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STABILITY ON INTERPOLATION OF SCATTERED DATA VIA KERNELS

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ABSTRACT. For an approximation, the inverse inequality can guarantee the smoothness of an approximant based on its rate approximation. The purpose of this paper is to present new inverse inequalities for scattered data interpolation on $\mathbb{R}^d$ and bounded domain Ω. Finally, some numerical experiments are given as well.

AMS (MOS) Subject Classification. 39A10.

1. Introduction

It is well known that the inverse inequality plays an important role in finite element method (FEM) analysis by estimating the condition number of stiffness matrix. However, for radial basis functions interpolation method, only a few papers discuss it. Narcowich et al. [1, 2] proposed a Bernstein-type inequality for RBFs on the whole domain $\mathbb{R}^d$ by introducing a band-limited approximation. In [3], the author obtained the same result on a bounded domain Ω. In this paper, we present some new inverse inequalities for scattered data interpolation on $\mathbb{R}^d$ and Ω.

At first, we are concerned with the RBF approximation to a function f in $W_2^\tau(\mathbb{R}^d)$, $\tau > d/2$. The approximation will be a sum of finite linear combinations of translates of an RBF Φ and the translates are from the set of data points $X = \{x_1, \dots, x_N\} \subset \mathbb{R}^d$. Therefore, given an RBF Φ and a set X, the RBF approximation is defined by

$$f(x) = \sum_{j=1}^{N} \lambda_j \Phi(x - x_j). \tag{1.1}$$

When the Fourier transform $\widehat{\Phi}$ of Φ satisfies

$$c_1(1 + \|\omega\|_2^2)^{-\tau} \le \widehat{\Phi}(\omega) \le c_2(1 + |\|\omega\|_2^2)^{-\tau}, \tag{1.2}$$

where positive constants $c_1 \le c_2$, the Native space $\mathcal{N}_\Phi(\mathbb{R}^d)$ corresponding to Φ coincides with the Sobolev space $W_2^\tau(\mathbb{R}^d)$ and norms are equivalent. Then, we study the

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relationship between $\|f\|_{W_2^\tau(\mathbb{R}^d)}$ and $\|f\|_{W_2^\beta(\mathbb{R}^d)}$, $\tau > \beta > 0$. We consider

$$\begin{aligned}
\|f\|_{W_2^\beta}(\mathbb{R}^d) &= \int_{\mathbb{R}^d} |\widehat{f}(\omega)|^2 (1+\|\omega\|_2^2)^\beta d\omega \\
&= \sum_{j,k=1}^N \lambda_j \lambda_k \int_{\mathbb{R}^d} e^{-i(x_j-x_k)^T\omega} |\widehat{\Phi}(\omega)|^2 (1+\|\omega\|_2^2)^\beta d\omega \\
&= \sum_{j,k=1}^N \lambda_j \lambda_k \int_{\mathbb{R}^d} e^{-i(x_j-x_k)^T\omega} (1+\|\omega\|_2^2)^{-2\tau+\beta} d\omega \\
&= \sum_{j,k=1}^N \lambda_j \lambda_k \Psi(x_j - x_k),
\end{aligned} \tag{1.3}$$

where Ψ is a new RBF and satisfies

$$c_1(1+\|\omega\|_2^2)^{-2\tau+\beta} \le \widehat{\Psi}(\omega) \le c_2(1+\|\omega\|_2^2)^{-2\tau+\beta} \tag{1.4}$$

and the Native space $\mathcal{N}_\Psi(\mathbb{R}^d)$ corresponding to Ψ coincides with the Sobolev space $W_2^\beta(\mathbb{R}^d)$.

The organization of this paper is given in the following. In Section 2, the relevant mathematical background of RBF approximations are given. A new inverse inequality on $\mathbb{R}^d$ is given in Section 3 while the inverse inequality on Ω is derived in Section 4. Some numerical examples for both 1D and 2D are then given in the final Section 5.

2. Mathematical Preliminaries

2.1. **Notation.** We start by introducing some notation. For a bounded domain $\Omega \subseteq \mathbb{R}^d$ (d is dimension) and the data centers $X = \{x_1, \ldots, x_N\} \subseteq \Omega$, the mesh norm h and separation distance q are defined as follows

$$h = \sup_{x\in\Omega} \min_{x_j \in X} \|x - x_j\|_2; \tag{2.1}$$

$$q = \frac{1}{2} \min_{i \neq j} \|x_i - x_j\|_2. \tag{2.2}$$

Moreover, for a non-negative integer k and $1 \le p < \infty$ let $W_p^k(\Omega)$ denote the Sobolev space with differentiability order k and integrability power p. Define for $u \in W_p^k(\Omega)$ and finite p the Sobolev (semi-)norms

$$|u|_{W_p^k(\Omega)} = \left(\sum_{|\alpha|=k} \|D^\alpha u\|_{L_p(\Omega)}^p \right)^{1/p} \quad \text{and} \quad \|u\|_{W_p^k(\Omega)} = \left(\sum_{|\alpha|\le k} \|D^\alpha u\|_{L_p(\Omega)}^p \right)^{1/p}. \tag{2.3}$$

In the case $p = 2$, we have a Hilbert space and can introduce a norm via the Fourier transforms which has the advantage that it can be generalized to non-integer values $0 < \tau < \infty$ and yields an equivalent norm to the one defined above if we choose τ to be an integer. We can describe the functions in the fractional Sobolev space $W_2^\tau(\mathbb{R}^d)$ as precisely square-integrable functions that are finite in the form

$$\|u\|_{W_2^\tau(\mathbb{R}^d)} = \|(1+\|\omega\|_2^2)^{\tau/2}\widehat{u}(\omega)\|_{L_2(\mathbb{R}^d)}. \tag{2.4}$$

Here, $\widehat{u}(\cdot)$ is the Fourier transform

$$\widehat{u}(\omega) = \int_{\mathbb{R}^d} u(x)e^{-2\pi i\omega\cdot x}dx. \tag{2.5}$$

In this paper, we also use the inverse Fourier transform in the form

$$u(\mathbf{x}) = \int_{\mathbb{R}^d} \widehat{u}(\omega)e^{2\pi i\omega\cdot x}dx. \tag{2.6}$$

2.2. **Radial Basis Functions and Native Space.** Let $r = \|\cdot\|$ be Euclidean norm on $\mathbb{R}^d$. A kernel function $\Phi(x, x_j) : \mathbb{R}^d \to \mathbb{R}$ is called radial if

$$\Phi(x, x_j) = \Phi(x - x_j) = \varphi(\|x - x_j\|) = \varphi(r),\ x \in \mathbb{R}^d. \tag{2.7}$$

$\varphi(r)$ is used as a basis function in the RBF method and the univariate function φ is independent from the number of dimensions d. Therefore, the RBF method can be easily adapted to solve higher dimensional problems. In recent applications, the RBFs given in Tables 1 and 2 are most commonly used.

Gaussian (GA)	e^{-cr^2}, $c > 0$
Multiquadric (MQ)	$\sqrt{r^2+c^2}$, $c > 0$
Inverse MQ	$1/\sqrt{r^2+c^2}$, $c > 0$
Thin-plate spline (TPS)	$(-1)^{1+\beta/2}r^\beta \log r$, $\beta \in 2N$

TABLE 1. Global RBFs

$\Phi_{l,0}$	$(1-r)_+^l$
$\Phi_{l,1}$	$(1-r)_+^{l+1}[(l+1)\mathbf{r}+1]$
$\Phi_{l,2}$	$(1-r)_+^{l+2}[(l^2+4l+3)r^2+(3l+6)r+3]$

TABLE 2. Compactly supported functions. $l = \lceil 2+k+1 \rceil$,$k = 0, 1, \dots$

General convergence results for RBF approximations on a domain $\Omega \in \mathbb{R}^d$ have been derived for functions on Native spaces $\mathcal{N}_\Phi(\Omega)$. The Native space is a reproducing kernel Hilbert space with RBFs, i.e., RBFs satisfy reproducing property in the Native space.

Difinition 1 (Reproducing property [6]). A Hilbert space $\mathcal{N}_\Phi(\Omega)$ of functions $f : \Omega \to \mathbb{R}$ is called a reproducing-kernel Hilbert space(RKHS) with a reproducing kernel $\Phi : \Omega \times \Omega \to \mathbb{R}$, if

- $\Phi(\cdot, y) \in \mathcal{N}_\Phi(\Omega)$;

- $f(y) = (\Phi(\cdot, y), f)_{\Phi(\Omega)}$,

for all $f \in \mathcal{N}_\Phi(\Omega)$ and all $y \in \Omega$.

For strictly positive definited basis functions (SPD), such as Gaussian and IMQ, these spaces can be defined as the completion of the pre-Hilbert space

$$F_\Phi(\Omega) := span\{\Phi(\cdot, \mathbf{y}) : \mathbf{y} \in \Omega\} \tag{2.8}$$

and equip this space with the inner product

$$\left(\sum_{i=1}^{N} \lambda_i \Phi(\cdot, \mathbf{x}_i), \sum_{j=1}^{N} \lambda_j \Phi(\cdot, \mathbf{x}_j)\right)_\Phi := \sum_{i,j=1}^{N} \lambda_i \lambda_j \Phi(\mathbf{x}_i - \mathbf{x}_j). \tag{2.9}$$

The Native space for conditionally positive definite basis functions can be defined in a similar form [6]. It is worth pointing out that, the native space $\mathcal{N}_\Phi(\mathbb{R}^d)$ can be characterized using Fourier transforms,

$$\mathcal{N}_\Phi(\mathbb{R}^d) := \left\{ f \in L_2(\mathbb{R}^d) \cap C(\mathbb{R}^d) : \widehat{f}/\sqrt{\widehat{\Phi}} \in L_2(\mathbb{R}^d) \right\}. \tag{2.10}$$

3. Inverse Inequality on $\mathbb{R}^d$

Throughout the paper we assume the set of data points X is quasi-uniform and a generic constant C represents all constants independent of q.

Theorem 3.1. *Assuming an RBF Φ satisfies* (1.2)*, then for any $f = \sum_{j=1}^N \lambda_j \Phi(\cdot - x_j)$ and two real numbers β and τ, the following inequality holds.*

$$\|f\|_{W_2^\tau(\mathbb{R}^d)} \le C q^{\beta - 2\tau} \|f\|_{W_2^\beta(\mathbb{R}^d)}, \quad 0 < \beta < \tau, \tag{3.1}$$

where C is a positive constant independent of q.

Proof. Since Φ satisfies (1.2), the Native space norm and the Sobolev space norm are equivalent. According to the reproducing property, we have

$$\|f\|^2_{W_2^\tau(\mathbb{R}^d)} = \|f\|^2_{\mathcal{N}_\Phi(\mathbb{R}^d)} = \left(\sum_{j=1}^{N} \lambda_j \Phi(\cdot - x_j), \sum_{k=1}^{N} \lambda_j \Phi(\cdot - x_k)\right)_\Phi = \sum_{j,k=1}^{N} \lambda_j \lambda_k \Phi(x_j - x_k). \tag{3.2}$$

Furthermore

$$
\begin{aligned}
(3.3) \qquad \|f\|^2_{W_2^\beta(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} |\widehat{f}|^2 (1+\|\omega\|_2^2)^\beta d\omega \\
&= \int_{\mathbb{R}^d} \sum_{j,k=1}^N \lambda_j \lambda_k e^{-2\pi i (x_j - x_k)^T \omega} (1+||\omega||_2^2)^{-2\tau} (1+\|\omega\|_2^2)^\beta d\omega \\
&= \int_{\mathbb{R}^d} \sum_{j,k=1}^N \lambda_j \lambda_k e^{-2\pi i (x_j - x_k)^T \omega} (1+\|\omega\|_2^2)^{-2\tau+\beta} d\omega \\
&= \sum_{j,k=1}^N \lambda_j \lambda_k \Psi(x_j - x_k),
\end{aligned}
$$

where $\widehat{\Psi}(\omega) \sim (1+\|\omega\|_2^2)^{-2\tau+\beta}$.

According to [6, Theorem 12.3], we have:

$$
\begin{aligned}
(3.4) \qquad C_1 \widehat{\Phi}(C_d/q) q^{-d} \|\lambda\|_2^2 &\leq \sum_{j,k=1}^N \lambda_j \lambda_k \Phi(x_j - x_k) \leq C_2 q^{-d} \|\lambda\|_2^2 \\
C_3 \widehat{\Psi}(C_d/q) q^{-d} \|\lambda\|_2^2 &\leq \sum_{j,k=1}^N \lambda_j \lambda_k \Psi(x_j - x_k) \leq C_4 q^{-d} \|\lambda\|_2^2
\end{aligned}
$$

where $\|\lambda\|_2 = \sqrt{\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_N^2}$.

Since $\widehat{\Phi}(\omega) \sim (1+\|\omega\|_2^2)^{-\tau}$ and $\widehat{\Psi}(\omega) \sim (1+\|\omega\|_2^2)^{-2\tau+\beta}$, so

$$
\begin{aligned}
(3.5) \qquad C_5 q^{2\tau-d} \|\lambda\|_2^2 &\leq \|f\|^2_{W_2^\tau(\mathbb{R}^d)} \leq C_6 q^{-d} \|\lambda\|_2^2 \\
C_7 q^{4\tau-2\beta-d} \|\lambda\|_2^2 &\leq \|f\|^2_{W_2^\beta(\mathbb{R}^d)} \leq C_6' q^{-d} \|\lambda\|_2^2
\end{aligned}
$$

we have:

$$
(3.6) \qquad \|f\|_{W_2^\tau(\mathbb{R}^d)} \leq C q^{\beta-2\tau} \|f\|_{W_2^\beta(\mathbb{R}^d)}, \quad 0 < \beta < \tau.
$$

□

4. **Inverse Inequality on Bounded Domain**

At first, we obtain the relationships between reproducing-kernel Hilbert spaces and the Sobolev Space on a close bounded domain, which is playing an important role in estimating the inverse inequality.

Theorem 4.1. *Assuming an RBF Φ satisfies (1.2), then for any $f = \sum_{j=1}^N \lambda_j \Phi(\cdot - x_j)$ and a real number τ, the following inequality holds.*

$$
(4.1) \qquad \|f\|_{W_2^\tau(\Omega)} \leq C q^{-\tau} \|f\|_{\mathcal{N}_\Phi(\Omega)}.
$$

Proof. Since the set of centers $X = \{x_1, x_2, \ldots, x_N\}$ is quasi-uniform, then $N \sim C_8 q^{-d}$, C is constant, q is the separation distance.

$$\begin{aligned}
\|f\|^2_{W_2^\tau(\Omega)} &= \|\sum_{j=1}^N \lambda_j \Phi(x - x_j)\|^2_{W_2^\tau(\Omega)} \\
&\le \max_j \|\Phi(x - x_j)\|^2_{W_2^\tau(\Omega)} \left(\sum_{j=1}^N |\lambda_j|\right)^2 \\
&\le N \max_j \|\Phi(x - x_j)\|^2_{W_2^\tau(\Omega)} \|\lambda\|_2^2 \\
&\le C_8 q^{-d} \|\lambda\|_2^2.
\end{aligned} \tag{4.2}$$

According to [6, Theorem 12.3], we can obtain

$$C_1 \widehat{\Phi}(C_d/q) q^{-d} \|\lambda\|_2^2 \le \sum_{j,k=1}^N \lambda_j \lambda_k \Phi(x_j - x_k) \le C_2 q^{-d} \|\lambda\|_2^2. \tag{4.3}$$

Since Φ satisfies (1.2), we have

$$\|f\|^2_{\mathcal{N}_\Phi(\Omega)} \ge C_9 q^{2\tau - d} \|\lambda\|_2^2. \tag{4.4}$$

Thus

$$\|f\|_{W_2^\tau(\Omega)} \le C_{10} q^{-\tau} \|f\|_{\mathcal{N}_\Phi(\Omega)}. \tag{4.5}$$

Thus, we obtain the relationship between Sobolev space and Native space on bounded domain Ω. □

Theorem 4.2. *Assuming an RBF Φ satisfies* (1.2)*, then for any $f = \sum_{j=1}^N \lambda_j \Phi(\cdot - x_j)$, there exists a function $g = \sum_{j=1}^N \lambda_j \Psi(\cdot - x_j)$ and the translation invariant kernel Ψ satisfies*

$$c_1(1 + \|\omega\|_2^2)^{-\tau + \beta/2} \le \widehat{\Psi}(\omega) \le c_2(1 + \|\omega\|_2^2)^{-\tau + \beta/2}, \quad \tau - \beta/2 > d/2, \tau > \beta > 0. \tag{4.6}$$

such that

$$\|f\|_{W_2^\beta(\Omega)} \ge C_{11} q^{2\tau - \beta} \|g\|_{\mathcal{N}_\Psi(\Omega)} \tag{4.7}$$

Proof. Since RBFs are translation invariant kernels, we can obtain the result

(4.8)

$$\sum_{|t|\le\beta}|D^t f(x)|^2 = \sum_{|t|\le\beta}|\sum_{j=1}^N \lambda_j D^t\Phi(x-x_j)|^2 \ge C_{11}|\sum_{j=1}^N \lambda_j \sum_{|t|\le\beta} D^t\Phi(x-x_j)|^2$$
$$\ge C_{11}|\sum_{j=1}^N \lambda_j \int_{\mathbb{R}^d}\sum_{|t|\le\beta}|i\omega|^t(1+\|\omega\|_2^2)^{-\tau}e^{2\pi i(x-x_j)^T\omega}d\omega|^2$$

Using $\sum_{|t|\le\beta}(|\omega|^2)^{t/2}\gtrsim(1+|\omega|^2)^{\beta/2}$, we have

$$\ge C_{11}|\sum_{j=1}^N \lambda_j \int_{\mathbb{R}^d}(1+\|\omega\|_2^2)^{-\tau+\beta/2}e^{2\pi i(x-x_j)^T\omega}d\omega|^2$$
$$\ge C_{11}|\sum_{j=1}^N \lambda_j \int_{\mathbb{R}^d}\widehat{\Psi}(\omega)e^{2\pi i(x-x_j)^T\omega}d\omega|^2$$
$$\ge C_{11}|\sum_{j=1}^N \lambda_j \Psi(x-x_j)|^2 \ge C_{11}|g(x)|^2,$$

where $C_{11}=\frac{1}{\beta}$. Integrating both sides of (4.8) with respect to the x variable on Ω, we obtain

$$\|f\|_{W_2^\beta(\Omega)} \ge C_{11}\|g\|_{L_2(\Omega)}. \tag{4.9}$$

Furthermore,

(4.10)

$$\|g\|_{L_2(\Omega)}^2 = \int_\Omega|\sum_{j=1}^N\lambda_j\Psi(x-x_j)|^2dx = \frac{1}{N}\sum_{i=1}^N|\sum_{j=1}^N\lambda_j\Psi(x_i-x_j)|^2$$
$$= \frac{1}{N}\sum_{i=1}^N[\sum_{j,k=1}^N\lambda_j\lambda_k\Psi(x_i-x_j)\Psi(x_i-x_k)]$$

Since $X=\{x_1,\dots,x_N\}$ is quasi-uniform, then $N\sim C_8q^{-d}$.

$$= C_8q^d\sum_{i,j,k=1}^N\lambda_j\lambda_k\int_{R^d}\widehat{\Psi_j\cdot\Psi_k}(\omega)e^{2\pi ix_i^T\omega}d\omega$$
$$= C_8q^d\sum_{i,j,k=1}^N\lambda_j\lambda_k\int_{R^d}\widehat{\Psi_j}*\widehat{\Psi_k}(\omega)e^{2\pi ix_i^T\omega}d\omega$$
$$= C_8q^d\sum_{i,j,k=1}^N\lambda_j\lambda_k\int_{R^d}\int_{R^d}\widehat{\Psi_j}(\omega-\eta)\widehat{\Psi_k}(\eta)d\eta e^{2\pi ix_i^T\omega}d\omega$$
$$= C_8q^d\sum_{i,j,k=1}^N\lambda_j\lambda_k\int_{R^d}\int_{R^d}\widehat{\Psi}(\omega-\eta)\widehat{\Psi}(\eta)e^{2\pi i(x_i-x_j)^T\omega}e^{2\pi i(x_i-x_k)^T\eta}d\omega d\eta.$$

Since the RBF is smooth, its Fourier transform is decreasing and tends to zero at infinity. At first, we define a characteristic function $\chi_{B(0,M)}$, $M > 0$, where $B(0, M)$ is the ball, M is radius, i.e.,

$$\chi_{B(0,M)}(x) = \begin{cases} 1, & x \in B(0, M) \\ 0, & x \notin B(0, M). \end{cases} \tag{4.11}$$

According to the property of the convolution, we have:

$$\begin{aligned} \chi_{B(0,M)} * \chi_{B(0,M)}(x) &= \int_{R^d} \chi_{B(0,M)}(x-y)\chi_{B(0,M)}(y)dy \\ &= \int_{\|y\|\le M} \chi_{B(0,M)}(x-y)dy \\ &= \int_{\|x-y\|\le M, \|y\|\le M} \chi_{B(0,2M)}(x)dx \\ &\le \chi_{B(0,2M)}(x) vol(B(0,2M)). \end{aligned} \tag{4.12}$$

Then

$$\chi_{B(0,2M)}(x) \ge \frac{\chi_{B(0,M)} * \chi_{B(0,M)}(x)}{vol(B(0,2M))} \tag{4.13}$$

Let

$$\begin{aligned} \gamma(x) &= \int_{R^d} (\chi_{B(0,M)} * \chi_{B(0,M)}(\xi)) e^{-2\pi i x^T \xi} d\xi \\ &= |\widehat{\chi_{B(0,M)} * \chi_{B(0,M)}}(x)| \\ &= |\widehat{\chi_{B(0,M)}}(x)|^2, \end{aligned} \tag{4.14}$$

then

$$\begin{aligned} &\sum_{i,j,k=1}^{N} \lambda_j \lambda_k \Psi_j(x_i) \Psi_k(x_i) \\ &= \sum_{i,j,k=1}^{N} \lambda_j \lambda_k \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \widehat{\Psi}(\xi-\eta)\widehat{\Psi}(\eta) e^{-2\pi i (x_i - x_j)^T \eta} e^{-2\pi i (x_i - x_k)^T \xi} d\eta d\xi \\ &\ge \sum_{i,j,k=1}^{N} \lambda_j \lambda_k \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \chi_{B(0,2M)}(\xi)\chi_{B(0,2M)}(\eta)\widehat{\Psi}(\xi-\eta)\widehat{\Psi}(\eta) \\ &\quad e^{-2\pi i (x_i - x_j)^T \eta} e^{-2\pi i (x_i - x_k)^T \xi} d\eta d\xi \\ &\ge \underbrace{\inf_{\|\xi\|\le 2M, \|\eta\|\le 2M} \widehat{\Psi}(\xi-\eta)\widehat{\Psi}(\eta)(vol(B(0,2M)))^{-2}}_{\text{A1}} \underbrace{\sum_{i,j,k=1}^{N} \lambda_i \lambda_j \gamma(x_i - x_k)\gamma(x_i - x_j)}_{\text{A2}} \end{aligned} \tag{4.15}$$

At first, we discuss A_2.

$$\sum_{i,j,k=1}^{N} \lambda_j \lambda_k \gamma(x_i - x_k)\gamma(x_i - x_j) \geq \gamma^2(0)\|\lambda\|_2^2 - \gamma(0)\|\lambda\|_2^2 \sum_{j\neq k} |\gamma(x_i - x_k)| - \gamma(0)\|\lambda\|_2^2 \sum_{i\neq j} |\gamma(x_i - x_j)| - \|\lambda\|_2^2 \sum_{i\neq j} |\gamma(x_i - x_j)| \sum_{i\neq j} |\gamma(x_i - x_j)| - \|\lambda\|_2^2 \sum_{i\neq j} |\gamma(x_i - x_j)| \sum_{i\neq k} |\gamma(x_i - x_k)|. \tag{4.16}$$

For $\sum_{i\neq j} |\gamma(x_i - x_j)|$, we can use the same technique in [6, Theorem 12.3] to evaluate its upper bound.

$$\sum_{i\neq j} |\gamma(x_i - x_j)| \leq \gamma(0)\frac{\Gamma^2(d/2+1)\pi}{18} \cdot \left(\frac{12}{Mq}\right)^{d+1}. \tag{4.17}$$

Thus

$$\sum_{i,j,k=1}^{N} \lambda_i \lambda_j \gamma(x_i - x_k)\gamma(x_j - x_i) \geq \gamma^2(0)\|\lambda\|_2^2 \left\{1 - 2\frac{\Gamma^2(d/2+1)\pi}{18} \cdot (\frac{12}{Mq})^{d+1} - 2(\frac{\Gamma^2(d/2+1)\pi}{18} \cdot (\frac{12}{Mq})^{d+1})^2\right\}. \tag{4.18}$$

Let $\frac{\Gamma^2(d/2+1)\pi}{18} \cdot \left(\frac{12}{Mq}\right)^{d+1} = \frac{1}{4}$, then $M = \frac{C_{13}}{q}$.

We obtain the following inequality

$$\sum_{i,j,k=1}^{N} \lambda_i \lambda_j \gamma(x_i - x_k)\gamma(x_j - x_i) \geq C_{14} q^{-4d} \|\lambda\|_2^2. \tag{4.19}$$

Since

$$\begin{aligned} \widehat{\Psi}(\xi - \eta)\widehat{\Psi}(\eta) &\geq c_1^2 (1 + \|\xi - \eta\|^2)^{-\tau} (1 + \|\eta\|^2)^{-\tau} \\ &= c_1^2 \frac{1}{((1 + \|\xi - \eta\|^2)(1 + \|\eta\|^2))^{\tau}} \\ &\geq c_1^2 \frac{1}{(((1 + \|\xi - \eta\|^2)^2 + (1 + \|\eta\|^2)^2)/2)^{\tau}} \end{aligned} \tag{4.20}$$

and the last expression can and will be equal if and only if $\|\xi - \eta\| = \|\eta\|$. Thus, for A1, we have

$$\inf_{\|\xi\|\leq 2M, \|\eta\|\leq 2M} \widehat{\Psi}(\xi - \eta)\widehat{\Psi}(\eta) = |\widehat{\Psi}(2M)|^2. \tag{4.21}$$

$$\|g\|_{L_2(\Omega)}^2 \geq C_{14} q^d |\widehat{\Phi}(\frac{C_{15}}{q})|^2 q^{-2d} \|\lambda\|_2^2 = C_{14} q^{4\tau - 2\beta - d} \|\lambda\|_2^2,\ C_{15} = 2C_{13}. \tag{4.22}$$

Since

$$\|g\|_{\mathcal{N}_\Psi(\Omega)}^2 = \sum_{j,k=1}^{N} \lambda_j \lambda_k \Phi(x_j - x_k) \leq C_{16} q^{-d} \|\lambda\|_2^2, \tag{4.23}$$

so, we obtain

$$\|g\|_{\mathcal{N}_\Psi(\Omega)} \le C_{17} q^{\beta-2\tau} \|g\|_{L_2(\Omega)} \tag{4.24}$$

Then using (4.9), (4.24) becomes:

$$\|f\|_{W_2^\beta(\Omega)} \ge C_{18} q^{2\tau-\beta} \|g\|_{\mathcal{N}_\Psi(\Omega)} \tag{4.25}$$

□

Lemma 4.3. *Assuming an RBF Φ satisfies* (1.2), *then for any $f = \sum_{j=1}^N \lambda_j \Phi(\cdot - x_j)$ and two real numbers τ and β, the following inequality holds.*

$$\|f\|_{W_2^\tau(\Omega)} \le C q^{3\beta/2-4\tau} \|f\|_{W_2^\beta(\Omega)} \tag{4.26}$$

where $\tau > \beta > 0$,and $\tau - \beta/2 > d/2$.

Proof. According to the following two inequalities

$$\|f\|_{\mathcal{N}_\Phi(\Omega)} = \sum_{j,k=1}^N \lambda_j \lambda_k \Phi(x_j - x_k) \le C_2 q^{-d} \|\lambda\|_2^2 \tag{4.27}$$

$$\|g\|_{\mathcal{N}_\Psi(\Omega)} = \sum_{j,k=1}^N \lambda_j \lambda_k \Psi(x_j - x_k) \ge C_1 \widehat{\Psi}(C_d/q) q^{-d},$$

we have

$$\|f\|_{\mathcal{N}_\Phi(\Omega)} \le C_{19} q^{\beta-2\tau} \|g\|_{\mathcal{N}_\Psi(\Omega)}. \tag{4.28}$$

Combining (4.5), (4.7) with (4.28), we can obtain

$$\|f\|_{W_2^\tau(\Omega)} \le C_{20} q^{3\beta/2-4\tau} \|f\|_{W_2^\beta(\Omega)} \tag{4.29}$$

□

5. **Numerical Experiments**

In these experiments, we use the global functions and compactly supported functions. We present six experiments. The first two experiments aim to test (3.1) in 1D and 2D. The other four experiments focus on verifying (4.5).

Example 5.1. This example aims to test the convergence rate $O(q^{rate})$ in (3.1). It is computed using the formula

$$rate_k = \frac{\ln(e_{k-1}/e_k)}{\ln(q_{k-1}/q_k)}, k \in \mathbb{N}^+ > 1, \tag{5.1}$$

where $e_k = \dfrac{\|f\|_{W_2^\tau(\mathbb{R}^d)}}{\|f\|_{W_2^\beta(\mathbb{R}^d)}}$ and q_k is the separation distance of the k-th computation mesh, $rate_k$ is the order of q_k.

1D case: The first experiment is set up as follows: the basis function is a compactly supported RBF $\Phi(r) = (1-r)_+^3(3r+1)$ and the compactly supported RBF

$\Psi(r) = (1-r)_+^5(8r^2+5r+1)$ will be chosen as another RBF. Since $\widehat{\Phi}(\omega) \sim \|\omega\|^{-4}$, then $\tau = 2$. For Ψ, its Fourier transform satisfies $\widehat{\Psi}(\omega) \sim \|\omega\|^{-6}$. According to (3.1), there exists the inequality

$$\|f\|_{W_2^2(\mathbb{R}^d)} \le Cq^{-3}\|f\|_{W_2^1(\mathbb{R}^d)}. \tag{5.2}$$

The numerical results are given in Table 3.

$\Phi(r) = (1-r)_+^3(3r+1)$ and $\Psi(r) = (1-r)_+^5(8r^2+5r+1)$

k	N	e_k	$rate_k$	k	N	e_k	$rate_k$
1	20	39.4224		9	300	1.5519e+005	-3.1053
2	25	76.9539	-2.8631	10	350	2.5010e+005	-3.0862
3	50	649.0227	-2.9873	11	400	3.7591e+005	-3.0435
4	100	5.2269e+003	-2.9663	12	450	5.3430e+005	-2.9782
5	150	1.7962e+004	-3.0195	13	500	7.2559e+005	-2.8984
6	200	4.3848e+004	-3.0843	14	550	9.4949e+005	-2.8164
8	250	8.7919e+004	-3.1037	15	600	1.2061e+006	-2.7445

TABLE 3. $\|f\|_{W_2^2(\mathbb{R}^d)} \le Cq^{-3}\|f\|_{W_2^1(\mathbb{R}^d)}$, the exact *rate*=-3

The second experiment is set up as follows: the basis function is a compactly supported RBF $\Phi(r) = (1-r)_+^6(35r^2+18r+3)$ and the compactly supported RBF $\Psi(r) = (1-r)_+^8(32r^3+25r^2+8r+1)$ will be chosen as another RBF. Since $\widehat{\Phi}(\omega) \sim \|\omega\|^{-6}$, then $\tau = 3$. For Ψ, its Fourier transform satisfies $\widehat{\Psi}(\omega) \sim \|\omega\|^{-8}$. According to (3.1), there exists the inequality

$$\|f\|_{W_2^3(\mathbb{R}^d)} \le Cq^{-4}\|f\|_{W_2^2(\mathbb{R}^d)}. \tag{5.3}$$

The numerical results are given in Table 4.

$\Phi(r) = (1-r)_+^6(35r^2+18r+3)$ and $\Psi(r) = (1-r)_+^8(32r^3+25r^2+8r+1)$

k	N	e_k	$rate_k$	k	N	e_k	$rate_k$
1	20	37.8880		8	300	2.2839e+006	-3.9989
2	25	96.5191	-4.0028	9	350	4.2385e+006	-3.9988
3	50	1.6558e+003	-3.9821	10	400	7.2404e+006	-3.9993
4	100	2.7502e+004	-3.9954	11	450	1.1607e+007	-3.9973
5	150	1.4099e+005	-3.9979	12	500	1.7703e+007	-3.9980
6	200	4.4838e+005	-3.9983	13	550	9.4949e+005	-2.8164
7	250	1.0987e+006	-3.9984	14	600	1.2061e+006	-2.7445

TABLE 4. $\|f\|_{W_2^3(\mathbb{R}^d)} \le Cq^{-4}\|f\|_{W_2^2(\mathbb{R}^d)}$, the exact *rate*=-4.

2D case: The third experiment is set up as follows: the basis function is a global RBF $\Phi(r) = r^3$ and the global RBF $\Psi(r) = r^5$ will be chosen as another RBF. Since

$\widehat{\Phi}(\omega) \sim \|\omega\|^{-5}$, then $\tau = 5/2$. For Ψ, its Fourier transform satisfies $\widehat{\Psi}(\omega) \sim \|\omega\|^{-7}$. According to (3.1), there exists the inequality

$$\|f\|_{W_2^{5/2}(\mathbb{R}^d)} \leq Cq^{-7/2}\|f\|_{W_2^{3/2}(\mathbb{R}^d)}. \tag{5.4}$$

The numerical results are given in Table 5.

$\Phi(r) = r^3$ and $\Psi(r) = r^5$

k	N	e_k	$rate_k$	k	N	e_k	$rate_k$
1	6^2	152.2112		11	16^2	6.2468e+003	-3.3904
2	7^2	281.9554	-3.3813	12	17^2	7.7765e+003	-3.3939
3	8^2	476.7675	-3.4076	13	18^2	9.5554e+003	-3.3980
4	9^2	747.1584	-3.3644	14	19^2	1.1606e+004	-3.4013
5	10^2	1.1120e+003	-3.3760	15	20^2	1.3952e+004	-3.4050
6	11^2	1.5857e+003	-3.3681	16	21^2	1.6617e+004	-3.4079
7	12^2	2.1874e+003	-3.3752	17	22^2	1.9627e+004	-3.4122
8	13^2	2.9345e+003	-3.3768	18	23^2	2.3005e+004	-3.4137
9	14^2	3.8469e+003	-3.3823	19	24^2	2.6778e+004	-3.4165
10	15^2	4.9439e+003	-3.3854	20	25^2	3.0973e+004	-3.4196

TABLE 5. $\|f\|_{W_2^{5/2}(\mathbb{R}^d)} \leq Cq^{-7/2}\|f\|_{W_2^{3/2}(\mathbb{R}^d)}$, the exact $rate$=-7/2

Example 5.2. This example aims to test the convergence rate $O(q^{rate})$ in (4.26). It is computed using the formula

$$rate_k = \frac{\ln(e_{k-1}/e_k)}{2\ln(q_{k-1}/q_k)}, k \in \mathbb{N}^+ > 1, \tag{5.5}$$

where $e_k = \frac{T1_k}{T2_k}$ and $T1 = \max \frac{\|f\|^2_{W_2^{\tau}(\Omega)}}{\|\lambda\|_2^2}$, $T2 = \min \frac{\|f\|_{W_2^{\beta}(\Omega)}}{\|\lambda\|_2^2}$.

1D case: We chose global RBF $\Phi(r) = r^5$, $\Phi(r) = r^3$ and compactly supported RBF $\Phi(r) = (1-0.5r)_+^4(4*0.5r+1)$ as test basis functions. According to (4.26), the convergence rates will be -3, -2 and -2, respectively. The numerical results can be seen in Tables 6, 7 and 8, respectively.

$\Phi(r) = r^5$

k	N	$T1/T2$	$rate_k$	k	N	$T1/T2$	$rate_k$
1	31	2.7876e+010		11	41	1.5216e+011	-2.9526
2	32	3.3823e+010	-2.9487	12	42	1.7607e+011	-2.9553
3	33	4.0779e+010	-2.9454	13	43	2.0300e+011	-2.9531
4	34	4.8896e+010	-2.9496	14	44	2.3330e+011	-2.9562
5	35	5.8302e+010	-2.9468	15	45	2.6725e+011	-2.9548
6	36	6.9182e+010	-2.9514	16	46	3.0524e+011	-2.9572
7	37	8.1686e+010	-2.9488	17	47	3.4761e+011	-2.9570
8	38	9.6029e+010	-2.9520	18	48	3.9477e+011	-2.9578
9	39	1.1240e+011	-2.9514	19	49	4.4713e+011	-2.9579
10	40	1.3103e+011	-2.9520	20	50	5.0517e+011	-2.9595

TABLE 6. the exact $rate = -3$.

$\Phi(r) = r^3$

k	N	$T1/T2$	$rate_k$	k	N	$T1/T2$	$rate_k$
1	31	1.2203e+007		11	41	3.7772e+007	-1.9666
2	32	1.3878e+007	-1.9613	12	42	4.1628e+007	-1.9683
3	33	1.5718e+007	-1.9608	13	43	4.5771e+007	-1.9686
4	34	1.7736e+007	-1.9627	14	44	5.0213e+007	-1.9682
5	35	1.9941e+007	-1.9627	15	45	5.4971e+007	-1.9690
6	36	2.2345e+007	-1.9633	16	46	6.0062e+007	-1.9707
7	37	2.4960e+007	-1.9643	17	47	6.5498e+007	-1.9711
8	38	2.7798e+007	-1.9652	18	48	7.1294e+007	-1.9713
9	39	3.0870e+007	-1.9652	19	49	7.7465e+007	-1.9715
10	40	3.4192e+007	-1.9673	20	50	8.4028e+007	-1.9721

TABLE 7. the exact $rate = -2$.

$\Phi(r) = (1-0.5r)_+^4(4*0.5r+1)$

k	N	$T1/T2$	$rate_k$	k	N	$T1/T2$	$rate_k$
1	31	5.0927e+005		11	41	1.6123e+006	-2.0219
2	32	5.8148e+005	-2.0219	12	42	1.7792e+006	-1.9946
3	33	6.5957e+005	-1.9845	13	43	1.9579e+006	-1.9858
4	34	7.4681e+005	-2.0185	14	44	2.1532e+006	-2.0204
5	35	8.4046e+005	-1.9787	15	45	2.3590e+006	-1.9853
6	36	9.4515e+005	-2.0249	16	46	2.5827e+006	-2.0157
7	37	1.0580e+006	-2.0019	17	47	2.8182e+006	-1.9852
8	38	1.1793e+006	-1.9808	18	48	3.0712e+006	-1.9988
9	39	1.3124e+006	-2.0050	19	49	3.3427e+006	-2.0118
10	40	1.4554e+006	-1.9908	20	50	3.6266e+006	-1.9767

TABLE 8. the exact $rate = -2$.

2D case: We still chose global RBFs $\Phi(r) = r^5$, $\Phi(r) = r^3$ and compactly supported RBF $\Phi(r) = (1-0.5r)_+^4(4*0.5r+1)$ as test basis functions. According to (4.26), the convergence rates will be -3.5, -2.5 and -2.5, respectively. The numerical results can be seen in Tables 9, 10 and 11, respectively.

$\Phi(r) = r^5$

k	N	$T1/T2$	$rate_k$	k	N	$T1/T2$	$rate_k$
1	4	38.2379		7	64	4.3731e+007	-3.3439
2	9	1.0261e+004	-4.0340	8	81	1.0469e+008	-3.2687
3	16	1.3460e+005	-3.1741	9	100	2.3015e+008	-3.3440
4	25	9.5440e+005	-3.4044	10	121	4.6348e+008	-3.3221
5	36	4.6578e+006	-3.5520	11	144	8.7497e+008	-3.3335
6	49	1.5598e+007	-3.3144	12	169	1.5639e+009	-3.3372

TABLE 9. the exact $rate = -3.5$.

$\Phi(r) = r^3$

k	N	$T1/T2$	$rate_k$	k	N	$T1/T2$	$rate_k$
1	4	52.2650		7	64	5.4827e+005	-2.3081
2	9	1.3989e+003	-2.3712	8	81	1.0082e+006	-2.2809
3	16	1.1714e+004	-2.6206	9	100	1.7670e+006	-2.3820
4	25	4.2304e+004	-2.2318	10	121	2.8989e+006	-2.3493
5	36	1.1707e+005	-2.2808	11	144	4.5218e+006	-2.3323
6	49	2.6912e+005	-2.2828	12	169	6895600	-2.4248

TABLE 10. the exact $rate = -2.5$.

$\Phi(r) = (1-0.5r)_+^4(4*0.5r+1)$

k	N	$T1/T2$	$rate_k$	k	N	$T1/T2$	$rate_k$
1	4	1.2163		7	64	1.3804e+003	-2.5088
2	9	4.2965	-0.9103	8	81	2.6670e+003	-2.4660
3	16	17.2410	-1.7134	9	100	4.9895e+003	-2.6591
4	25	83.1422	-2.7344	10	121	8.0933e+003	-2.2955
5	36	242.9310	-2.4026	11	144	1.3791e+004	-2.7961
6	49	636.9264	-2.6433	12	169	2.0444e+004	-2.2622

TABLE 11. the exact $rate = -2.5$.

6. **Conclusion**

In this paper, new inverse inequalities on interpolation of scattered data via RBFs are presented. New inequalities are based on translation invariant and smoothness of RBFs. Comparing with existing inverse inequalities on $\mathbb{R}^d$ and Ω, the results in this paper can be easily verified by numerical experiments.

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