeta} \end{pmatrix} = \mathbb{F} \begin{pmatrix} \dfrac{\partial\phi}{\partial y_1} \\ \dfrac{\partial\phi}{\partial y_2} \\ \dfrac{\partial\phi}{\partial y_3} \end{pmatrix}, \quad \text{where} \quad \mathbb{F} = \begin{pmatrix} \dfrac{\partial y_1}{\partial\xi} & \dfrac{\partial y_2}{\partial\xi} & \dfrac{\partial y_3}{\partial\xi} \\ \dfrac{\partial y_1}{\partial\eta} & \dfrac{\partial y_2}{\partial\eta} & \dfrac{\partial y_3}{\partial\eta} \\ \dfrac{\partial y_1}{\partial\zeta} & \dfrac{\partial y_2}{\partial\zeta} & \dfrac{\partial y_3}{\partial\zeta} \end{pmatrix}$$

is the Jacobian matrix of the coordinate matrix. The entries of the latter are easily computed from the connection between the two coordinate systems. After the components of matrix $\mathbb{F}$ are identified, one needs to compute the inverse and we denote $\mathbb{G} = \mathbb{F}^{-1}$. Then we can use the relation

$$\begin{pmatrix} \dfrac{\partial\phi}{\partial y_1} \\ \dfrac{\partial\phi}{\partial y_2} \\ \dfrac{\partial\phi}{\partial y_3} \end{pmatrix} = \mathbb{G} \begin{pmatrix} \dfrac{\partial\Phi}{\partial\xi} \\ \dfrac{\partial\Phi}{\partial\eta} \\ \dfrac{\partial\Phi}{\partial\zeta} \end{pmatrix}.$$

Using Mathematica, we find that:

$$\mathbb{G} = \begin{pmatrix} \dfrac{\cos\zeta(\cos\xi\cosh\eta - 1)}{c} & -\dfrac{\cos\zeta\sin\xi\sinh\eta}{c} & \dfrac{(\cos\xi - \cosh\eta)\sin\zeta}{c\sin\xi} \\ \dfrac{\sin\zeta(\cos\xi\cosh\eta - 1)}{c} & -\dfrac{\sin\zeta\sin\xi\sinh\eta}{c} & -\dfrac{\cos\zeta(\cos\xi - \cosh\eta)}{c\sin\xi} \\ -\dfrac{\sin\xi\sinh\eta}{c} & \dfrac{1-\cos\xi\cosh\eta}{c} & 0 \end{pmatrix}$$

Now, we are ready to find the time derivative as follows:

$$\frac{\partial \Phi}{\partial t} = \frac{\partial \phi}{\partial t} + \dot{\boldsymbol{x}} \cdot \nabla \phi, \quad \Rightarrow \quad \frac{\partial \phi}{\partial t} = \frac{\partial \Phi}{\partial t} - \dot{\boldsymbol{x}} \cdot \big(\mathbb{G} \nabla_\xi \Phi\big),$$

where $\nabla \phi$ has been computed in the previous step, and $\boldsymbol{x} = \boldsymbol{x}_p + y_1 \boldsymbol{i} + y_2 \boldsymbol{j} + y_3 \boldsymbol{k}$ is the position vector of the point (y_1, y_2, y_3) and $\boldsymbol{x}_P$ is the above defined position vector of the point P. The latter does not influence the solution in the bi-spherical coordinates, but does contribute to the velocities of the fluid particles. Respectively ∇_ξ is a notation for the gradient in the bi-spherical coordinate system.

4. THE BOUNDARY VALUE PROBLEM FOR THE POTENTIAL IN TERMS OF BI-SPHERICAL COORDINATES

The Laplace equation has the following form in terms of bi-spherical coordinates

$$\Delta \Phi \equiv \frac{(\cosh \eta - \cos \xi)^3}{c^2 \sin \xi} \Big[\frac{\partial}{\partial \eta} \Big(\frac{\sin \xi}{\cosh \eta - \cos \xi} \frac{\partial \Phi}{\partial \eta} \Big) + \frac{\partial}{\partial \xi} \Big(\frac{\sin \xi}{\cosh \eta - \cos \xi} \frac{\partial \Phi}{\partial \xi} \Big) + \frac{1}{\sin \xi (\cosh \eta - \cos \xi)} \frac{\partial^2 \Phi}{\partial \zeta^2} \Big]. \tag{4.1}$$

Let us define $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ to be the unit vectors of the coordinate axes Oy_1, Oy_2, and Oy_3, respectively. Then for the unit vector along tangential to the coordinate line η one has

$$\boldsymbol{e}_\eta = -c \frac{\sin \xi \sinh \eta \cos \zeta}{(\cosh \eta - \cos \xi)^2} \boldsymbol{i} - c \frac{\sin \xi \sinh \eta \sin \zeta}{(\cosh \eta - \cos \xi)^2} \boldsymbol{j} - c \frac{\cosh \eta \cos \xi - 1}{(\cosh \eta - \cos \xi)^2} \boldsymbol{k}. \tag{4.2}$$

Note that the vector $\boldsymbol{e}_\eta$ is not a vector with unit norm. When (if) necessary we will normalize it and will use so-called "physical components". The unit vector can be obtained using the modulus $|\boldsymbol{e}_\eta| \equiv h_\eta = c/(\cosh \eta - \cos \xi)$.

To pose the boundary conditions we observe that the outward normal derivative $\frac{\partial}{\partial n}$ is in fact a partial derivative with respect to η at $\eta = \eta_a$ and with respect to $(-\eta)$ at $\eta = \eta_b$. Respectively, the outward normal vector $\boldsymbol{n}$ is equal to $\boldsymbol{e}_\eta$ or $-\boldsymbol{e}_\eta$.

The boundary condition at infinity is that the fluid is at rest, hence the potential must be constant. Since the velocity potential is defined up to a constant, we are free to select this to be the constant zero. Then

$$\Phi \to 0 \quad \text{as} \quad \eta, \xi \to 0. \tag{4.3a}$$

The boundary conditions at the spheres stem from (2.1) when $\boldsymbol{V}^a = \dot{\boldsymbol{z}}_a$ and $\boldsymbol{V}^b = \dot{\boldsymbol{z}}_b$ are the velocity vectors of the centers of the spheres, respectively.

$$\frac{\partial \Phi}{\partial \eta}\Big|_{\eta=\eta_a} = -c \frac{\sin \xi \sinh \eta_a \cos \zeta}{(\cosh \eta_a - \cos \xi)^2} V_1^a - c \frac{\sin \xi \sinh \eta_a \sin \zeta}{(\cosh \eta_a - \cos \xi)^2} V_2^a - c \frac{\cosh \eta_a \cos \xi - 1}{(\cosh \eta_a - \cos \xi)^2} V_3^a. \tag{4.3b}$$

$$\frac{\partial \Phi}{\partial \eta}\Big|_{\eta=\eta_b} = c \frac{\sin \xi \sinh \eta_b \cos \zeta}{(\cosh \eta_b - \cos \xi)^2} V_1^b + c \frac{\sin \xi \sinh \eta_b \sin \zeta}{(\cosh \eta_b - \cos \xi)^2} V_2^b + c \frac{\cosh \eta_b \cos \xi - 1}{(\cosh \eta_b - \cos \xi)^2} V_3^b. \tag{4.3c}$$

Here the subscripts denote the respective components of the velocity vectors, which are functions of time only:

$$\boldsymbol{V}^a(t) := \dot{\boldsymbol{z}}_a(t), \qquad \boldsymbol{V}^b(t) := \dot{\boldsymbol{z}}_b(t). \tag{4.4}$$

The purpose of this work is to find an analytical solution (albeit in infinite series) for arbitrary velocities and positions of the centers of the spheres.

5. DUAL FORMULATION USING STREAM FUNCTION

It can be observed that for the spheres moving along the line connecting their centers of spheres, the potential satisfies

$$\Delta\phi \equiv \frac{(\cosh\eta-\cos\xi)^3}{c^2\sin\xi}\Big[\frac{\partial}{\partial\eta}\Big(\frac{\sin\xi}{\cosh\eta-\cos\xi}\frac{\partial\Phi}{\partial\eta}\Big)+\frac{\partial}{\partial\xi}\Big(\frac{\sin\xi}{\cosh\eta-\cos\xi}\frac{\partial\Phi}{\partial\xi}\Big)\Big]=0. \tag{5.1}$$

We observe that the Laplace equation (5.1) for the potential is satisfied automatically, if we introduce the following auxiliary function ψ, which is the stream function, via the relations

$$\frac{\partial\psi}{\partial\xi} := \frac{\sin\xi}{\cosh\eta-\cos\xi}\frac{\partial\Phi}{\partial\eta}, \quad \frac{\partial\psi}{\partial\eta} := -\frac{\sin\xi}{\cosh\eta-\cos\xi}\frac{\partial\Phi}{\partial\xi}. \tag{5.2}$$

It can be rewritten as follows

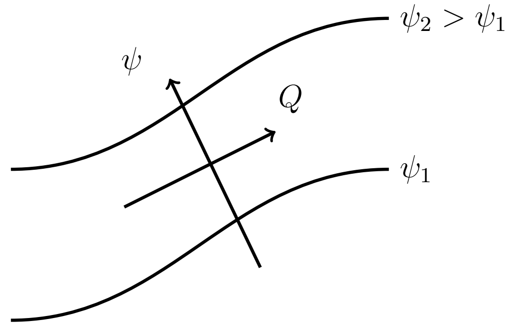
$$\frac{\cosh\eta-\cos\xi}{\sin\xi}\frac{\partial\psi}{\partial\xi} =: \frac{\partial\Phi}{\partial\eta}, \quad \frac{\cosh\eta-\cos\xi}{\sin\xi}\frac{\partial\psi}{\partial\eta} =: -\frac{\partial\Phi}{\partial\xi}. \tag{5.3}$$

The last form of the relations allows us to derive the equation for ψ, namely

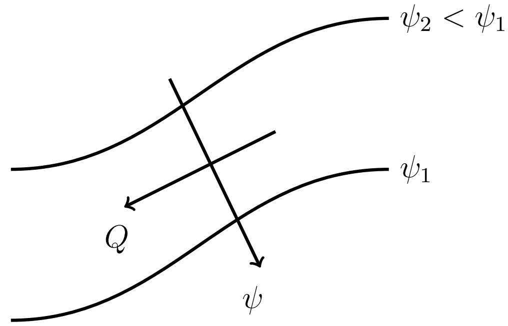
$$\frac{\partial}{\partial\eta}\Big(\frac{\sin\xi}{\cosh\eta-\cos\xi}\frac{\partial\psi}{\partial\eta}\Big)+\frac{\partial}{\partial\xi}\Big(\frac{\sin\xi}{\cosh\eta-\cos\xi}\frac{\partial\psi}{\partial\xi}\Big)=0, \tag{5.4}$$

which has the same form as the Laplace equation for the potential. This means that we can use the same Legendre series expansion to solve eq. (5.4). The difference is in the boundary conditions. We will show here that for ψ one obtains a Dirichlet b.v.p., which has some decisive advantages. The isolines of stream functions are known as streamlines. These are lines such that at any given time they are tangent to the velocity vector. It should be noted that, by definition, the component of the velocity normal to a streamline is always zero so that there is no mass flux across a streamline. What it means is that it is possible to represent every solid body/boundary by a streamline. Another important aspect of introducing the stream function is that it is relatively easier to demonstrate the fluid flow by using stream function as change in ψ can be used to determine the flow direction as shown in Fig. 2.

Now, in order to obtain a boundary condition for $\partial\phi/\partial\eta$, we need to take into account the last terms in each of eqs. (4.3). Though we wish to obtain the potential flow model for arbitrary velocities and positions of the spheres, for the sake of better understanding, we need to consider the velocity vector components of the centers of the spheres which are passing along the line joining the centers of the spheres (albeit



(a) When $\psi_2 > \psi_1$, the volume flow (Q) is positive, since, $Q = \int_{\psi_1}^{\psi_2} \mathrm{d}\psi = \psi_2 - \psi_1$. This indicates that flow is to the right.



(b) When $\psi_2 < \psi_1$, the volume flow (Q) is negative, which indicates that flow is to the left.

FIGURE 2. Determining the direction of flow from two neighboring streamlines.

to opposite directions) to be nontrivial. In other words, it is assumed the spheres are moving along the line connecting the centers of the spheres which are approaching towards each other, which means except V_3^a (and V_3^b) in eqs. (4.3), other velocity vector components will be trivial. Thus

$$\begin{aligned}\left.\frac{\partial\psi}{\partial\xi}\right|_{\eta=\eta_a} &= \frac{\sin\xi}{\cosh\eta - \cos\xi}\frac{\partial\Phi}{\partial\eta} = \frac{-\sin\xi}{\cosh\eta - \cos\xi}\frac{\cosh\eta_a\cos\xi - 1}{(\cosh\eta_a - \cos\xi)^2} \\ &= -\sin\xi\frac{\cosh\eta_a\cos\xi - 1}{(\cosh\eta_a - \cos\xi)^3},\end{aligned}$$

where the sign of the r.h.s. depends on the sphere under consideration.

Since the stream function is defined up to a constant, we can integrate the last definitive equality from 0 to ξ. Actually, the stream function is defined up to a constant. The rigorous way to do it is to find the first indefinite integral:

$$\psi(\eta_a,\xi) = -\int \sin\xi \frac{\cosh\eta_a\cos\xi - 1}{(\cosh\eta_a - \cos\xi)^3}\mathrm{d}\xi = -\frac{1+\cosh^2\eta_a - 2\cosh\eta_a\cos\xi}{2(\cosh\eta_a - \cos\xi)^2} + C$$
$$= -\frac{1-\mu^2}{2(\cosh\eta_a - \cos\xi)^2} - \frac{1}{2} + C,$$

where $\cos\xi = \mu$. Then, by choosing $C = \frac{1}{2}$, it can be ensured that $\psi(\eta_a, \pm\pi) = 0$, which is the natural symmetry of the problem. Hence

$$\psi(\eta_a,\xi) = -\frac{1-\mu^2}{2(\cosh\eta_a - \mu)^2}.$$

Using the same substitution $\psi = \sqrt{2(\cosh\eta - \mu)}B(\eta,\mu)$ we can show that the solution for B is a Legendre series:

$$B(\eta,\mu) = \sum_{n=0}^{\infty}[L_n e^{(n+\frac{1}{2})\eta} + M_n e^{-(n+\frac{1}{2})\eta}]P_n(\mu). \tag{5.5}$$

The stream function formulation offers a very concise way to solve the boundary value problem. One has to insert the expression of type eq. (5.5) into the boundary condition and solve the subsequent algebraic systems.

For further convenience, we will specify two different solutions for B according to the two sets of boundary conditions:

$$B_a(\eta_a,\mu) = -\frac{1-\mu^2}{2^{3/2}(\cosh\eta_a - \mu)^{5/2}}, \qquad B_a(\eta_b,\mu) = 0. \tag{5.6a}$$

$$B_b(\eta_a,\mu) = 0, \qquad B_b(\eta_b,\mu) = \frac{1-\mu^2}{2^{3/2}(\cosh\eta_b - \mu)^{5/2}}. \tag{5.6b}$$

To complete the solution we need the above boundary functions expanded into Legendre series. To this end we use the method of generating function as outlined in [6, 7]. For the sake of completeness, the derivation is provided below:

The expression on the right-hand side of the last equation can be expanded into Legendre series, if the generating function of Legendre polynomials (cf. [17,§3.6]) is used. The latter is defined as follows

$$G(t,\mu) \stackrel{\text{def}}{=} (1 - 2t\mu + t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(\mu) \quad \text{for} \quad t < 1. \tag{5.7}$$

For the first and second derivatives of the generating function one gets

$$\frac{\partial G(t,\mu)}{\partial t} = \frac{(\mu - t)}{(1 - 2t\mu + t^2)^{3/2}} = \sum_{n=0}^{\infty} n t^{n-1} P_n(\mu), \tag{5.8}$$

and,

$$\frac{\partial^2 G(t,\mu)}{\partial t^2} = \frac{(3\mu^2 - 4\mu t + 2t^2 - 1)}{(1 - 2t\mu + t^2)^{5/2}} = \sum_{n=0}^{\infty} n(n-1) t^{n-2} P_n(\mu), \tag{5.9}$$

respectively. A linear combination of eqs. (5.7), (5.8) and (5.9) gives the Legendre series of the r.h.s. of eq. (5.5), i.e.,

$$\beta G(t,\mu)+\gamma G_t(t,\mu)+\delta G_{tt}(t,\mu)=\frac{-(1-\mu^2)}{2^{3/2}(\cosh\eta-\mu)^{5/2}}\equiv\frac{-2(1-\mu^2)t^{5/2}}{(1-2t\mu+t^2)^{5/2}},\ t\stackrel{\text{def}}{=}e^{\eta_a}<1$$

$$\frac{\beta}{(1-2t\mu+t^2)^{1/2}}+\frac{\gamma(\mu-t)}{(1-2t\mu+t^2)^{3/2}}+\frac{\delta(3\mu^2-4\mu t+2t^2-1)}{(1-2t\mu+t^2)^{5/2}}=-\frac{2(1-\mu^2)t^{5/2}}{(1-2t\mu+t^2)^{5/2}}$$

$$\beta(1-2t\mu+t^2)^2+\gamma(\mu-t)(1-2t\mu+t^2)+\delta(3\mu^2+2t^2-4\mu t-1)=-2(1-\mu^2)t^{5/2}. \tag{5.10}$$

After collecting and comparing the like terms from both sides of eq. (5.10), we can write the following system of equations for solving the unknowns β, γ, and δ:

$$\begin{bmatrix}1+2t^2+t^4 & -(t+t^3) & 2t^2-1\\ -4(t+t^3) & 3t^2+1 & -4t\\ 4t^2 & -2t & 3\end{bmatrix}\begin{bmatrix}\beta\\ \gamma\\ \delta\end{bmatrix}=\begin{bmatrix}-2t^{5/2}\\ 0\\ 2t^{5/2}\end{bmatrix} \tag{5.11}$$

Solving the system of equations (5.11), we arrive at:

$$\beta=\frac{4t^{5/2}}{3(-1+t^2)},\ \gamma=\frac{8t^{7/2}}{3(-1+t^2)},\ \text{and}\ \delta=\frac{2t^{5/2}}{3},$$

which allow us to recast the above boundary conditions as follows:

$$B_a(\eta_a,\mu)=\sum_{n=0}^{\infty}\frac{2\left[(n+1)(n+2)e^{(n+\frac{5}{2})\eta_a}-n(n-1)e^{(n+\frac{1}{2})\eta_a}\right]}{3(e^{2\eta_a}-1)}P_n(\mu), \tag{5.12a}$$

$$B_a(\eta_b,\mu)=0, \tag{5.12b}$$

$$B_b(\eta_a,\mu)=0, \tag{5.13a}$$

$$B_b(\eta_b,\mu)=\sum_{n=0}^{\infty}\frac{2\left[(n+1)(n+2)e^{-(n+\frac{5}{2})\eta_b}-n(n-1)e^{-(n+\frac{1}{2})\eta_b}\right]}{3(-e^{2\eta_b}+1)}P_n(\mu). \tag{5.13b}$$

Then eq. (5.12) gives the following system for each n

$$L_n^a e^{(n+\frac{1}{2})\eta_a}+M_n^a e^{-(n+\frac{1}{2})\eta_a}=\frac{2\left[(n+1)(n+2)e^{(n+\frac{5}{2})\eta_a}-n(n-1)e^{(n+\frac{1}{2})\eta_a}\right]}{3(e^{2\eta_a}-1)},$$

$$L_n^a e^{(n+\frac{1}{2})\eta_b}+M_n^a e^{-(n+\frac{1}{2})\eta_b}=0.$$

Respectively, eq. (5.13) gives the following system for each n

$$L_n^b e^{(n+\frac{1}{2})\eta_a}+M_n^b e^{-(n+\frac{1}{2})\eta_a}=0,$$

$$L_n^b e^{(n+\frac{1}{2})\eta_b}+M_n^b e^{-(n+\frac{1}{2})\eta_b}=\frac{2\left[(n+1)(n+2)e^{-(n+\frac{5}{2})\eta_b}-n(n-1)e^{-(n+\frac{1}{2})\eta_b}\right]}{3(-e^{2\eta_b}+1)}.$$

Solving for L_n^a and M_n^a, we have

$$L_n^a = \frac{2e^{(1+2n)\eta_a}\left[1+n+n^2+(1+2n)\coth\eta_a\right]}{3\left[e^{(1+2n)\eta_a}-e^{(1+2n)\eta_b}\right]},$$

$$M_n^a = \frac{2\left[1+n+n^2+(1+2n)\coth\eta_a\right]}{3\left[e^{-(1+2n)\eta_a}-e^{-(1+2n)\eta_b}\right]},$$

and for L_n^b and M_n^b, we have

$$L_n^b = \frac{2e^{-2\eta_b}\left[1+n+n^2-(1+2n)\coth\eta_b\right]}{3\left[-e^{(1+2n)\eta_a}+e^{(1+2n)\eta_b}\right]},$$

$$M_n^b = \frac{2e^{(2n+1)\eta_a-2\eta_b}\left[1+n+n^2-(1+2n)\coth\eta_b\right]}{3\left[e^{(1+2n)\eta_a}-e^{(1+2n)\eta_b}\right]}.$$

5.1. **Recovering the potential from Stream Function.** After the stream function is known, one can ideally integrate eq. $(5.3)_1$ to obtain the potential function. Similarly to ψ, the potential is also defined up to a constant, so one can see $\phi(0,\xi)=0$. Before manipulating the said equation we observe that

$$\frac{\partial\psi}{\partial\xi} = \frac{\partial\psi}{\partial\mu}\frac{\partial\mu}{\partial\xi} = \sin\xi\frac{\partial\psi}{\partial\mu}.$$

Now, introducing this in eq. $(5.3)_1$ we obtain

$$(5.14)\quad \frac{\partial\Phi}{\partial\eta} = (\cosh\eta-\mu)\frac{\partial\psi}{\partial\mu} = -\sqrt{\frac{(\cosh\eta-\mu)}{2}}\sum_{n=0}^{N-1}\left[L_n e^{(n+\frac{1}{2})\eta}+M_n e^{-(n+\frac{1}{2})\eta}\right]P_n(\mu)$$
$$+\sqrt{2(\cosh\eta-\mu)^3}\sum_{n=0}^{N-1}\left[L_n e^{(n+\frac{1}{2})\eta}+M_n e^{-(n+\frac{1}{2})\eta}\right]P_n'(\mu).$$

Recovery of Φ from eq. (5.14) entirely depends of finding suitable close forms for the products duo – $(\cosh\eta-\mu)^{1/2}P_n(\mu)$ and $(\cosh\eta-\mu)^{3/2}P_n'(\mu)$. This problem will be treated elsewhere.

6. RESULTS AND DISCUSSIONS

Using the Rodrigues formula (see, e.g., [13]), it can be shown that for the standardized Legendre polynomials ($P_n(0)=1$) the following expressions hold forever for the coefficients of the Legendre series (see, e.g., [14])

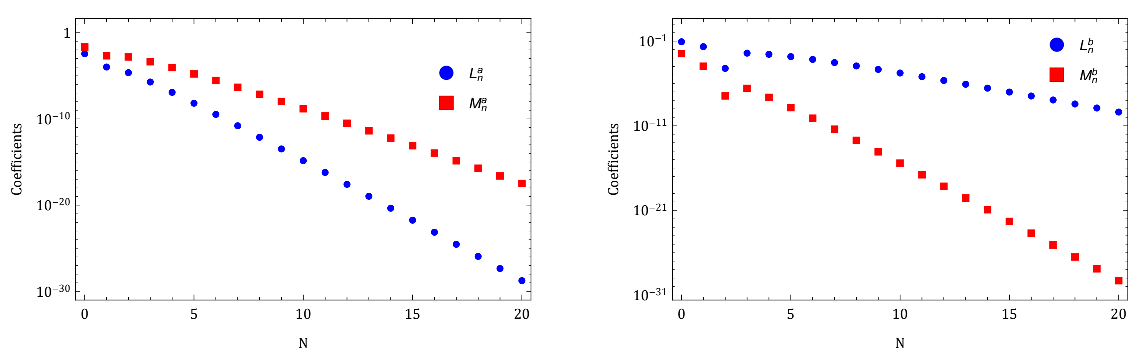
$$f(z) = \sum_{n=0}^{\infty} a_n\,P_n(z), \quad a_n = \frac{2n+1}{2^{n+1}n!}\int_{-1}^{1} f^{(n)}(x)\,(1-x^2)^n dx.$$

The last formula ensures exponential convergence for the series when the sought function is analytic (all derivative up to infinite order exist). We should stress the point here that the functions $B(\eta,\mu)$ will remain analytic even if a situation arise where one has to deal with an equation with discontinuous coefficients, e.g., for the case of identifying effective coefficient of heat conductivity [7, 15] or electric conductivity. This is due to the advantage presented by the bi-spherical coordinates, which are structured in a way that makes the discontinuity to take place only in the first derivatives of functions that depend on the variable η. The first task to perform here is to verify the practical convergence of our spectral method.

The exponential convergence ensures that retaining 10-20 terms in the series should prove to be a sufficient number for quantitatively very good approximation. For the sake of testing the practical convergence, we focus on the following three main cases:

1. $z = 4$, $a = 1$, $b = 2$;
2. $z = 10$, $a = 1$, $b = 2$;
3. $z = 3.1$, $a = 1$, $b = 2$.

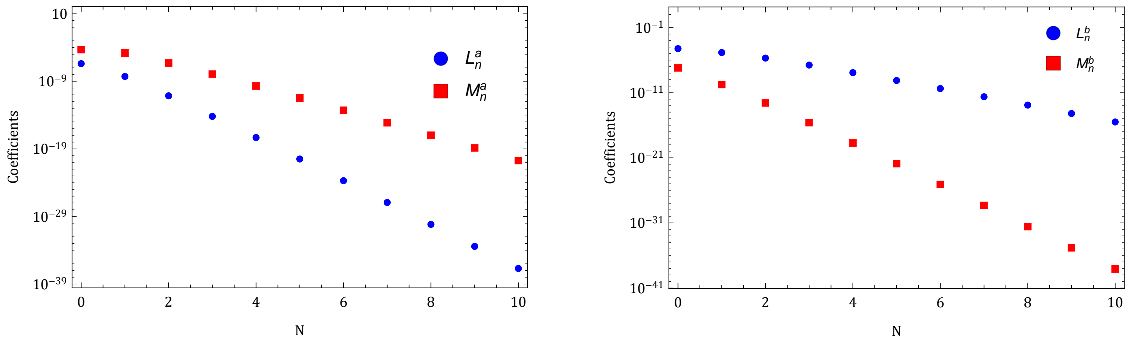
For the in-depth validation of the method, it is important to have spheres of different radii. In the left panel of Fig. 3, we present the computed coefficients, L_n^a, M_n^a, versus their number(N). Similarly, in the right panel, we present coefficients, L_n^b, M_n^b, versus their number. In both panels of Fig. 3, the distance between the centers of the spheres is fixed as $z = 4$, (case 1). Without even comparing them with suitable best-fit curves, one can recognize the exponential nature of the coefficients. In the similar manner, Fig. 4 shows the convergence for large distance between the spheres (case 2), while Fig. 5 shows the result for case 3 when the spheres are almost touching each other.



(a) For sphere with radius 1 (b) For sphere with radius 2

FIGURE 3. Exponential decay of the computed coefficients when $z = 4$.

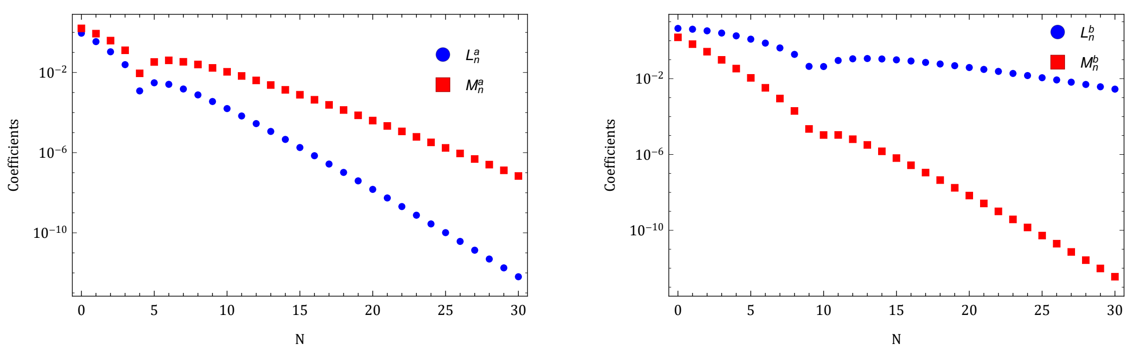
In case 1, the distance between the closest points of the spheres is one unit which is twice smaller than the radius of the bigger sphere and is equal to the radius of the smaller sphere. In terms of the dimensionless distance $a/z = 1/4$ we see that the small parameter is not anymore really small. In case 2, the distance between the spheres is large relative to their radii. The expectation is that such a case will be easier in some sense and that the solution will resemble closely the mere superposition of the



(a) For sphere with radius 1 (b) For sphere with radius 2

FIGURE 4. Exponential decay of the computed coefficients when $z = 10$.

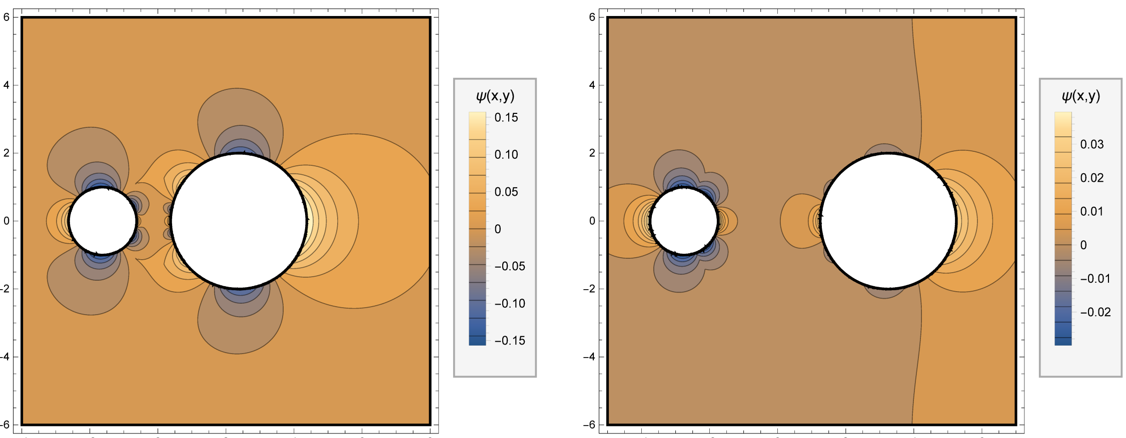
solutions originated from the introduction of two spheres individually. In case 3, the dimensionless distance a/z is equal to $1/3.01$, which bigger than the dimensionless distance from the case 1. This relatively "bigger" parameter makes this case 3 a probing one, which means, even to achieve reasonably good approximation in the range of 10^{-15}, we need to retain more than 30 terms. Similarly, for case 2, one can obtain high order of approximation (in the range of 10^{-40}) by merely retaining 10 terms for the faster converging coefficients – L_n^a and M_n^b in this case.



(a) For sphere with radius 1 (b) For sphere with radius 2

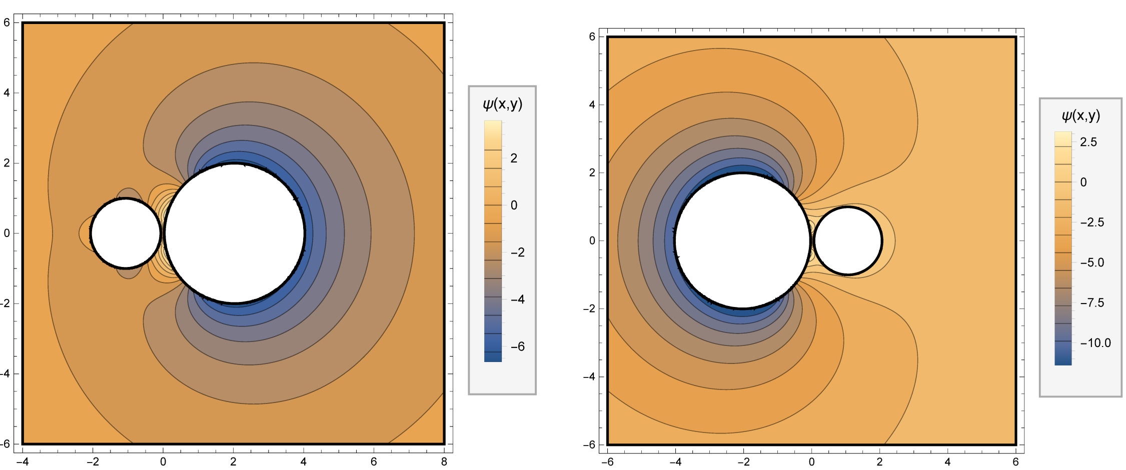
FIGURE 5. Exponential decay of the computed coefficients when $z = 3.01$.

Next, we present in Fig. 6 and in Fig. 7 the contour plots of stream functions. In order to verify the method, we consider the case of two spheres moving against each other along the axis that connects their centers. The left sphere has radius $a = 1$ and moves with unit speed to the right ($V^a = -1$). The right sphere is of radius $b = 2$ which is moving with unit speed to the left ($V^b = 1$). In the left panel of Fig. 6, the centers of the spheres are kept 4 unit away from each other when spheres are allowed to move towards each other with equal speed. It can be noticed that the overall maximum of the modulus of the stream function is of order of 0.15 in Fig. 6(a). This modulus goes down to the order of 0.03 when the distance between the centers of the spheres is set to be 6 units in Fig. 6(b). This is quite reasonable since close interaction between the spheres will create more disturbance in the streamline profiles.



(a) For $z = 4$ (b) For $z = 6$

FIGURE 6. Contour plots for profiles of Stream functions for various proximity between the centers of the spheres.



(a) Smaller sphere in the left (b) Smaller sphere in the right

FIGURE 7. Contour plots for profiles of Stream functions for spheres almost touching each other.

Now, let's consider the fluid flow between two neighboring streamlines in order to understand the profile of the velocity potential of the fluid in somewhat vicarious way. We know that the change in the value of ψ from one streamline to another is equal to the net flux of the fluid between them. Also, being reminded the connection between the two neighboring streamlines and the flow direction between them as shown in Fig. 2, it can be determined that in the lee side of the bigger spheres in both panels of Fig. 6, the flow direction is counterclockwise. Moreover, since the value of the velocity potential (ϕ) drops along the direction of flow, in this case, the value of ϕ

decreases in the counterclockwise direction. As we shift our attention to the sides of the spheres facing each other, we can observe more interesting case. In the left side of the bigger sphere, we can notice the flow direction to be counterclockwise. In the right side of the smaller sphere, there exists a flow direction which is clockwise. When two opposite streams come in contact to each other, disturbance is created. When the spheres are away from each other, the magnitude of disturbance is negligible as can be seen on the right panel of Fig. 6. Note that these are the streamlines and subsequently, the potentials created due to the presence of the spheres. Furthermore, it indicates that there are inflow/outflow velocity components at the boundaries of the spheres. This is explained that motion of the boundary creates these components.

As spheres approach very close to each other, the profile becomes more chaotic as can be seen on Fig. 7. One noticeable feature in contour plots of Fig. 7 is that the flows in the lee side of the bigger spheres are clockwise in direction. Also, albeit unsurprisingly, the order of magnitude of the stream functions increases reasonably. Moreover, if we allow the bigger sphere to go to the direction opposite to that of the fluid flow, the order of magnitude becomes even bigger. Since the fluid is incompressible, the rate of flow at any point must be same. This implies that if v is the velocity and $d\psi$ is the distance between the streamlines, then $v d\psi$ = constant. Therefore, the closer the streamlines are, the greater the velocity of the fluid between the two neighboring streamlines or *stream tubes*.

7. CONCLUSION

In this paper, we employ a semi-analytical approach to solve the problem of distribution of velocity potential around two, non-intersecting, unequal spheres moving arbitrarily in an ideal fluid flowing with constant velocity. The equivalent stream function formulation is introduced in order to attack the boundary value problem with more straight-forward approach. We show that the general solution can be expressed in series in Legendre polynomials. To this end, we use the method of generating functions to expand the boundary conditions into Legendre series and to obtain a closed algebraic system for the coefficients. Thus a fast spectral method based on expansion in Legendre series of stream functions is devised, which has exponential convergence. Our numerical results confirm the exponential convergence, and we are able to obtain solutions of accuracy of 10^{-20} with 10-20 terms, which makes the proposed method very efficient. The solution is thoroughly validated for different distances between the spheres. Various cases are treated and shown graphically in order to analyze the distribution of streamlines and subsequently the corresponding velocity potential profiles for different proximities between the spheres.

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