Neural, Parallel, and Scientific Computations 25 (2017) 229-250

DETECTING CHANGE-POINTS FOR POISSON PROCESSES WITH FUNCTIONAL DATA

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ABSTRACT. In this paper, we consider i.i.d. Poisson processes with some change-point at $t_0 \in]0,1[$. We study existence and intensity of jump at t_0 and we want to detect this point. We consider the case of either continuous or discretely observed data. For continuous data, our obtained exponential bounds imply that t_0 is detectable (almost surely). In the case of sampled data, a consistent estimator of t_0 is given in two cases: fixed sampling rate or high frequency data. Also, estimation of intensities of the Poisson processes and of the jump at t_0 are discussed and exponential bounds are derived. Finally, the case of a random change-point T_0 is also examined. Estimators of each Poisson's intensity are proposed and studied with almost sure rate of convergence.

Key Words Functional process, change-point, detecting position of jump, intensity of jump, exponential rate

2010 MSC Primary 62M, 60F17

1. INTRODUCTION

In this paper, our purpose is to detect the presence of a change-point t_0, fixed or random, in the context of i.i.d. Poisson processes on $[0,1]$ and to study the intensity of jump at this point. The functional autoregressive linear process appear in the context of Poisson processes that model their innovation. We refer for example to [4, 5], [8] for general $D[0,1]$-valued linear processes as well to [9] for jumps in derivatives.

Works dedicated to jumps in stochastic processes are numerous, we may only give recent and limited references. First, fixed jumps may be considered per day with, for example, financial data, cf [25, p. 208], for intra-day cumulative returns, or electricity consumption, see e.g. [21] where jumps are present early in the morning and in the evening. Random jumps can be also envisaged in the context of Poisson or, more generally, Levy processes. Applications appear in various domains such as epidemic models [31], financial modeling [12, 13, 20, 24], pollution damage [16], chemical reaction network [1]. Also, a deterministic model appears in the context of dengue disease, cf [22], or actuarial risk with rare events [23].

Received April 4, 2017

In this paper, we consider Poisson processes on [0, 1] with an inner change-point at t_0. More precisely, let $(N_t,\, t \in [0,1[)$ (resp. $(N'_t,\, t \in [0,1[)$) be a Poisson process with intensity λt (resp. $\lambda' t$) and such that:

$$Z_0(t) = N_t\, \mathbb{1}_{\{t<t_0\}} + N'_t\, \mathbb{1}_{\{t\geq t_0\}}, \quad 0 \leq t < 1,\ 0 < t_0 < 1$$

We suppose that (N_t) and (N'_t) are independent and we try to detect t_0 from the copies $Z_i(t)$:

$$Z_i(t) = (N_{i+t} - N_i)\mathbb{1}_{\{t<t_0\}} + (N'_{i+t} - N'_i)\mathbb{1}_{\{t\geq t_0\}}, \quad 0 \leq t < 1,\ 1 \leq i \leq n.$$

Literature about the change point models is extensive and is developed around several approaches: either parametric or nonparametric with, for example, location of a shift in the mean or variance of the distribution. The methodologies often assume a single or known number of change points and then, the likelihood function plays a major role, see for example the results collected in the book of [14]. In the context of Poisson processes with detection of abrupt change-points, we may refer to [18] and for more recent advances to [11], [17], [28] and [29]. As in [15] and [26], our approach is based of i.i.d processes with change-point in their intensities. However, the latter references consider the case of non-homogeneous Poisson processes while our model allows, for example, negative jumps. Also our methodology is different as our $Z'_i s$ correspond to the functional innovation of an autoregressive processes with values in $D[0, 1]$.

The plan of the paper is the following. In Section 2, first we introduce functional autoregressive processes, see [7] and [10] for usual properties of functional linear processes. We observe that the jumps of the process coincide with those of the innovation $(Z_i,\, i = 1, \ldots, n)$. We study existence and intensity of jumps at change-point t_0 and set

$$J_0(t_0) = N'(t_0) - N(t_0-).$$

Note that $J_0(t_0)$ can have a value which may be zero, positive or negative (cf. Proposition 2.7). Next, we study $P(\cap_{i=1}^n (J_i(t_0) = k_i))$ where k_i is an integer and

$$J_i(t_0) = Z_i(t_0) - Z_i(t_0-),$$

as well as $J_i(t_0)$ which belongs to $\mathbb{Z}$, for a jump at $D([0, 1])$, see [3]. In this context we derive exponential bounds for probabilities of the occurrence of jumps at t_0.

In Section 3, we consider change-points corresponding to no jump or jump equal to 1. We derive bounds for these probabilities implying conditions under which t_0 is detectable. We study the case of Poisson processes with either different or the same intensity (see Lemma 3.3).

In Section 4, one obtains consistency results in mean square and almost surely for estimating the intensities $(\lambda,\, \lambda')$ of Poisson processes and the jump's intensity

$E(J_0(t_0))$. For the latter quantity, we also derive a Bernstein's inequality (cf. Proposition 4.2).

In Section 5, a central limit theorem holds in an Hilbertian context.

Section 6 is devoted to detection of t_0 in the case of discretely observed processes. In the first step, we suppose that the sampling rate of observation, $\delta > 0$, is fixed. In the second, we consider the high frequency case, with $\delta_n \to 0$, and one obtains consistency at t_0 (cf. Proposition 6.2) Also, a little simulation study is given: some cases are envisaged with various λ, λ' and δ.

Finally, in Section 7 we consider the case of a random change-point and we obtain the estimation of the Poisson processes intensities with an exponential rate (cf. Propositions 7.3 and 7.4). Technical proofs are postponed to the appendix.

2. STUDY OF JUMPS

2.1. **Functional autoregressive processes.** In order to study the jumps of a functional continuous time process $X = (X_t,\ 0 \le t \le 1)$, the space $D = D([0,1])$ of càdlàg real functions defined over $[0,1]$ is well adapted, see [3]. Consider for example the ARD(1) process defined as

$$X_n = m_Z + \rho(X_{n-1} - m_Z) + (Z_n - m_Z), \quad n \in \mathbb{Z},$$

where ρ is a bounded linear operator, the sequence (Z_n) is i.i.d. and such that $E\|Z_n\|^2 < \infty$, $m_Z = E(Z_n)$. If there exists $\ell \ge 1$ such that $\|\rho^\ell\|_{\mathcal{L}} < 1$, then X_n is stationary and

$$X_n - m_Z =_{L^2_D} \sum_{j=0}^{\infty} \rho^j (Z_{n-j} - m_Z) = Z_n - m_Z + \sum_{j=1}^{\infty} \rho^j (Z_{n-j} - m_Z),$$

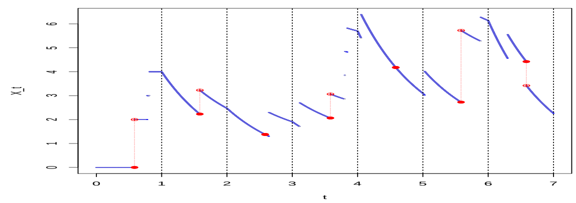
see Lemma 2.1 in [5]. Next, the following proposition holds.

Proposition 2.1. If $\rho(D) \subset C$, we get

$$X_n(t) - X_n(t-) = Z_n(t) - Z_n(t-),\ 0 \le t \le 1,\ n \in \mathbb{Z}.$$

Proof. Clear. □

Examples of processes fulfilling conditions of Proposition 2.1, such as the Ornstein-Uhlenbeck process driven by a Levy process or operator ρ with integral representation, are developed in [5] and [9]. Note that (X_n) are correlated functional variables but with jumps similar to (Z_n). In the sequel, we model (Z_n) as in Section 1 by two independent Poisson processes with intensities (λ, λ') and a change-point occurring at time $t_0 \in]0,1[$. Below, we represent an illustration of a simulated sample of $X_n(t)$, $n = 1, \dots, 7$, with the choices $\lambda = 1$ and $\lambda' = 3$. We may observe that either jumps may not exist or be positive or negative at points t_0 (represented by circles).



Note that Proposition 2.1 gives that $X_n(t) - X_n(t-) = Z_n(t) - Z_n(t-) \neq 0$ if t is a time of jump. In this case, $X_n(t) - X_n(t-)$ are observed i.i.d random variables and we may focus to the study of the i.i.d. $(Z_n(t) - Z_n(t-),\ n \geq 1)$.

2.2. **Notation and Assumptions.** Now we consider a sequence of i.i.d random processes, defined as

$$Z_n(t) = (N_{n+t} - N_n)\mathbb{1}_{\{t<t_0\}} + (N'_{n+t} - N'_n)\mathbb{1}_{\{t\geq t_0\}}, \quad 0 \leq t < 1,\ 0 < t_0 < 1,\ n \in \mathbb{Z}.$$

Remark 2.2. Note that it is also possible to get a jump per day for $t_0 \in \mathbb{Z}$, see [25, p. 208]. In fact, since

$$P(\cap_{i=1}^{n}(N'_{i+1(-)} - N'_i) = 0) = \exp(-n\lambda'), \quad n \geq 1$$

a jump appears almost surely for n large enough. Now, by using the empirical mean, we get

$$\bar{Z}_n(1-) = \frac{1}{n}\sum_{i=1}^{n}(N'_{i+1(-)} - N'_i)$$

and one obtains an exponential rate from Bernstein's inequality (see Lemma 2.4).

We envisage the following assumption:

Assumption A1. $(N_{n+t} - N_n, 0 \leq t < 1, n \in \mathbb{Z})$ *and* $(N'_{n+t} - N'_n, 0 \leq t < 1, n \in \mathbb{Z})$ *are globally independent.*

Lemma 2.3. We get $EZ_n(t) = \lambda t\,\mathbb{1}_{\{t<t_0\}} + \lambda' t\,\mathbb{1}_{\{t\geq t_0\}}$ and if A1 holds, the same result holds for the variance.

Proof. Clear. □

This lemma implies that EX_n, defined by $m_Z(\cdot)$, is also such that $m_Z(t) = \lambda t\,\mathbb{1}_{\{t<t_0\}} + \lambda' t\,\mathbb{1}_{\{t\geq t_0\}}$.

2.3. **Some useful lemmas.** We begin with two classical exponential inequalities for sums of independent variables.

Lemma 2.4. Let $\zeta_1, \dots, \zeta_n$ be independent zero-mean real-valued random variables and set $S_n = \sum_{i=1}^{n}\zeta_i$. The following inequalities hold:

- Hoeffding's inequality: If $a_i \le \zeta_i \le b_i, 1 \le i \le n$ where $a_1, b_1, \dots, a_n, b_n$ are constant, then

$$P(|S_n| \ge t) \le 2 \exp\left(-\frac{2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right), \; t > 0.$$

- Bernstein's inequality: If there exists $c > 0$ s.t. $E|\zeta_i|^k \le c^{k-2} k! E\zeta_i^2 < \infty$, $1 \le i \le n$, $k \ge 3$ (Cramer's condition), then

$$P(|S_n| \ge t) \le 2 \exp\Big(-\frac{t^2}{4\sum_{i=1}^n E\zeta_i^2 + 2ct}\Big).$$

Proof. See [6, p. 24–26]. □

In this part, we examine the behaviour of $J_0(t_0) = Z_0(t_0) - Z_0(t_0-)$ for detecting possible jumps at t_0.

Lemma 2.5. Under A1,

$$P(J_0(t_0) = k) = \exp(-(\lambda + \lambda')\, t_0)\, (\lambda' t_0)^k \sum_{h=0}^{\infty} \frac{(\lambda\lambda' t_0^2)^h}{h!(k+h)!}, \; k \ge 0$$

and

$$P(J_0(t_0) = -k) = \exp(-(\lambda + \lambda')\, t_0)\, (\lambda t_0)^k \sum_{h=0}^{\infty} \frac{(\lambda\lambda' t_0^2)^h}{h!(k+h)!}, \; k \ge 0.$$

Proof. We have

$$P(J_0(t_0) = k) = \sum_{h=0}^{\infty} P(N'_{t_0} = k + h, \, N_{t_0} = h)$$

next, from A1,

$$P(J_0(t_0) = k) = \sum_{h=0}^{\infty} \exp(-\lambda' t_0)\, \frac{(\lambda' t_0)^{h+k}}{(h+k)!}\, \exp(-\lambda t_0)\, \frac{(\lambda t_0)^h}{h!}$$

and the first result follows. The second result is similar since $P(J_0(t_0) = -k) = P(N(t_0-) - N'(t_0) = k)$ so λ and λ' are inverted. □

Now, we get

$$P(J_0(t_0) = k) = p_k, \quad 0 < p_k < 1, \; k \in \mathbb{N}$$

with $\sum_{k\in\mathbb{N}} p_k = 1$. For $J_i(t_0) = Z_i(t_0) - Z_i(t_0-)$, this implies that

$$P\left(\cap_{i=1}^n \{J_i(t_0) = 0 \cup J_i(t_0) = 1\}\right) = (p_0 + p_1)^n$$

with $p_0 + p_1 < 1$. Next, from the BC lemma, almost surely for n large enough, there exist at least one sample path Z_i with a jump distinct from 1 at t_0. By this way, for n large enough, t_0 is observed. But from Lemma 2.5, it appears that the p_k's are rather intricate. Then, we need a more simple exponential inequality. First, in the following statement, we examine the case where no jump holds:

Lemma 2.6. Under A1, we get

$$P(J_0(t_0)=0)=\exp(-(\lambda+\lambda')\,t_0)\ \sum_{h=0}^{\infty}\frac{(\lambda\lambda' t_0^2)^h}{(h!)^2}$$

and, if $\lambda\neq\lambda'$, we obtain the bound:

$$P(J_0(t_0)=0)\le \exp(-(\sqrt{\lambda}-\sqrt{\lambda'})^2\ t_0).$$

Proof. From Lemma 2.5, we obtain

$$P(J_0(t_0)=0)=\exp(-(\lambda+\lambda')\,t_0)\ \sum_{h=0}^{\infty}\frac{(\lambda\lambda' t_0^2)^h}{(h!)^2}.$$

Now, by using the Gamma function we get for some $c>0$:

$$h!=\int_0^{\infty} t^h\,\exp(-t)\,dt\ge\int_{ct_0}^{\infty} t^h\,\exp(-t)\,dt\ge (ct_0)^h\exp(-ct_0).$$

Reporting in the bound, we obtain

$$P(J_0(t_0)=0)\le \exp\left(-(\lambda+\lambda')\,t_0+ct_0\right)\sum_{h=0}^{\infty}\frac{(\lambda\lambda' t_0)^h}{c^h(h!)}$$

thus

$$P(J_0(t_0)=0)\le \exp\left(-t_0\left[\lambda+\lambda'-c-\frac{\lambda\lambda'}{c}\right]\right).$$

The result follows with the choice $c=\sqrt{\lambda\lambda'}$. □

Proposition 2.7. If A1 holds and $\lambda\neq\lambda'$ we have

$$P(J_i(t_0)=0,\,1\le i\le n)\le\exp(-n\,(\sqrt{\lambda}-\sqrt{\lambda'})^2\ t_0)$$

then, the probability that $\{J_i(t_0)=0,\,1\le i\le n\}$ occurs infinitely often is zero.

Proof. Under A1 we have

$$P(J_i(t_0)=0,\,1\le i\le n)=\prod_{i=1}^{n}P(J_i(t_0)=0)$$

and Lemma 2.6 entails the result.

Finally $\sum_{n\ge1}\exp(-n\,(\sqrt{\lambda}-\sqrt{\lambda'})^2\ t_0)<\infty$ gives the Borel-Cantelli lemma. □

Remark 2.8. The probability of having at least one jump is strong. For example, if $t_0=\frac{1}{2}$, $\lambda=1$, $\lambda'=2$, $n=100$ the probability of at least one jump is about $1-1.88\,10^{-4}$. If $t_0=\frac{1}{2}$, $\lambda=1$, $\lambda'=11$, $n=10$, the probability of at least one jump is about $1-2.22.10^{-12}$.

We may derive a lower bound for the probability of having a jump at t_0 in the following statement.

Corollary 2.9. We have

$$P(\bigcup_{i=1}^{n}[(Z_i(t_0) - Z_i(t_0-) \geq 1]) \geq 1 - \exp(-n(\sqrt{\lambda} - \sqrt{\lambda'})^2\, t_0).$$

Proof. Note that $P(\cup_{i=1}^n A_i) = 1 - \prod_{i=1}^n P(\bar{A}_i)$ hence

$$P(\bigcup_{i=1}^{n}[(Z_i(t_0) - Z_i(t_0-) \geq 1]) = 1 - \prod_{i=1}^{n} P(Z_i(t_0) - Z_i(t_0-) = 0)$$

and Proposition 2.7 gives the result. □

Finally, we consider the general case of jumps with value k or $-k$.

Proposition 2.10. Under A1 and if $\lambda \neq \lambda'$, one obtains

$$P(J_0(t_0) = k) \leq \Big(\sqrt{\frac{\lambda'}{\lambda}}\Big)^k \exp(-(\sqrt{\lambda} - \sqrt{\lambda'})^2 t_0), \quad k \geq 0$$

and

$$P(J_0(t_0) = -k) \leq \Big(\sqrt{\frac{\lambda}{\lambda'}}\Big)^k \exp(-(\sqrt{\lambda} - \sqrt{\lambda'})^2\, t_0), \quad k \geq 0.$$

Proof. Similarly as in the proof of Lemma 2.6, we may write

$$(k+h)! = \int_0^\infty t^{k+h} \exp(-t)\, dt \geq \int_{ct_0}^\infty t^{k+h} \exp(-t)\, dt \geq (ct_0)^{h+k} \exp(-c\, t_0)$$

and by using Lemma 2.5, we find

$$P(J_0(t_0) = k) \leq \exp(-(\lambda + \lambda')\, t_0)\, (\lambda' t_0)^k (ct_0)^{-k} \exp(ct_0) \sum_{h=0}^{\infty} \frac{(\lambda\lambda' t_0^2)^h}{(ct_0)^h} \frac{1}{h!}$$

then

$$P(J_0(t_0) = k) \leq \left(\sqrt{\frac{\lambda'}{\lambda}}\right)^k \exp\left(-t_0\left(\lambda + \lambda' - \frac{\lambda\lambda'}{c} - c\right)\right)$$

and the choice $c = \sqrt{\lambda\lambda'}$ gives the result. The second proof is similar. □

Corollary 2.11. Under A1 and if $k_1, \ldots, k_n \in \mathbb{Z}$,

$$P(J_i(t_0) = k_i,\, 1 \leq i \leq n) \leq \left(\frac{\lambda'}{\lambda}\right)^{\sum_{i=1}^n k_i (\mathbb{1}_{\{k_i > 0\}} - \mathbb{1}_{\{k_i < 0\}})} \exp(-n(\sqrt{\lambda} - \sqrt{\lambda'})^2\, t_0)$$

Proof. First, from A1 we get

$$P(J_i(t_0) = k_i,\, 1 \leq i \leq n) = \prod_{i=1}^{n} P(J_i(t_0) = k_i)$$

and the corollary follows from Proposition 2.10. □

Discussion of the Borel-Cantelli (BC) lemma. First remark that the condition $\lambda' < \lambda$ is sufficient to obtain bounds less than 1 in Proposition 2.10. Concerning Corollary 2.11, the BC lemma is not valid in some situations: for example, if $\lambda' > \lambda$ and $k_i = i$, the bound is $(\frac{\lambda'}{\lambda})^{\frac{n(n+1)}{4}} \exp(-n(\sqrt{\lambda} - \sqrt{\lambda'})^2\, t_0)$ and the series diverges.

If $\lambda' < \lambda$, BC holds provided $k_i \geq 0$, $1 \leq i \leq n$.

Finally, if $\sum_{i=1}^n k_i = \mathcal{O}(n\varepsilon_n)$, with $k_i \geq 0,\ 1 \leq i \leq n$ and $(\varepsilon_n) \to 0$, BC is valid since

$$\sum_{n\geq 1} \exp\left(\frac{1}{2}\log\frac{\lambda'}{\lambda}\ \sum_{i=1}^{n} k_i - n(\sqrt{\lambda} - \sqrt{\lambda'})^2\, t_0\right) < \infty.$$

Then, under the BC condition, the probability that $P(J_i(t_0) = k_i,\, 1 \leq i \leq n)$ occurs infinitely often is zero.

3. DETECTING THE CHANGE-POINT

3.1. **The case $\lambda \neq \lambda'$.** To detect the possibility of presence of zero or one jump, we may write:

$$\Delta_n = P((J_i(t_0) = 0 \cup J_i(t_0) = 1,\ 1 \leq i \leq n) = [P(J_1 = 0) + P(J_1 = 1)]^n.$$

Lemma 3.1. One obtains

$$\Delta_n \leq \exp\bigl(-n(\sqrt{\lambda} - \sqrt{\lambda'})^2\, t_0\bigr)\, \left(1 + \sqrt{\frac{\lambda'}{\lambda}}\right)^n \leq \exp\Bigl(-n\bigl((\sqrt{\lambda} - \sqrt{\lambda'})^2 t_0 - \sqrt{\frac{\lambda'}{\lambda}}\bigr)\Bigr)$$

and for λ and λ'such that $1 > t_0 > \frac{\sqrt{\lambda'/\lambda}}{(\sqrt{\lambda} - \sqrt{\lambda'})^2}$, the Borel-Cantelli lemma is valid.

Proof. Using Proposition 2.10 we get:

$$\Delta_n \leq \exp(-n(\sqrt{\lambda} - \sqrt{\lambda'})^2 t_0)\left(1 + \sqrt{\frac{\lambda'}{\lambda}}\right)^n$$

then

$$\Delta_n \leq \exp\left(-n\Bigl((\sqrt{\lambda} - \sqrt{\lambda'})^2\, t_0 - \log\Bigl(1 + \sqrt{\frac{\lambda'}{\lambda}}\Bigr)\Bigr)\right).$$

Now, from the bound $\log\left(1 + \sqrt{\frac{\lambda'}{\lambda}}\right) < \sqrt{\frac{\lambda'}{\lambda}}$, it follows that

$$\Delta_n \leq \exp\left(-n\left((\sqrt{\lambda} - \sqrt{\lambda'})^2\, t_0 - \sqrt{\frac{\lambda'}{\lambda}}\right)\right)$$

then, BC lemma holds with the condition given at t_0. □

Remark 3.2. Note that, for $\sqrt{\lambda} - \sqrt{\lambda'}$ large enough, the above condition holds. In particular, the conditions $1 > t_0 > \sqrt{\frac{\lambda'}{\lambda}}$ with $\sqrt{\lambda} - \sqrt{\lambda'} > 1$ are sufficient to detect the position of t_0.

3.2. **The case of equal intensities.** We now envisage the case where $\lambda = \lambda'$ and Proposition 2.7 does not work. Under a somewhat different assumption, we may obtain a jump at t_0 almost surely for n large enough:

Lemma 3.3. If $\lambda = \lambda'$ and $t_0 < \frac{2}{\lambda}$, we obtain

$$P(\cap_{i=1}^{n}(J_i(t_0) = 0)) \leq \exp(-n\lambda t_0(2 - \lambda t_0))$$

with $\sum_{n=1}^{\infty} \exp(-n\lambda t_0(2 - \lambda t_0)) < \infty$.

Proof. From Lemma 2.6, we have

$$P(J_0(t_0) = 0) = \exp(-(\lambda + \lambda')\, t_0) \sum_{h=0}^{\infty} \frac{(\lambda\lambda' t_0^2)^h}{(h!)^2}.$$

As $\lambda = \lambda'$ and $\frac{1}{h!} \leq 1$ one gets

$$\begin{aligned} P(J_i(t_0) = 0) &\leq \exp(-2\lambda t_0) \sum_{h=0}^{\infty} \frac{(\lambda^2 t_0^2)^h}{h!} \\ &\leq \exp(-\lambda t_0(2 - \lambda t_0)) \end{aligned}$$

and since $t_0 < \frac{2}{\lambda}$ the exponential is strictly less than 1. Finally we get the desired result since $J_1, \ldots, J_n$ are i.i.d. □

This Lemma is more elaborated.

Lemma 3.4. If $\lambda = \lambda'$ and $t_0 < \frac{1}{\lambda}$, we get

$$P(\cap_{i=1}^{n}(J_i(t_0) = k_i)) \leq (\lambda t_0)^{\sum_{i=1}^{n} k_i} \exp(-n\lambda t_0(2 - \lambda t_0))$$

and, if $\sum_{i=1}^{n} k_i \geq 0$,

$$\sum_{n\geq 1} (\lambda t_0)^{\sum_{i=1}^{n} k_i} \exp(-n\lambda t_0(2 - \lambda t_0)) < \infty.$$

Proof. We consider the proof in Lemma 2.5, then, since $\lambda = \lambda'$ and $\frac{1}{(k+h)!} \leq 1$, one obtains

$$P(J_i(t_0) = k_i) \leq \sum_{h=0}^{\infty} \exp(-2\lambda t_0)\; (\lambda t_0)^{2h+k_i} \frac{1}{h!}$$

thus

$$P(J_i(t_0) = k_i) \leq \exp(-2\lambda t_0)\, (\lambda t_0)^{k_i} \sum_{h=0}^{\infty} \frac{(\lambda^2 t_0^2)^h}{h!}.$$

Finally

$$P(J_i(t_0) = k_i) \leq \exp(-2\lambda t_0)\, (\lambda t_0)^{k_i} \exp(\lambda^2 t_0^2), \quad i = 1, \ldots, n.$$

and the result follows.

Now, if $t_0 < \frac{1}{\lambda}$ the series converges. □

Remark that for $\lambda > 2$ (Lemma 3.3) and $\lambda > 1$ (Lemma 3.4), the exponential inequality still holds in the case $\lambda = \lambda'$ but with $t_0 < \frac{2}{\lambda}$ or $t_0 < \frac{1}{\lambda}$. Namely, similarly to Lemma 3.1, we may derive the following corollary for the previous defined quantity $\Delta_n = P((J_i(t_0) = 0 \cup J_i(t_0) = 1,\ 1 \leq i \leq n)$.

Corollary 3.5. Under A1, we have $\Delta_n \leq \exp(-n\lambda t_0(1 - \lambda t_0))$ and Borel-Cantelli lemma holds for $0 < t_0 < \frac{1}{\lambda}$.

Remark 3.6. The previous lemma implies that t_0 is detectable at least for $0 < t_0 < \frac{1}{\lambda}$. since, almost surely for n large enough, there exist a sample path having a jump at t_0 and with intensity different from 1. Note also that methodology used in Lemma 3.3 and 3.4 allows us to obtain an alternative to Lemma 3.1. Namely, the obtained bound for Δ_n is $\exp(-nt_0\lambda(1 - \lambda' t_0))$. In this case, t_0 is detectable as soon as $0 < t_0 < \frac{1}{\lambda'}$ which is true if, in particular, $\lambda' \leq 1$.

Remark 3.7. Finally, one may express Δ_n in terms of modified Bessel functions of the first kind. Recall that $I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{h\geq 0} \frac{(z/2)^{2h}}{h!(\nu+h)!}$. From Proposition 2.10, we get

$$\Delta_n = \left(e^{-(\lambda+\lambda')t_0} I_0(2\sqrt{\lambda\lambda'}t_0) + e^{-(\lambda+\lambda')t_0}\sqrt{\frac{\lambda'}{\lambda}} I_1(2\sqrt{\lambda\lambda'}t_0)\right)^n.$$

Next, the following upper bound for $I_\nu(z)$ is given in [2, p. 583], for $z > 0$, $\nu > -1$:

$$I_v(z) < \frac{z^\nu}{2^\nu \Gamma(\nu+1)} \exp\left(\frac{z^2}{4(\nu+1)}\right).$$

This allows us to derive the new following bound for Δ_n:

$$\Delta_n \leq (1 + \lambda' t_0 e^{-\frac{\lambda\lambda' t_0^2}{2}})^n \exp\big(-nt_0(\lambda + \lambda' - \lambda\lambda' t_0)\big).$$

We may show that the bound is less than 1 for again $0 < t_0 < \frac{1}{\lambda'}$. Under this condition, the bound appears more accurate than our previous bound in Lemma 3.1. But it may takes big values for higher values of t_0. For example, for $n = 1$, $\lambda = 2$, $\lambda' = 10$ and $t_0 = 0.9$, one gets the bound 222.011 (...) while Lemma 3.1 gives 0.207. Remark also that the obtained bounds are respectively 0.519 and 0.702 for $t_0 = 0.5$.

4. CONSISTENCY

From Lemma 3.1, we know that a.s. for n large enough, there exist $i = 1, \ldots, n$, such that Z_i gets a jump at t_0 and with value different from 1. By this way, we may consider that t_0 is known. We set

$$\hat{J}_n(t_0) = \frac{1}{n}\sum_{i=1}^n \left[(N'_{i+t_0} - N'_i) - (N_{i+t_0} - N_i)\right]$$

thus, we have $E\hat{J}_n(t_0) = (\lambda' - \lambda)\, t_0$.

Then

Proposition 4.1. If A1 holds, we get $\hat{J}_n(t_0) \to (\lambda' - \lambda)\, t_0$ almost surely and in L^2.

Proof. The strong law of large number and the convergence in mean square is clear. □

It is again possible to obtain a exponential inequality:

Proposition 4.2. We have

$$P(|\hat{J}_n(t_0) - (\lambda' - \lambda)t_0| \geq \varepsilon) \leq 4 \exp\left(-\frac{n\varepsilon^2}{4c_\varepsilon}\right)$$

where $c_\varepsilon = \min(4\lambda t_0 + d\varepsilon, 4\lambda' t_0 + d'\varepsilon)$, $d > 0$, $d' > 0$, $\varepsilon > 0$.

Proof. We will use Bernstein's inequality (given in Lemma 2.4).

First, note that the Poisson process admits an exponential moment:

$$\begin{aligned} E(\exp aN_t) &= \sum_{k\geq 0} \exp(ak)\, \exp(-\lambda t)\, \frac{(\lambda t)^k}{k!}) = \exp(-\lambda t) \sum_{k \geq 0} \frac{(e^a \lambda t)^k}{k!} \\ &= \exp((e^a - 1)\lambda t)) < \infty. \end{aligned}$$

Now using notation $U_n = \frac{1}{n}\sum_{i=1}^n (N'_{i+t_0} - N'_i)$ and $V_n = \frac{1}{n}\sum_{i=1}^n (N_{i+t_0} - N_i)$, one obtains:

$$P(|U_n + V_n - \lambda\, t_0 - \lambda' t_0| \geq \varepsilon) \leq P\left(|U_n - \lambda t_0| \geq \frac{\varepsilon}{2}\right) + P\left(|V_n - \lambda t_0| \geq \frac{\varepsilon}{2}\right)$$

and Bernstein inequality entails

$$P(|\hat{J}_n(t_0) - (\lambda' - \lambda)t_0| \geq \varepsilon) \leq 2\exp\left(-\frac{n^{\varepsilon^2}/4}{4\sigma^2 + 2d^{\varepsilon}/2}\right) + 2\exp\left(-\frac{n^{\varepsilon^2}/4}{4\sigma'^2 + 2d'^{\varepsilon}/2}\right)$$

hence the result with $\sigma^2 = \lambda t_0$ and $\sigma'^2 = \lambda' t_0$. □

Now, we note that for $0 < t_0 < 1$, there exists s and s' such that $0 < s < t_0$ and $t_0 \leq s' < 1$. Actually, we may choose $s' - s$ not very far from 1. Then, we can estimate λ and λ':

Corollary 4.3. We have for $0 < s < t_0$:

$$\frac{1}{ns}\sum_{i=1}^n (N_{i+s} - N_i) \to \lambda,\ a.co.$$

and for $t_0 \leq s' < 1$

$$\frac{1}{ns'}\sum_{i=1}^n (N'_{i+s'} - N'_i) \to \lambda',\ a.co.$$

Proof. For $0 < s < t_0$, we get $EN_s = \lambda s$ and Bernstein inequality gives the result (see Lemma 2.4). The second result is similar. □

Remark 4.4. Note that N_s is observable for $s < t_0$ and that $N_s^{'}$ is observable for $s' \geq t_0$. Concerning the rate, we have

$$n\,E\left(\frac{1}{ns}\sum_{i=1}^{n}(N_{i+s}-N_i)-\lambda\right)^2=\frac{\lambda}{s}$$

and

$$n\,E\left(\frac{1}{ns'}\sum_{i=1}^{n}(N'_{i+s'}-N'_i)-\lambda'\right)^2=\frac{\lambda'}{s'}.$$

5. CENTRAL LIMIT THEOREM IN FUNCTION SPACE

In order to exhibit a jump at t_0 we consider the Hilbert space $H(t_0) = L^2([0,1]\,,\ B_{[0,1]},\ l+\delta_{(t_0)})$ where l is Lebesgue measure over $[0,1]$ and $\delta_{(t_0)}$ is Dirac at t_0.

Proposition 5.1. If $(Z_n\,, n \in \mathbb{Z})$ is a $H(t_0)$-strong white noise, then

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_i \Rightarrow N\sim \mathcal{N}\,(0,\, C_{Z_1})$$

where C_{Z_1} is a covariance operator.

Proof. See [7, p. 51]. □

Corollary 5.2. We also have

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}((Z_i(t_0)-Z_i(t_0-)-E(Z_i(t_0)-Z_i(t_0-))$$
$$\Rightarrow N\sim N\,(0,\, V(Z_1(t_0)-Z_1(t_0-)).$$

Proof. Clear, since the intensity of jumps are i.i.d. □

Remark 5.3. It is possible to consider the empirical variance. Thus, a confidence interval is clear.

6. DETECTING t_0 FOR DISCRETELY OBSERVED PROCESSES

In this part, we consider discretely observed processes at a grid of the form $\delta, 2\delta, \ldots, k\delta$ with $\delta > 0$ and $k\delta = 1$ in the fixed case and $\delta_n, 2\delta_n, \ldots, k_n\delta_n$ with $\delta_n \to 0$ and $k_n\delta_n \to 1$ in the high frequency one. For discretely observed sample paths, the assumption "t_0 known" is no longer realistic and so inference in this case is more intricate.

6.1. **The fixed case.** We suppose that, for some j_0,

$$0 < \delta \leq (j_0 - 1)\delta < t_0 \leq j_0\delta \leq (k-1)\delta < 1$$

thus $2 \leq j_0 \leq k-1$.

Let us set $\bar{Z}_n(j\delta) = \frac{1}{nj\delta}\sum_{i=1}^{n} Z_i(j\delta)$, $j = 1, \ldots, k-1$. We have

$$\bar{Z}_n(j\delta) = \frac{1}{nj\delta}\sum_{i=1}^{n}[(N_{i+j\delta} - N_i)\mathbb{1}_{\{j\delta < t_0\}} + (N'_{i+j\delta} - N'_i)\mathbb{1}_{\{j\delta \geq t_0\}}, \quad 1 \leq j \leq k-1.$$

Note that $\bar{Z}_n(j\delta)$ is observable but one cannot detect t_0 directly. Now, we put

$$V_{n,j} = \left|\bar{Z}_n(j\delta) - \bar{Z}_n((j-1)\delta)\right| \; 2 \leq j \leq k-1$$

and, for detecting t_0, we may set

$$\hat{j}_{0,n} = \arg\max_{2 \leq j \leq k-1} V_{n,j} \text{ and } \hat{t}_{0,n} = \hat{j}_{0,n}\delta.$$

Proposition 6.1. If A1 holds and $\lambda' \neq \lambda$, then almost surely for n large enough, we obtain $(j_0 - 1)\delta < \hat{t}_{0,n} \leq j_0\delta$.

Proof. From Corollary 4.3, we get $\bar{Z}_n(j\delta) \to \lambda$ a.s. if $j\delta < t_0$ while if $j\delta > t_0$, $\bar{Z}_n(j\delta) \to \lambda'$ a.s. Then for $j \leq j_0 - 1$

$$V_{n,j} \to |\lambda - \lambda| = 0 \text{ a.s.}$$

and for $j \geq j_0 + 1$

$$V_{n,j} \to |\lambda' - \lambda'| = 0 \text{ a.s..}$$

Now, for $j = j_0$, we get $(j_0 - 1)\delta < t_0 \leq j_0\delta$, and it follows that

$$V_{n,j_0} \to |\lambda' - \lambda| \neq 0 \text{ a.s.}$$

Consequently, since k and δ are fixed, we have almost surely for n large enough:

$$\max_{j=2,\ldots,k-1} V_{n,j} = V_{n,j_0}$$

so

$$\hat{j}_{0,n} = \arg\max_{2 \leq j \leq k-1} V_{n,j} = j_0 \text{ a.s.}$$

and $(j_0 - 1)\delta < \hat{t}_{0,n} \leq j_0\delta$ almost surely. ☐

6.2. **The high frequency case.** Finally, we modify δ by letting it tending to zero as $n \to \infty$. Then we have $\delta_n \to 0$ and $k_n \delta_n \to 1$. We suppose that there exists $j_0(n)$ such that

$$(j_0(n) - 1)\,\delta_n < t_0 \leq j_0(n)\,\delta_n$$

where $2 \leq j_0(n) \leq k_n - 1$. Finally, we have $j_0(n)\,\delta_n \to t_0(+)$.

Now, we set

$$V_{n,j} = \left|\bar{Z}_n(j\delta_n) - \bar{Z}_n((j-1)\delta_n)\right|, \quad 2 \leq j \leq k_n - 1$$

and again $\hat{j}_{0,n} = \arg\max_{2 \leq j \leq k_n - 1} V_{n,j}$.

Then we can obtain consistency:

Proposition 6.2. Under A1 and $\lambda \neq \lambda'$, and if $\delta_n \geq \frac{(\log n)^b}{n}$, $b > 1$, and $\delta_n \to 0$, then

$$\hat{j}_{0,n}\delta_n \to t_0(+)$$

almost surely as $n \to \infty$.

Proof. First observe that for $\varepsilon > 0$,

$$\left\{\left|\hat{j}_{0,n} - j_{0,n}\right| > \varepsilon\right\} \Rightarrow \left\{\exists\, j = 2, \ldots, k_n - 1,\ j \neq j_0(n) \text{ such that } V_{n,j} > V_{n,j_0(n)}\right\}.$$

Next for all $\eta > 0$,

$$\left\{V_{n,j} > V_{n,j_0(n)}\right\} = \left\{V_{n,j} > V_{n,j_0(n)},\, V_{n,j} > \eta\right\} \cup \left\{V_{n,j} > V_{n,j_0(n)},\, V_{n,j} \leq \eta\right\}$$

so that

$$\bigcup_{j=2, j\neq j_0(n)}^{k_n-1} \left\{V_{n,j} > V_{n,j_0(n)}\right\} \subset \left\{\bigcup_{j=2, j\neq j_0(n)}^{k_n-1} \left\{V_{n,j} > \eta\right\}\right\} \cup \left\{V_{n,j_0(n)} < \eta\right\}.$$

Consequently, for all $\varepsilon > 0$ and $\eta > 0$:

$$P\left(\limsup_{n\to\infty} \left|\hat{j}_{0,n} - j_{0,n}\right| > \varepsilon\right) \leq P\left(\limsup_{n\to\infty} \bigcup_{j=2, j\neq j_0(n)}^{k_n-1} \left\{V_{n,j} > \eta\right\}\right) + P\left(\limsup_{n\to\infty} V_{n,j_0(n)} < \eta\right).$$

This implies that the strong consistency of $\hat{j}_{0,n}$ can be derived from the almost sure behaviour of $\cup_{j=2,j\neq j_0(n)}^{k_n-1} V_{n,j}$ and $V_{n,j_0(n)}$. First, we establish the Kolmogorov theorem (see [30, p. 389]) for $V_{n,j_0(n)}$. We get for $j = j_0(n) - 1$:

$$\sum_{n\geq 1} \frac{V(N_{(j_0(n)-1)\delta_n})}{n^2 (j_0(n)-1)^2 \delta_n^2} = \sum_{n\geq 1} \frac{\lambda}{n^2 (j_0(n)-1)\delta_n} \leq \sum_{n\geq 1} \frac{\lambda}{n^2 \delta_n}$$

since $j_0(n) \geq 2$. Now, if $\delta_n \geq \frac{(\log n)^b}{n}$, $b > 1$, from the Bertrand series we obtain

$$\sum_{n\geq 1} \frac{V(N_{(j_0(n)-1)\delta_n})}{n^2 (j_0(n)-1)^2 \delta_n^2} \leq \lambda \sum_{n\geq 1} \frac{1}{n (\log n)^b} < \infty.$$

Then, the stronger law of large numbers gives

$$\bar{Z}_n((j_0(n)-1)\delta_n) \to \lambda \text{ a.s.}$$

For $j = j_0(n)$, we may write

$$\sum_{n\geq 1} \frac{\lambda'}{n^2 j_0(n)\delta_n} \leq \frac{\lambda'}{t_0} \sum_{n\geq 1} \frac{1}{n^2} < \infty$$

hence $\bar{Z}_n(j_0(n)\delta_n) \to \lambda'$ a.s.

We may conclude that:

$$V_{n,j_0(n)} \xrightarrow[n\to\infty]{\text{a.s.}} |\lambda' - \lambda| \neq 0.$$

Now, we turn to the almost complete convergence of $\cup_{j=2,j\neq j_0(n)}^{k_n-1} V_{n,j}$ to 0. We have for all $\eta > 0$:

$$P\left(\bigcup_{j=2,j\neq j_0(n)}^{k_n-1} V_{n,j} > \eta\right) \leq S_{1n} + S_{2n}$$

with

$$S_{1n} = \sum_{j=2}^{j_0(n)-1} P\Big(\Big|\frac{1}{n\delta_n j}\sum_{i=1}^{n}(N_{i+j\delta_n} - N_i - \lambda j\delta_n) - \frac{1}{n\delta_n(j-1)}\sum_{i=1}^{n}(N_{i+(j-1)\delta_n} - N_i - \lambda(j-1)\delta_n)\Big| > \eta\Big)$$

and

$$S_{2n} = \sum_{j=j_0(n)+1}^{k_n-1} P\Big(\Big|\frac{1}{n\delta_n j}\sum_{i=1}^{n}(N'_{i+j\delta_n} - N'_i - \lambda' j\delta_n) - \frac{1}{n\delta_n(j-1)}\sum_{i=1}^{n}(N'_{i+(j-1)\delta_n} - N'_i - \lambda'(j-1)\delta_n)\Big| > \eta\Big).$$

Next, for each term, we use again Bernstein inequality (see Lemma 2.4) to obtain after some easy calculation that

$$P\left(\bigcup_{j=2,j\neq j_0(n)}^{k_n-1} V_{n,j} > \eta\right) = \mathcal{O}\left(k_n \exp\left(-\frac{\eta^2 n\delta_n}{16\lambda + 4\eta d}\right)\right) + \mathcal{O}\left(k_n \exp(-\frac{\eta^2 n\delta_n}{16\lambda' + 4\eta d})\right).$$

Since $k_n\delta_n \to 1$, $\log k_n + \log \delta_n \to 0$ thus $\frac{\log k_n}{\log \delta_n} \to -1$ and the condition $\delta_n \geq \frac{(\log n)^b}{n}$, $b > 1$, implies for all $\eta > 0$ that

$$\sum_n P\left(\bigcup_{j=2,j\neq j_0(n)}^{k_n-1} V_{n,j} > \eta\right) < \infty.$$

Collecting the results, we obtain that for all $\eta > 0$

$$P\left(\limsup_{n\to\infty} \bigcup_{j=2, j\neq j_0(n)}^{k_n-1} \{V_{n,j} > \eta\}\right) = 0$$

and $P(\limsup_{n\to\infty} V_{n,j_0(n)} \leq \eta) = 0$ for η chosen small enough, e.g. $\eta = \frac{|\lambda-\lambda'|}{2}$. Finally, we have a.s. for n large enough:

$$\hat{\jmath}_{0,n} = j_0(n),$$

and the final result is derived from $j_0(n)\delta_n \to t_0(+)$. □

Remark 6.3. We may specify the integer k_n by writing $k_n = \left[\frac{1}{\delta_n}\right]$, thus we get $k_n\delta_n \to 1$.

6.3. **Simulation.** In this part, we compute our estimator of t_0 for Poisson processes with various intensities. First, for each intensity varying in $\{1, 2, 5, 10\}$, we simulate 10^5 independent and homogeneous Poisson processes on $[0, 1]$ with sampling rate equal to 0.001. Next, we fix $n = 100$ intervals so we have at our disposal $N = 10^3$ replications of the $\big(Z_i(t),\ i = 1, \ldots, n,\ 0 \leq t < 1\big)$ with the change-point t_0 randomly chosen in $[0, 1]$ for each of one. Finally, we compute our estimator of j_0 and compute the probabilities of $P(\hat{\jmath}_0 \neq j_0)$ with various of δ in $\{0.001, 0.01, 0.05, 0.1\}$.

δ \ (λ, λ')	(1, 2)	(1, 3)	(1, 5)	(1, 10)
0.001	0.644	0.394	0.179	0.016
0.01	0.244	0.034	0.021	0.019
0.05	0.121	0.091	0.094	0.098
0.1	0.2	0.19	0.2	0.208

TABLE 1. Case $\lambda < \lambda'$

δ \ (λ, λ')	(2, 1)	(3, 1)	(5, 1)	(10, 1)
0.001	0.863	0.786	0.644	0.277
0.01	0.407	0.132	0.032	0.02
0.05	0.153	0.11	0.107	0.083
0.1	0.202	0.218	0.205	0.183

TABLE 2. Case $\lambda > \lambda'$

From Tables 1 and 2, we observe that:

- for $\delta = 0.01$ or $\delta = 0.05$, results are globally quite satisfactory;

- for $\delta = 0.001$, one gets a quite bad estimation of j_0. This phenomenon is consistent with Proposition 6.2 and the condition $\delta_n \geq \frac{(\log n)^b}{n}$, $b > 1$ (for $n = 100$, one has $\frac{\log 100}{100} \simeq 0.046$ and $\frac{(\log 100)^{1.1}}{100} \simeq 0.054$);
- In both cases, the estimation is better for large values of $|\lambda - \lambda'|$;
- Comparing the cases $\lambda < \lambda'$ and $\lambda > \lambda'$, it appears that obtained results are close but with slightly smaller probabilities in the case $\lambda < \lambda'$, except in the case $\delta = 0.001$.

7. POISSON PROCESS WITH RANDOM CHANGE-POINTS

We now suppose that the position of jump is random and we set

$$Z_0(t) = N_t \, \mathbb{1}_{\{t<T_0\}} + N'_t \, \mathbb{1}_{\{t\geq T_0\}};\ 0 \leq t < 1.$$

We construct equidistributed random jumps (T_n) and we put

$$Z_n(t) = (N_{n+t} - N_n) \, \mathbb{1}_{\{t<T_n\}} + (N'_{n+t} - N'_n) \, \mathbb{1}_{\{t\geq T_n\}}, \quad 0 \leq t < 1,\ n \in \mathbb{Z}.$$

Now we make the following assumption:

Assumption A2. $(T_n,\, N_{n+t} - N_n,\, N'_{n+t} - N'_n)$ *are globally i.i.d. and we suppose that* (T_n) *is i.i.d. with distribution function* F *over* $[0, 1]$.

We set

$$F_n(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{T_i \leq t\}}, \quad 0 \leq t \leq 1,\, n \in \mathbb{Z}.$$

Then, the Glivenko-Cantelli theorem (1933) entails

$$\|F_n - F\|_\infty \to 0 \ \text{ a.s. } \ n \to \infty$$

and, from [19, 27]

$$P(\|F_n - F\|_\infty \geq \varepsilon) \leq 2\, \exp(-2n\varepsilon^2), \quad \varepsilon > 0,\, n \geq 1.$$

We now study expectation and variance of $Z_n(t)$:

Lemma 7.1. From A2 we get

$$EZ_0(t) = \lambda t\,(1 - F(t)) + \lambda' t\, F(t),\ 0 \leq t \leq 1$$

$$VZ_n(t) = \lambda t\,(1 + \lambda t)\,(1 - F(t)) + \lambda' t\,(1 + \lambda' t)\, F(t) - (EZ_0(t))^2,\ 0 \leq t \leq 1.$$

Proof. A2 gives

$$EZ_0(t) = EN_t P(t < T_0) + EN'_t P(T_0 \leq t)$$

and the result is straightforward.

Now, write $EZ_0^2(t) = E(N_t \, \mathbb{1}_{\{t<T_0\}} + N'_t \, \mathbb{1}_{\{t\geq T_0\}})^2$ and since the double product vanishes, we obtain

$$EZ_0^2(t) = \lambda t\,(1 + \lambda t)\,(1 - F(t)) + \lambda' t\,(1 + \lambda' t)\, F(t).$$

The variance is rather intricate but clear. □

Remark 7.2. Let us put $\bar{T}_n = \frac{1}{n}\sum_{i=1}^n T_i$ and since the T_i 's are bounded and i.i.d., the central limit theorem, the strong law of large number, the Hoeffding inequality, the Berry-Esseen bound and the law of the iterated logarithm are clear!

Now, Corollary 4.3 is not completely applicable but it is possible to use Lemma 7.1 to derive consistency results for the estimation of λ and λ'.

Proposition 7.3. If A2 holds, we may put

$$\bar{Z}_n(t_i) = \hat{\lambda}_n t_i(1 - F_n(t_i)) + \hat{\lambda}'_n t_i F_n(t_i),\ i = 1, 2,$$

where $\bar{Z}_n(t_i) = \frac{1}{n}\sum_{j=1}^n Z_j(t_i),\ i = 1, 2$, with $0 < t_1 < t_2 < 1,\ F(t_1) \neq F(t_2)$.

Then, almost surely for n large enough:

$$\hat{\lambda}_n = \frac{t_2 F_n(t_2)\,\bar{Z}_n(t_1) - t_1 F_n(t_1)\,\bar{Z}_n(t_2)}{t_1 t_2 (F_n(t_2) - F_n(t_1))}$$

and

$$\hat{\lambda}'_n = \frac{t_1(1 - F_n(t_1))\,\bar{Z}_n(t_2) - t_2(1 - F_n(t_2))\,\bar{Z}_n(t_1)}{t_1 t_2 (F_n(t_2) - F_n(t_1))}.$$

Finally, we have $\hat{\lambda}'_n \to \lambda'$ and $\hat{\lambda}_n \to \lambda$ almost surely.

Proof. Using the Cramer's rule, we set

$$M_n = \begin{pmatrix} t_1(1 - F_n(t_1)) & t_1 F_n(t_1) \\ t_2(1 - F_n(t_2) & t_2 F_n(t_2) \end{pmatrix}$$

where $\det(M_n) \neq 0$ almost surely for n large enough. Inverting M_n, we obtain the desired result.

Now, the Glivenko-Cantelli theorem entails

$$F_n(t_j) \to F(t_j), \quad j = 1, 2,\ \text{a.s.}$$

and the result follows from Lemma 7.1 and

$$\bar{Z}_n(t_j) \xrightarrow[n\to\infty]{\text{a.s.}} \lambda t_j\,(1 - F(t_j)) + \lambda' t_j\,F(t_j), \quad j = 1, 2.$$

□

We now envisage an exponential rate, we obtain

Proposition 7.4. For some constant $d > 0$,

$$P(|\hat{\lambda}_n - \lambda| > \varepsilon) \leq 4\exp\left(-\frac{nb^2\varepsilon^2}{d^2}\right) \varepsilon > 0.$$

Also, we get

$$|\hat{\lambda}_n - \lambda| = \mathcal{O}\left(\left(\frac{\log n}{n}\right)^{1/2}\right) \text{ almost surely.}$$

Proof. See Appendix and Lemma 2.4. □

Remark 7.5. A similar result holds for λ'.

Remark 7.6. From Assumption A2, one may show that

$$\hat{J}_n = \frac{1}{n} \sum_{i=1}^{n} \left[(N'_{i+T_i} - N'_i) - (N_{i+T_i} - N_i) \right]$$

is a consistent estimator of $E(J(T_0)) = (\lambda' - \lambda)E(T_0)$. Also, the alternative estimator of $E(J(T_0))$ given by $(\hat{\lambda}'_n - \hat{\lambda}_n)\overline{T}_n$ is consistent.

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APPENDIX

Proof of Proposition 7.4

Notation. Set

$$Z_i(t) = (N_{i+t} - N_i)\,\mathbb{1}_{\{t<T_i\}} + (N'_{i+t} - N'_i)\,\mathbb{1}_{\{t\geq T_i\}}\ 1 \leq i \leq n,$$

then, from independence of A2, we get

$$EZ_i(t) = \lambda t\,(1 - F(t)) + \lambda' t\,F(t),\ 1 \leq i \leq n.$$

we also put

$$\lambda = \frac{a}{b} := \frac{t_2 F(t_2)\,EZ(t_1) - t_1 F(t_1)\,EZ(t_2)}{t_1 t_2 (F(t_2) - F(t_1))},$$

where $F(t_2) > F(t_1)$. Finally, we set

$$V_n = t_2 F_n(t_2)\, \bar{Z}_n(t_1) - t_1 F_n(t_1)\, \bar{Z}_n(t_2)$$

and

$$U_n = t_1 t_2 \left(F_n(t_2) - F_n(t_1)\right)$$

where $U_n > 0$ almost surely for n large enough.

First exponential inequality.

$$P(|\hat{\lambda}_n - \lambda| > \varepsilon) = P(|\frac{V_n}{U_n} - \frac{a}{b}| > \varepsilon)$$

then

$$\begin{aligned} P(|\frac{V_n}{U_n} - \frac{a}{b}| > \varepsilon) &= P(|\frac{V_n}{U_n} - \frac{a}{b}| > \varepsilon,\, |U_n - b| > \eta) \\ &\quad + P(|\frac{V_n}{U_n} - \frac{a}{b}| > \varepsilon,\, |U_n - b| \leq \eta) \\ &\leq P(|U_n - b| > \eta) + P(|\frac{V_n}{U_n} - \frac{a}{b}| > \varepsilon,\, |U_n - b| \leq \eta) \end{aligned}$$

Now, by using Hoeffding inequality (see Lemma 2.4), one obtains

$$\begin{aligned} P(|U_n - b| > \eta) &= P(|(F_n(t_2) - F(t_2)) - (F_n(t_1) - F(t_1))| > \frac{\eta}{t_1 t_2}) \\ &\leq P(|F_n(t_2) - F(t_2)| > \frac{\eta}{2t_1t_2}) + P(|F_n(t_1) - F(t_1)| > \frac{\eta}{2t_1t_2}) \\ &\leq 4\exp(-\frac{n\eta^2}{2t_1^2 t_2^2}). \end{aligned}$$

Second exponential inequality.

$$\begin{aligned} &P(|\frac{V_n}{U_n} - \frac{a}{b}| > \varepsilon,\, |U_n - b| \leq \eta) \\ &\leq P(\frac{V_n}{U_n} - \frac{a}{b} > \varepsilon,\, -\eta \leq U_n - b \leq \eta) \\ &\quad + P(\frac{V_n}{U_n} - \frac{a}{b} < -\varepsilon,\, -\eta \leq U_n - b \leq \eta) \\ &= P(V_n > U_n(\varepsilon + \frac{a}{b}), -\eta + b \leq U_n \leq \eta + b) \\ &\quad + P(V_n < U_n(-\varepsilon + \frac{a}{b}), -\eta + b \leq U_n \leq \eta + b) \\ &\leq P(|V_n - a + \varepsilon\eta| > b\varepsilon - \frac{a}{b}\eta) \\ &\leq P(|V_n - a| > b\varepsilon - (\lambda + \varepsilon)\eta) \end{aligned}$$

with $0 < \eta < \frac{b\varepsilon}{\lambda+\varepsilon}$ and we envisage an exponential inequality for $P(|V_n - a| > \xi)$ with $\xi = b\varepsilon - (\lambda + \varepsilon)\eta > 0$. Now we get

$$\begin{aligned} |V_n - a| &\leq t_2|\bar{Z}_n(t_1) - EZ(t_1)| + t_1|\bar{Z}_n(t_2) - EZ(t_2)| \\ &\quad + t_2 EZ(t_1)|F_n(t_2) - F(t_2)| + t_1 EZ(t_2)|F_n(t_1) - F(t_1)| \end{aligned}$$

then

$$P(|V_n - a| > \xi) \leq$$
$$P(|\bar{Z}_n(t_1) - EZ(t_1)| > \frac{\xi}{4t_2}) + P(|\bar{Z}_n(t_2) - EZ(t_2)| > \frac{\xi}{4t_1})$$
$$+ P(|F_n(t_2) - F(t_2)| > \frac{\xi}{4t_2 EZ(t_1)}) + P(|F_n(t_1) - F(t_1)| > \frac{\xi}{4t_1 EZ(t_2)}).$$

The first part involves Bernstein's inequality (see Lemma 2.4) and the second part the Hoeffding's one. Connecting U_n and V_n one obtains, for $\varepsilon > 0$ and the choice $\eta = \frac{b\varepsilon}{2(\lambda+\varepsilon)}$

$$P(|\hat{\lambda}_n - \lambda| > \varepsilon) \leq 4\exp(-\frac{nb^2\varepsilon^2}{2(\lambda+\varepsilon)^2 t_1^2 t_2^2}) + 2\exp(-\frac{nb^2\varepsilon^2}{16t_2^2(EZ(t_1))^2})$$
$$+ 2\exp(-\frac{nb^2\varepsilon^2}{16t_1^2(EZ(t_2))^2}) + 2\exp(-\frac{nb^2\varepsilon^2}{256t_2^2 VZ(t_1) + 16t_2 bc\varepsilon})$$
$$+ 2\exp(-\frac{nb^2\varepsilon^2}{256t_1^2 VZ(t_2) + 16t_1 bc\varepsilon})$$

Now, putting $\varepsilon = c_0(\frac{\log n}{n})^{1/2}$ and for c_0 large enough, we get, almost surely,

$$|\hat{\lambda}_n - \lambda| = \mathcal{O}\left((\frac{\log n}{n})^{1/2}\right).$$

and a similar form may be derived for λ'.