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ARCTANGENT FUNCTION BASED BANACH SPACE VALUED NEURAL NETWORK APPROXIMATION

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ABSTRACT. Here we study the univariate quantitative approximation of Banach space valued continuous functions on a compact interval or all the real line by quasi-interpolation Banach space valued neural network operators. We perform also the related Banach space valued fractional approximation. These approximations are derived by establishing Jackson type inequalities involving the modulus of continuity of the engaged function or its Banach space valued high order derivative or fractional derivatives. Our operators are defined by using a density function induced by the arctangent function. The approximations are pointwise and with respect to the uniform norm. The related Banach space valued feed-forward neural networks are with one hidden layer.

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1. Introduction

The author in [1] and [2], see Chapters 2–5, was the first to establish neural network approximation to continuous functions with rates by very specifically defined neural network operators of Cardaliagnet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact suport. Also in [2] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see Chapters 4–5 there.

The author inspired by [13], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [3], [4], [5],[6], [7], by treating both

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the univariate and multivariate cases. He did also the corresponding fractional case [8].

The author here performs arctangent function based neural network approximations to continuous functions over compact intervals of the real line or over the whole $\mathbb{R}$ with values to an arbitrary Banach space $(X, \|\cdot\|)$. Finally he treats completely the related X-valued fractional approximation. All convergences here are with rates expressed via the modulus of continuity of the involved function or its X-valued high order derivative, or X-valued fractional derivatives and given by very tight Jackson type inequalities.

Our compact intervals are not necessarily symmetric to the origin. Some of our upper bounds to error quantity are very flexible and general. In preparation to prove our results we establish important properties of the basic density function defining our operators which is induced by arctangent.

Feed-forward X-valued neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle a_j \cdot x\rangle + b_j\right), \quad x \in \mathbb{R}^s, \ s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in X$ are the coefficients, $\langle a_j \cdot x\rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental neural network models, the activation function is the arctangent based. About neural networks in general read [14], [16], [18]. See also [9] for a complete study of real valued approximation by neural network operators.

2. **Basics**

We consider the

$$\arctan x = \int_0^x \frac{dz}{1+z^2}, \quad x \in \mathbb{R}. \tag{1}$$

We will be using

$$h(x) := \frac{2}{\pi}\arctan\left(\frac{\pi}{2}x\right) = \frac{2}{\pi}\int_0^{\frac{\pi x}{2}} \frac{dz}{1+z^2}, \quad x \in \mathbb{R}, \tag{2}$$

which is a sigmoidal type of function and is a strictly increasing function. We have that

$$h(0) = 0, \quad h(-x) = -h(x), \quad h(+\infty) = 1, \quad h(-\infty) = -1,$$

and

$$h'(x) = \frac{2}{\pi}\left(\frac{1}{1+\frac{\pi^2 x^2}{4}}\right)\frac{\pi}{2} = \frac{4}{4+\pi^2 x^2} > 0, \quad \text{all } x \in \mathbb{R}. \tag{3}$$

We consider the activation function

$$\psi(x) := \frac{1}{4}(h(x+1) - h(x-1)), \quad x \in \mathbb{R}, \tag{4}$$

and we notice that

$$\begin{aligned}\psi(-x) &= \frac{1}{4}(h(-x+1) - h(-x-1)) \\ &= \frac{1}{4}(-h(x-1) + h(x+1)) = \frac{1}{4}(h(x+1) - h(x-1)) = \psi(x),\end{aligned} \tag{5}$$

thus ψ is an even function. Since $x+1 > x-1$, then $h(x+1) > h(x-1)$, and $\psi(x) > 0$, all $x \in \mathbb{R}$.

We see that

$$\psi(0) = \frac{1}{4}(h(1) - h(-1)) = \frac{1}{4}(h(1) + h(1)) = \frac{h(1)}{2} = \frac{1}{\pi}\arctan\frac{\pi}{2}$$

($\arctan\frac{\pi}{2} \cong \arctan 1.57 \cong 57.505$)

$$= \frac{57.505}{3.14} \cong 18.31, \tag{6}$$

thus

$$\psi(0) \cong 18.31. \tag{7}$$

Let $x > 0$, we have that

$$\begin{aligned}\psi'(x) &= \frac{1}{4}(h'(x+1) - h'(x-1)) \\ &= \frac{-4\pi^2 x}{\left(4+\pi^2(x+1)^2\right)\left(4+\pi^2(x-1)^2\right)} < 0.\end{aligned} \tag{8}$$

That is

$$\psi'(x) < 0, \quad \text{for } x > 0.$$

That is ψ is strictly decreasing on $[0,\infty)$ and clearly is strictly increasing on $(-\infty, 0]$, and $\psi'(0) = 0$.

See that

$$\lim_{x\to+\infty}\psi(x) = \frac{1}{4}(h(+\infty) - h(+\infty)) = 0,$$
$$\lim_{x\to-\infty}\psi(x) = \frac{1}{4}(h(-\infty) - h(-\infty)) = 0.$$

That is the x-axis is the horizontal asymptote on ψ.

Conclusion, ψ is a bell symmetric function with maximum

$$\psi(0) \cong 18.31.$$

We need

Theorem 1. *We have that*

$$\sum_{i=-\infty}^{\infty}\psi(x-i) = 1, \quad \forall\, x \in \mathbb{R}. \tag{9}$$

Proof. We observe that

$$\sum_{i=-\infty}^{\infty}\left(h\left(x-i\right)-h\left(x-1-i\right)\right)=\sum_{i=0}^{\infty}\left(h\left(x-i\right)-h\left(x-1-i\right)\right)$$

$$+\sum_{i=-\infty}^{-1}\left(h\left(x-i\right)-h\left(x-1-i\right)\right). \tag{10}$$

Furthermore ($\lambda \in \mathbb{Z}^{+}$)

$$\sum_{i=0}^{\infty}\left(h\left(x-i\right)-h\left(x-1-i\right)\right)=\lim_{\lambda\to\infty}\sum_{i=0}^{\lambda}\left(h\left(x-i\right)-h\left(x-1-i\right)\right)= \tag{11}$$

(telescoping sum)

$$\lim_{\lambda\to\infty}\left(h\left(x\right)-h\left(x-\left(\lambda+1\right)\right)\right)=1+h\left(x\right).$$

Similarly

$$\sum_{i=-\infty}^{-1}\left(h\left(x-i\right)-h\left(x-1-i\right)\right)=\lim_{\lambda\to\infty}\sum_{i=-\lambda}^{-1}\left(h\left(x-i\right)-h\left(x-1-i\right)\right) \tag{12}$$

$$=\lim_{\lambda\to\infty}\left(h\left(x+\lambda\right)-h\left(x\right)\right)=1-h\left(x\right).$$

So adding the last two limits we obtain

$$\sum_{i=-\infty}^{\infty}\left(h\left(x-i\right)-h\left(x-1-i\right)\right)=2, \quad \forall\ x\in\mathbb{R}. \tag{13}$$

Therefore

$$\sum_{i=-\infty}^{\infty}\left(h\left(x+1-i\right)-h\left(x-i\right)\right)=2, \quad \forall\ x\in\mathbb{R}. \tag{14}$$

Consequently, by adding (13), (14) we get

$$\sum_{i=-\infty}^{\infty}\left(h\left(x+1-i\right)-h\left(x-1-i\right)\right)=4, \quad \forall\ x\in\mathbb{R}, \tag{15}$$

proving the claim. □

Furthermore we give:

Because ψ is even it holds

$$\sum_{i=-\infty}^{\infty}\psi\left(i-x\right)=1, \quad \forall\ x\in\mathbb{R}. \tag{16}$$

Hence

$$\sum_{i=-\infty}^{\infty}\psi\left(i+x\right)=1, \quad \forall\ x\in\mathbb{R},$$

and

$$\sum_{i=-\infty}^{\infty} \psi(x+i) = 1, \quad \forall\, x \in \mathbb{R}. \tag{17}$$

We give

Theorem 2. *It holds* $\int_{-\infty}^{\infty} \psi(x)\, dx = 1$.

Proof. We observe that

$$\int_{-\infty}^{\infty} \psi(x)\, dx = \sum_{j=-\infty}^{\infty} \int_{j}^{j+1} \psi(x)\, dx = \sum_{j=-\infty}^{\infty} \int_{0}^{1} \psi(x+j)\, dx \tag{18}$$
$$= \int_0^1 \left(\sum_{j=-\infty}^{\infty} \psi(x+j)\, dx \right) = \int_0^1 1dx = 1.$$

□

So that $\psi(x)$ is a density function on $\mathbb{R}$.

We need

Theorem 3. *Let* $0 < \alpha < 1$, *and* $n \in \mathbb{N}$ *with* $n^{1-\alpha} > 2$. *It holds*

$$\sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi(nx-k) < \frac{2}{\pi^2 \left(n^{1-\alpha} - 2\right)}. \tag{19}$$

Proof. Let $x \geq 1$. That is $0 \leq x - 1 < x + 1$. Applying the mean value theorem we get

$$\psi(x) = \frac{1}{4} \cdot 2 \cdot f'(\xi) = \frac{1}{2} \left(\frac{4}{4 + \pi^2 \xi^2} \right) = \frac{2}{4 + \pi^2 \xi^2}, \tag{20}$$

for some $x - 1 < \xi < x + 1$.

Notice that $\xi^2 > (x-1)^2$, hence $4 + \pi^2 \xi^2 > 4 + \pi^2 (x-1)^2$, and

$$\frac{2}{4 + \pi^2 \xi^2} < \frac{2}{4 + \pi^2 (x-1)^2}. \tag{21}$$

Thus

$$\psi(x) < \frac{2}{4 + \pi^2 (x-1)^2}, \quad \forall\, x \geq 1. \tag{22}$$

Therefore we have

$$\sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi(nx-k) = \sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi(|nx-k|)$$

$$< 2 \sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \frac{1}{\left[4+\pi^2\left(|nx-k|-1\right)^2\right]} \tag{23}$$

$$\leq 2 \int_{(n^{1-\alpha}-1)}^{\infty} \frac{dx}{\left[4+\pi^2\left(x-1\right)^2\right]} = (*)$$

(notice that: $4+\pi^2(x-1)^2 > \pi^2(x-1)^2$, hence $\frac{1}{4+\pi^2(x-1)^2} < \frac{1}{\pi^2(x-1)^2}$, for $x \neq 1$, and see here, if $x \geq n^{1-\alpha}-1$, then $x-1 \geq n^{1-\alpha}-2 > 0$).

Then

$$(*) < 2 \int_{n^{1-\alpha}-1}^{\infty} \frac{dx}{\pi^2(x-1)^2} = \frac{2}{\pi^2} \int_{n^{1-\alpha}-1}^{\infty} (x-1)^{-2}\, dx \tag{24}$$

(if $n^{1-\alpha}-1 \leq x < \infty$, then $n^{1-\alpha}-2 \leq x-1 < \infty$)

$$\begin{aligned} &= \frac{2}{\pi^2} \int_{n^{1-\alpha}-2}^{\infty} z^{-2} dz = \frac{2}{\pi^2} \lim_{r\to+\infty} \int_{n^{1-\alpha}-2}^{r} z^{-2} dz \\ &= \frac{2}{\pi^2} \left\{ \lim_{r\to+\infty} \left(\frac{z^{-2+1}}{-2+1} \Big|_{n^{1-\alpha}-2}^{r} \right) \right\} = \frac{2}{\pi^2} \left\{ \lim_{r\to+\infty} \left(\frac{1}{z} \Big|_{r}^{n^{1-\alpha}-2} \right) \right\} \\ &= \frac{2}{\pi^2} \left\{ \lim_{r\to+\infty} \left(\frac{1}{n^{1-\alpha}-2} - \frac{1}{r} \right) \right\} = \frac{2}{\pi^2} \left\{ \frac{1}{n^{1-\alpha}-2} - \frac{1}{\infty} \right\} \\ &= \frac{2}{\pi^2} \left(\frac{1}{n^{1-\alpha}-2} - 0 \right) = \frac{2}{\pi^2\left(n^{1-\alpha}-2\right)}, \end{aligned} \tag{25}$$

proving the claim. □

Denote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.

We further give

Theorem 4. *Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)} < \frac{1}{\psi(1)} \cong 0.0868, \quad \forall\, x \in [a,b]. \tag{26}$$

Proof. We observe that

$$1 = \sum_{k=-\infty}^{\infty} \psi(nx-k) > \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(|nx-k|) > \psi(|nx-k_0|), \tag{27}$$

$\forall\, k_0 \in [\lceil na \rceil, \lfloor nb \rfloor] \cap \mathbb{Z}$.

We can choose $k_0 \in [\lceil na \rceil, \lfloor nb \rfloor] \cap \mathbb{Z}$ such that $|nx-k_0| < 1$.

Therefore

$$\begin{aligned} \psi(|nx-k_0|) > \psi(1) &= \frac{1}{4}(h(2)-h(0)) = \frac{1}{4}h(2) = \frac{1}{4}\frac{2}{\pi}\arctan\left(\frac{\pi}{2}2\right) \\ &= \frac{1}{2\pi}\arctan(\pi) \cong \frac{72.3348}{6.28} \cong 11.5182. \end{aligned} \tag{28}$$

So that $\psi(1) \cong 11.5182$. Consequently we obtain

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k) > 11.5182, \tag{29}$$

and

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)} < 0.0868, \tag{30}$$

proving the claim. $\square$

We make

Remark 5. We also notice that

$$1 - \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nb-k) = \sum_{k=-\infty}^{\lceil na \rceil - 1} \psi(nb-k) + \sum_{k=\lfloor nb \rfloor + 1}^{\infty} \psi(nb-k) > \psi(nb - \lfloor nb \rfloor - 1) \tag{31}$$

(call $\varepsilon := nb - \lfloor nb \rfloor$, $0 \leq \varepsilon < 1$)

$$= \psi(\varepsilon - 1) = \psi(1-\varepsilon) \geq \psi(1) > 0.$$

Therefore

$$\lim_{n\to\infty} \left(1 - \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nb-k)\right) > 0. \tag{32}$$

Similarly

$$1 - \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(na-k) = \sum_{k=-\infty}^{\lceil na \rceil - 1} \psi(na-k) + \sum_{k=\lfloor nb \rfloor + 1}^{\infty} \psi(na-k) > \psi(na - \lceil na \rceil + 1) \tag{33}$$

(call $\eta := \lceil na \rceil - na$, $0 \leq \eta < 1$)

$$= \psi(1-\eta) \geq \psi(1) > 0.$$

Therefore again

$$\lim_{n\to\infty} \left(1 - \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(na-k)\right) > 0. \tag{34}$$

Therefore we find that

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k) \neq 1, \tag{35}$$

for at last some $x \in [a,b]$.

Note 6. For large enough n we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. In general it holds (by (9)) that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k) \leq 1. \tag{36}$$

Let $(X, \|\cdot\|)$ be a Banach space.

Definition 7. Let $f \in C([a,b], X)$ and $n \in \mathbb{N} : \lceil na \rceil \leq \lfloor nb \rfloor$. We introduce and define the X-valued linear neural network operators

$$A_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \psi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)}, \quad x \in [a,b]. \tag{37}$$

Clearly here $A_n(f,x) \in C([a,b], X)$. For convenience we use the same A_n for real valued function when needed. We study here the pointwise and uniform convergence of $A_n(f,x)$ to $f(x)$ with rates.

For convenience also we call

$$A_n^*(f,x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \psi(nx-k), \tag{38}$$

(similarly A_n^* can be defined for real valued function) that is

$$A_n(f,x) = \frac{A_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)}. \tag{39}$$

So that

$$A_n(f,x) - f(x) = \frac{A_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)} - f(x)$$

$$= \frac{A_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)}. \tag{40}$$

Consequently we derive

$$\|A_n(f,x) - f(x)\| \leq 0.0868 \left\| A_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)\right) \right\|. \tag{41}$$

That is

$$\|A_n(f,x) - f(x)\| \leq 0.0868 \left\| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x) \right) \psi(nx-k) \right\|. \tag{42}$$

We will estimate the right hand side of (42).

For that we need, for $f \in C([a,b], X)$ the first modulus of continuity

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in [a,b] \\ |x-y| \le \delta}} \|f(x) - f(y)\|, \quad \delta > 0. \tag{43}$$

Similarly, it is defined ω_1 for $f \in C_{uB}(\mathbb{R}, X)$ (uniformly continuous and bounded functions from $\mathbb{R}$ into X), for $f \in C_B(\mathbb{R}, X)$ (continuous and bounded X-valued) and for $f \in C_u(\mathbb{R}, X)$ (uniformly continuous).

The fact $f \in C([a,b], X)$ or $f \in C_u(\mathbb{R}, X)$, is equivalent to $\lim_{\delta \to 0} \omega_1(f,\delta) = 0$, see [11].

Definition 8. When $f \in C_{uB}(\mathbb{R}, X)$, or $f \in C_B(\mathbb{R}, X)$, we define

$$\overline{A}_n(f,x) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \psi(nx - k), \quad n \in \mathbb{N}, \ x \in \mathbb{R}, \tag{44}$$

the X-valued quasi-interpolation neural network operator.

Remark 9. We have that

$$\left\| f\left(\frac{k}{n}\right) \right\| \le \|f\|_{\infty,\mathbb{R}} < +\infty,$$

and

$$\left\| f\left(\frac{k}{n}\right) \right\| \psi(nx-k) \le \|f\|_{\infty,\mathbb{R}} \psi(nx-k), \tag{45}$$

and

$$\sum_{k=-\lambda}^{\lambda} \left\| f\left(\frac{k}{n}\right) \right\| \psi(nx-k) \le \|f\|_{\infty,\mathbb{R}} \left(\sum_{k=-\lambda}^{\lambda} \psi(nx-k) \right),$$

and finally

$$\sum_{k=-\infty}^{\infty} \left\| f\left(\frac{k}{n}\right) \right\| \psi(nx-k) \le \|f\|_{\infty,\mathbb{R}}, \tag{46}$$

a convergent in $\mathbb{R}$ series.

So the series $\sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \psi(nx-k)$ is absolutely convergent in X, hence it is convergent in X and $\overline{A}_n(f,x) \in X$.

We denote by $\|f\|_{\infty} := \sup_{x \in [a,b]} \|f(x)\|$, for $f \in C([a,b], X)$, similarly is defined for $f \in C_B(\mathbb{R}, X)$.

3. **Main Results**

We present a series of X-valued neural network approximations to a function given with rates.

We first give

Theorem 10. *Let* $f \in C([a,b], X)$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in [a,b]$. *Then*

i)

$$\|A_n(f,x) - f(x)\| \leq 0.0868 \left[\omega_1 \left(f, \frac{1}{n^\alpha} \right) + \frac{4\|f\|_\infty}{\pi^2 (n^{1-\alpha} - 2)} \right] =: \rho, \tag{47}$$

and

ii)

$$\|A_n(f) - f\|_\infty \leq \rho. \tag{48}$$

We notice $\lim_{n\to\infty} A_n(f) = f$, *pointwise and uniformly.*

Proof. We see that

$$\left\| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x) \right) \psi(nx-k) \right\|$$
$$\leq \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| \psi(nx-k)$$
$$= \sum_{\substack{k=\lceil na \rceil \\ \left|\frac{k}{n} - x\right| \leq \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| \psi(nx-k)$$
$$+ \sum_{\substack{k=\lceil na \rceil \\ \left|\frac{k}{n} - x\right| > \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| \psi(nx-k) \tag{49}$$
$$\leq \sum_{\substack{k=\lceil na \rceil \\ \left|\frac{k}{n} - x\right| \leq \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \omega_1\left(f, \left|\frac{k}{n} - x\right|\right) \psi(nx-k)$$
$$+ 2\|f\|_\infty \sum_{\substack{k=\lceil na \rceil \\ |k-nx| > n^{1-\alpha}}}^{\lfloor nb \rfloor} \psi(nx-k)$$

$$\leq \omega_1\left(f,\frac{1}{n^{\alpha}}\right)\sum_{\begin{cases} k=-\infty \\ \left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}\end{cases}}^{\infty}\psi(nx-k)$$

$$+2\|f\|_{\infty}\sum_{\begin{cases} k=-\infty \\ |k-nx|>n^{1-\alpha}\end{cases}}^{\infty}\psi(nx-k)$$

$$\underset{\text{(by Theorem 3)}}{\leq}\omega_1\left(f,\frac{1}{n^{\alpha}}\right)+\frac{4\|f\|_{\infty}}{\pi^2\left(n^{1-\alpha}-2\right)}. \tag{50}$$

That is

$$\left\|\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(f\left(\frac{k}{n}\right)-f(x)\right)\psi(nx-k)\right\|\leq\omega_1\left(f,\frac{1}{n^{\alpha}}\right)+\frac{4\|f\|_{\infty}}{\pi^2\left(n^{1-\alpha}-2\right)}. \tag{51}$$

Using (51) we derive (47). □

Next we give

Theorem 11. *Let $f\in C_B(\mathbb{R},X)$, $0<\alpha<1$, $n\in\mathbb{N}:n^{1-\alpha}>2$, $x\in\mathbb{R}$. Then*

i)

$$\left\|\overline{A}_n(f,x)-f(x)\right\|\leq\omega_1\left(f,\frac{1}{n^{\alpha}}\right)+\frac{4\|f\|_{\infty}}{\pi^2\left(n^{1-\alpha}-2\right)}=:\mu, \tag{52}$$

and

ii)

$$\left\|\overline{A}_n(f)-f\right\|_{\infty}\leq\mu. \tag{53}$$

For $f\in C_{uB}(\mathbb{R},X)$ we get $\lim_{n\to\infty}\overline{A}_n(f)=f$, pointwise and uniformly.

Proof. We observe that

$$\left\|\overline{A}_n(f,x)-f(x)\right\|=\left\|\sum_{k=-\infty}^{\infty}f\left(\frac{k}{n}\right)\psi(nx-k)-f(x)\sum_{k=-\infty}^{\infty}\psi(nx-k)\right\|$$

$$=\left\|\sum_{k=-\infty}^{\infty}\left(f\left(\frac{k}{n}\right)-f(x)\right)\psi(nx-k)\right\|$$

$$\leq\sum_{k=-\infty}^{\infty}\left\|f\left(\frac{k}{n}\right)-f(x)\right\|\psi(nx-k)$$

$$=\sum_{\begin{cases} k=-\infty \\ \left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}\end{cases}}^{\infty}\left\|f\left(\frac{k}{n}\right)-f(x)\right\|\psi(nx-k)$$

$$+\sum_{\begin{cases} k=-\infty \\ \left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}} \end{cases}}^{\infty} \left\| f\left(\frac{k}{n}\right)-f(x)\right\| \psi(nx-k) \tag{54}$$

$$\leq \sum_{\begin{cases} k=-\infty \\ \left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}} \end{cases}}^{\infty} \omega_1\left(f,\left|\frac{k}{n}-x\right|\right)\psi(nx-k)$$

$$+2\|f\|_{\infty} \sum_{\begin{cases} k=-\infty \\ \left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}} \end{cases}}^{\infty} \psi(nx-k)$$

$$\leq \omega_1\left(f,\frac{1}{n^{\alpha}}\right) \sum_{\begin{cases} k=-\infty \\ \left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}} \end{cases}}^{\infty} \psi(nx-k)+\frac{4\|f\|_{\infty}}{\pi^2\left(n^{1-\alpha}-2\right)}$$

$$\leq \omega_1\left(f,\frac{1}{n^{\alpha}}\right)+\frac{4\|f\|_{\infty}}{\pi^2\left(n^{1-\alpha}-2\right)}, \tag{55}$$

proving the claim. □

We need the X-valued Taylor's formula in an appropriate form:

Theorem 12 ([10], [12])**.** *Let $N\in\mathbb{N}$, and $f\in C^N([a,b],X)$, where $[a,b]\subset\mathbb{R}$ and X is a Banach space. Let any $x,y\in[a,b]$. Then*

$$f(x)=\sum_{i=0}^{N}\frac{(x-y)^i}{i!}f^{(i)}(y)+\frac{1}{(N-1)!}\int_y^x (x-t)^{N-1}\left(f^{(N)}(t)-f^{(N)}(y)\right)dt. \tag{56}$$

The derivatives $f^{(i)}$, $i\in\mathbb{N}$, are defined like the numerical ones, see [19, p. 83]. The integral $\int_y^x$ in (56) is of Bochner type, see [17].

By [12], [15] we have that: if $f\in C([a,b],X)$, then $f\in L_{\infty}([a,b],X)$ and $f\in L_1([a,b],X)$.

In the next we discuss high order neural network X-valued approximation by using the smoothness of f.

Theorem 13. *Let $f\in C^N([a,b],X)$, $n,N\in\mathbb{N}$, $0<\alpha<1$, $x\in[a,b]$ and $n^{1-\alpha}>2$. Then*

i)

$$\|A_n(f,x)-f(x)\|\leq 0.0868\left\{\sum_{j=1}^{N}\frac{\left\|f^{(j)}(x)\right\|}{j!}\left[\frac{1}{n^{\alpha j}}+\frac{2(b-a)^j}{\pi^2\left(n^{1-\alpha}-2\right)}\right]\right. \tag{57}$$

$$+\left[\omega_1\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\frac{4\left\|f^{(N)}\right\|_{\infty}(b-a)^N}{N!\pi^2\left(n^{1-\alpha}-2\right)}\right]\Bigg\}$$

ii) assume further $f^{(j)}(x_0)=0$, $j=1,\dots,N$, *for some* $x_0\in[a,b]$, *it holds*

$$\|A_n(f,x_0)-f(x_0)\|\leq 0.0868$$

$$\times\left\{\omega_1\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\frac{4\left\|f^{(N)}\right\|_{\infty}(b-a)^N}{N!\pi^2\left(n^{1-\alpha}-2\right)}\right\}, \tag{58}$$

and

iii)

$$\|A_n(f)-f\|_{\infty}\leq 0.0868\left\{\sum_{j=1}^{N}\frac{\left\|f^{(j)}\right\|_{\infty}}{j!}\left[\frac{1}{n^{\alpha j}}+\frac{2(b-a)^j}{\pi^2\left(n^{1-\alpha}-2\right)}\right]\right.$$

$$\left.+\left[\omega_1\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\frac{4\left\|f^{(N)}\right\|_{\infty}(b-a)^N}{N!\pi^2\left(n^{1-\alpha}-2\right)}\right]\right\}. \tag{59}$$

Again we obtain $\lim_{n\to\infty}A_n(f)=f$, *pointwise and uniformly.*

Proof. Next we apply the X-valued Taylor's formula with Bochner integral remainder (56). We have (here $\frac{k}{n}$, $x\in[a,b]$)

$$f\left(\frac{k}{n}\right)=\sum_{j=0}^{N}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^j+\int_x^{\frac{k}{n}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt. \tag{60}$$

Then

$$f\left(\frac{k}{n}\right)\psi(nx-k)=\sum_{j=0}^{N}\frac{f^{(j)}(x)}{j!}\psi(nx-k)\left(\frac{k}{n}-x\right)^j$$

$$+\psi(nx-k)\int_x^{\frac{k}{n}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt. \tag{61}$$

Hence

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\psi(nx-k)-f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k) \tag{62}$$

$$=\sum_{j=0}^{N}\frac{f^{(j)}(x)}{j!}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)\left(\frac{k}{n}-x\right)^j$$

$$+\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)\int_x^{\frac{k}{n}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt.$$

Thus

$$A_n^*(f,x)-f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)\right)$$

$$= \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} A_n^* \left((\cdot - x)^j \right) + \Lambda_n(x), \tag{63}$$

where

$$\Lambda_n(x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k) \int_x^{\frac{k}{n}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!} dt. \tag{64}$$

We assume that $b - a > \frac{1}{n^\alpha}$, which is always the case for large enough $n \in \mathbb{N}$, that is when $n > \left\lceil (b-a)^{-\frac{1}{n}} \right\rceil$.

Thus $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha}$ or $\left| \frac{k}{n} - x \right| > \frac{1}{n^\alpha}$.

Let

$$\gamma := \int_x^{\frac{k}{n}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!} dt, \tag{65}$$

in the case of $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha}$, we find that

$$\|\gamma\| \leq \omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N} N!} \tag{66}$$

for $x \leq \frac{k}{n}$ or $x \geq \frac{k}{n}$.

We prove it next.

i) Indeed, for the case of $x \leq \frac{k}{n}$, we have

$$\begin{aligned} \|\gamma\| &= \left\| \int_x^{\frac{k}{n}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!} dt \right\| \\ &\leq \int_x^{\frac{k}{n}} \left\| f^{(N)}(t) - f^{(N)}(x) \right\| \frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!} dt \\ &\leq \int_x^{\frac{k}{n}} \omega_1 \left(f^{(N)}, |t-x| \right) \frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!} dt \leq \omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \int_x^{\frac{k}{n}} \frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!} dt \\ &= \omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \frac{\left(\frac{k}{n}-x\right)^{N}}{N!} \leq \omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N} N!}. \end{aligned} \tag{67}$$

ii) for the case of $x > \frac{k}{n}$, we have

$$\begin{aligned} \|\gamma\| &= \left\| \int_x^{\frac{k}{n}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!} dt \right\| \\ &= \left\| \int_{\frac{k}{n}}^{x} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(t-\frac{k}{n}\right)^{N-1}}{(N-1)!} dt \right\| \\ &\leq \int_{\frac{k}{n}}^{x} \left\| f^{(N)}(t) - f^{(N)}(x) \right\| \frac{\left(t-\frac{k}{n}\right)^{N-1}}{(N-1)!} dt \\ &\leq \int_{\frac{k}{n}}^{x} \omega_1 \left(f^{(N)}, |t-x| \right) \frac{\left(t-\frac{k}{n}\right)^{N-1}}{(N-1)!} dt \leq \omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \int_{\frac{k}{n}}^{x} \frac{\left(t-\frac{k}{n}\right)^{N-1}}{(N-1)!} dt \end{aligned} \tag{68}$$

$$= \omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \frac{\left(x - \frac{k}{n}\right)^N}{N!} \leq \omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N} N!}.$$

We have proved (66).

We treat again γ, see (65), but differently:

Notice also for $x \leq \frac{k}{n}$ that

$$\left\| \int_x^{\frac{k}{n}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{k}{n} - t\right)^{N-1}}{(N-1)!} dt \right\|$$

(69)
$$\leq \int_x^{\frac{k}{n}} \left\| f^{(N)}(t) - f^{(N)}(x) \right\| \frac{\left(\frac{k}{n} - t\right)^{N-1}}{(N-1)!} dt$$

$$\leq 2 \left\| f^{(N)} \right\|_\infty \int_x^{\frac{k}{n}} \frac{\left(\frac{k}{n} - t\right)^{N-1}}{(N-1)!} dt = 2 \left\| f^{(N)} \right\|_\infty \frac{\left(\frac{k}{n} - x\right)^N}{N!}$$

$$\leq 2 \left\| f^{(N)} \right\|_\infty \frac{(b-a)^N}{N!}.$$

Next assume $\frac{k}{n} \leq x$, then

$$\left\| \int_x^{\frac{k}{n}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{k}{n} - t\right)^{N-1}}{(N-1)!} dt \right\|$$

$$= \left\| \int_{\frac{k}{n}}^x \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(t - \frac{k}{n}\right)^{N-1}}{(N-1)!} dt \right\|$$

(70)
$$\leq \int_{\frac{k}{n}}^x \left\| f^{(N)}(t) - f^{(N)}(x) \right\| \frac{\left(t - \frac{k}{n}\right)^{N-1}}{(N-1)!} dt$$

$$\leq 2 \left\| f^{(N)} \right\|_\infty \int_{\frac{k}{n}}^x \frac{\left(t - \frac{k}{n}\right)^{N-1}}{(N-1)!} dt = 2 \left\| f^{(N)} \right\|_\infty \frac{\left(x - \frac{k}{n}\right)^N}{N!}$$

$$\leq 2 \left\| f^{(N)} \right\|_\infty \frac{(b-a)^N}{N!}.$$

Thus

(71)
$$\|\gamma\| \leq 2 \left\| f^{(N)} \right\|_\infty \frac{(b-a)^N}{N!}.$$

in the two cases.

Therefore

(72)
$$\Lambda_n(x) = \sum_{\begin{cases} k = \lceil na \rceil \\ \left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha} \end{cases}}^{\lfloor nb \rfloor} \psi(nx - k)\, \gamma + \sum_{\begin{cases} k = \lceil na \rceil \\ \left| \frac{k}{n} - x \right| > \frac{1}{n^\alpha} \end{cases}}^{\lfloor nb \rfloor} \psi(nx - k)\, \gamma.$$

Hence

$$\|\Lambda_n(x)\| \le \sum_{\substack{k=\lceil na\rceil \\ \left|\frac{k}{n}-x\right| \le \frac{1}{n^\alpha}}}^{\lfloor nb\rfloor} \psi(nx-k)\left(\omega_1\left(f^{(N)},\frac{1}{n^\alpha}\right)\frac{1}{N!n^{\alpha N}}\right) \tag{73}$$

$$+\left(\sum_{\substack{k=\lceil na\rceil \\ \left|\frac{k}{n}-x\right| > \frac{1}{n^\alpha}}}^{\lfloor nb\rfloor} \psi(nx-k)\right) 2\left\|f^{(N)}\right\|_\infty \frac{(b-a)^N}{N!}$$

$$\le \omega_1\left(f^{(N)},\frac{1}{n^\alpha}\right)\frac{1}{N!n^{\alpha N}} + \left(\frac{2}{\pi^2\left(n^{1-\alpha}-2\right)}\right) 2\left\|f^{(N)}\right\|_\infty \frac{(b-a)^N}{N!}$$

$$= \omega_1\left(f^{(N)},\frac{1}{n^\alpha}\right)\frac{1}{N!n^{\alpha N}} + \frac{4\left\|f^{(N)}\right\|_\infty (b-a)^N}{N!\pi^2\left(n^{1-\alpha}-2\right)}.$$

That is

$$\|\Lambda_n(x)\| \le \frac{\omega_1\left(f^{(N)},\frac{1}{n^\alpha}\right)}{N!n^{\alpha N}} + \frac{4\left\|f^{(N)}\right\|_\infty (b-a)^N}{N!\pi^2\left(n^{1-\alpha}-2\right)}, \tag{74}$$

$\forall\ x \in [a,b]$.

We further see that

$$A_n^*\left((\cdot - x)^j\right) = \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \psi(nx-k)\left(\frac{k}{n}-x\right)^j, \tag{75}$$

where A_n^* is defined similarly for real valued functions.

Therefore

$$\left|A_n^*\left((\cdot - x)^j\right)\right| \le \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \psi(nx-k)\left|\frac{k}{n}-x\right|^j$$

$$= \sum_{\substack{k=\lceil na\rceil \\ \left|\frac{k}{n}-x\right| \le \frac{1}{n^\alpha}}}^{\lfloor nb\rfloor} \psi(nx-k)\left|\frac{k}{n}-x\right|^j$$

$$+ \sum_{\substack{k=\lceil na\rceil \\ \left|\frac{k}{n}-x\right| > \frac{1}{n^\alpha}}}^{\lfloor nb\rfloor} \psi(nx-k)\left|\frac{k}{n}-x\right|^j \tag{76}$$

$$\le \frac{1}{n^{\alpha j}} + (b-a)^j \frac{2}{\pi^2\left(n^{1-\alpha}-2\right)}.$$

That is

$$\left|A_n^*\left((\cdot - x)^j\right)\right| \leq \frac{1}{n^{\alpha j}} + (b-a)^j \frac{2}{\pi^2\left(n^{1-\alpha}-2\right)}, \tag{77}$$

for $j = 1, \ldots, N$.

Putting things together we have proved

$$\left\| A_n^*(f,x) - f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)\right)\right\| \leq \sum_{j=1}^{N}\frac{\left\|f^{(j)}(x)\right\|}{j!} \tag{78}$$

$$\left[\frac{1}{n^{\alpha j}} + \frac{2(b-a)^j}{\pi^2\left(n^{1-\alpha}-2\right)}\right] + \left[\omega_1\left(f^{(N)}, \frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!} + \frac{4\left\|f^{(N)}\right\|_{\infty}(b-a)^N}{N!\pi^2\left(n^{1-\alpha}-2\right)}\right],$$

that is establishing the theorem. □

All integrals from now on are of Bochner type [17].

We need

Definition 14 ([12])**.** Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$; $m = \lceil\alpha\rceil \in \mathbb{N}$, ($\lceil\cdot\rceil$ is the ceiling of the number), $f : [a,b] \to X$. We assume that $f^{(m)} \in L_1([a,b], X)$. We call the Caputo-Bochner left fractional derivative of order α:

$$\left(D_{*a}^{\alpha}f\right)(x) := \frac{1}{\Gamma(m-\alpha)}\int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t)\,dt, \quad \forall\, x \in [a,b]. \tag{79}$$

If $\alpha \in \mathbb{N}$, we set $D_{*a}^{\alpha}f := f^{(m)}$ the ordinary X-valued derivative (defined similar to numerical one, see [19, p. 83]), and also set $D_{*a}^{0}f := f$.

By [12], $\left(D_{*a}^{\alpha}f\right)(x)$ exists almost everywhere in $x \in [a,b]$ and $D_{*a}^{\alpha}f \in L_1([a,b], X)$.

If $\left\|f^{(m)}\right\|_{L_\infty([a,b],X)} < \infty$, then by [12], $D_{*a}^{\alpha}f \in C([a,b], X)$, hence $\left\|D_{*a}^{\alpha}f\right\| \in C([a,b])$.

We mention

Lemma 15 ([11])**.** *Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, $m = \lceil\alpha\rceil$, $f \in C^{m-1}([a,b], X)$ and $f^{(m)} \in L_\infty([a,b], X)$. Then $D_{*a}^{\alpha}f(a) = 0$.*

We mention

Definition 16 ([10])**.** Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$, $m := \lceil\alpha\rceil$. We assume that $f^{(m)} \in L_1([a,b], X)$, where $f : [a,b] \to X$. We call the Caputo-Bochner right fractional derivative of order α:

$$\left(D_{b-}^{\alpha}f\right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)}\int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z)\,dz, \quad \forall\, x \in [a,b]. \tag{80}$$

We observe that $\left(D_{b-}^{m}f\right)(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $\left(D_{b-}^{0}f\right)(x) = f(x)$.

By [10], $\left(D_{b-}^{\alpha}f\right)(x)$ exists almost everywhere on $[a,b]$ and $\left(D_{b-}^{\alpha}f\right) \in L_1([a,b],X)$.

If $\left\|f^{(m)}\right\|_{L_\infty([a,b],X)} < \infty$, and $\alpha \notin \mathbb{N}$, by [10], $D_{b-}^{\alpha}f \in C([a,b],X)$, hence $\left\|D_{b-}^{\alpha}f\right\| \in C([a,b])$.

We need

Lemma 17 ([11]). *Let $f \in C^{m-1}([a,b],X)$, $f^{(m)} \in L_\infty([a,b],X)$, $m = \lceil\alpha\rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$. Then $D_{b-}^{\alpha}f(b) = 0$.*

We mention the left fractional Taylor formula

Theorem 18 ([12]). *Let $m \in \mathbb{N}$ and $f \in C^m([a,b],X)$, where $[a,b] \subset \mathbb{R}$ and X is a Banach space, and let $\alpha > 0 : m = \lceil\alpha\rceil$. Then*

$$f(x) = \sum_{i=0}^{m-1} \frac{(x-a)^i}{i!} f^{(i)}(a) + \frac{1}{\Gamma(\alpha)} \int_a^x (x-z)^{\alpha-1} \left(D_{*a}^{\alpha}f\right)(z)\,dz, \tag{81}$$

$\forall\ x \in [a,b]$.

We also mention the right fractional Taylor formula

Theorem 19 ([10]). *Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$, $m = \lceil\alpha\rceil$, $f \in C^m([a,b],X)$. Then*

$$f(x) = \sum_{i=0}^{m-1} \frac{(x-b)^i}{i!} f^{(i)}(b) + \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} \left(D_{b-}^{\alpha}f\right)(z)\,dz, \tag{82}$$

$\forall\ x \in [a,b]$.

Convention 20. We assume that

$$D_{*x_0}^{\alpha}f(x) = 0, \quad \text{for } x < x_0, \tag{83}$$

and

$$D_{x_0-}^{\alpha}f(x) = 0, \quad \text{for } x > x_0, \tag{84}$$

for all $x, x_0 \in [a,b]$.

We mention

Proposition 21 ([11]). *Let $f \in C^n([a,b],X)$, $n = \lceil\nu\rceil$, $\nu > 0$. Then $D_{*a}^{\nu}f(x)$ is continuous in $x \in [a,b]$.*

Proposition 22 ([11]). *Let $f \in C^m([a,b],X)$, $m = \lceil\alpha\rceil$, $\alpha > 0$. Then $D_{b-}^{\nu}f(x)$ is continuous in $x \in [a,b]$.*

We also mention

Proposition 23 ([11]). *Let $f \in C^{m-1}([a,b],X)$, $f^{(m)} \in L_{\infty}([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and*

$$D_{*x_0}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{85}$$

for all $x, x_0 \in [a,b] : x \geq x_0$.

Then $D_{*x_0}^{\alpha} f(x)$ is continuous in x_0.

Proposition 24 ([11]). *Let $f \in C^{m-1}([a,b],X)$, $f^{(m)} \in L_{\infty}([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and*

$$D_{x_0-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_{x}^{x_0} (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta)\, d\zeta, \tag{86}$$

for all $x, x_0 \in [a,b] : x_0 \geq x$.

Then $D_{x_0-}^{\alpha} f(x)$ is continuous in x_0.

Corollary 25 ([11]). *Let $f \in C^{m}([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $x, x_0 \in [a,b]$. Then $D_{*x_0}^{a} f(x)$, $D_{x_0-}^{a} f(x)$ are jointly continuous functions in (x, x_0) from $[a,b]^2$ into X, X is a Banach space.*

We need

Theorem 26 ([11]). *Let $f : [a,b]^2 \to X$ be jointly continuous, X is a Banach space. Consider*

$$G(x) = \omega_1 (f(\cdot, x), \delta, [x,b]), \tag{87}$$

$\delta > 0$, $x \in [a,b]$.

Then G is continuous on $[a,b]$.

Theorem 27 ([11]). *Let $f : [a,b]^2 \to X$ be jointly continuous, X is a Banach space. Then*

$$H(x) = \omega_1 (f(\cdot, x), \delta, [a,x]), \tag{88}$$

$x \in [a,b]$, is continuous in $x \in [a,b]$, $\delta > 0$.

We make

Remark 28 ([11]). Let $f \in C^{n-1}([a,b])$, $f^{(n)} \in L_{\infty}([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$, $\nu \notin \mathbb{N}$. Then

$$\| D_{*a}^{\nu} f(x) \| \leq \frac{\| f^{(n)} \|_{L_{\infty}([a,b],X)}}{\Gamma(n-\nu+1)} (x-a)^{n-\nu}, \quad \forall\, x \in [a,b]. \tag{89}$$

Thus we observe

$$\omega_1 (D_{*a}^{\nu} f, \delta) = \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} \| D_{*a}^{\nu} f(x) - D_{*a}^{\nu} f(y) \| \tag{90}$$

$$\leq \sup_{\substack{x,y\in[a,b]\\|x-y|\leq\delta}} \left(\frac{\left\|f^{(n)}\right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)} (x-a)^{n-\nu} + \frac{\left\|f^{(n)}\right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)} (y-a)^{n-\nu} \right)$$

$$\leq \frac{2\left\|f^{(n)}\right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)} (b-a)^{n-\nu}.$$

Consequently

$$\omega_1\left(D_{*a}^{\nu} f, \delta\right) \leq \frac{2\left\|f^{(n)}\right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)} (b-a)^{n-\nu}. \tag{91}$$

Similarly, let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, then

$$\omega_1\left(D_{b-}^{\alpha} f, \delta\right) \leq \frac{2\left\|f^{(m)}\right\|_{L_\infty([a,b],X)}}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}. \tag{92}$$

So for $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, we find

$$\sup_{x_0\in[a,b]} \omega_1\left(D_{*x_0}^{\alpha} f, \delta\right)_{[x_0,b]} \leq \frac{2\left\|f^{(m)}\right\|_{L_\infty([a,b],X)}}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}, \tag{93}$$

and

$$\sup_{x_0\in[a,b]} \omega_1\left(D_{x_0-}^{\alpha} f, \delta\right)_{[a,x_0]} \leq \frac{2\left\|f^{(m)}\right\|_{L_\infty([a,b],X)}}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}. \tag{94}$$

By [12] we get that $D_{*x_0}^{\alpha} f \in C([x_0,b],X)$, and by [10] we obtain that $D_{x_0-}^{\alpha} f \in C([a,x_0],X)$.

We present the following X-valued fractional approximation result by neural networks.

Theorem 29. *Let $\alpha > 0$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $f \in C^N([a,b],X)$, $0 < \beta < 1$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. Then*

i)

$$\left\| A_n(f,x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n\left((\cdot - x)^j\right)(x) - f(x) \right\|$$

$$\leq \frac{(0.0868)}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D_{x-}^{\alpha} f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha} f, \frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\alpha\beta}} \right.$$

$$\left. + \frac{2}{\pi^2\left(n^{1-\beta}-2\right)} \left(\left\|D_{x-}^{\alpha} f\right\|_{\infty,[a,x]} (x-a)^{\alpha} + \left\|D_{*x}^{\alpha} f\right\|_{\infty,[x,b]} (b-x)^{\alpha} \right) \right\}, \tag{95}$$

ii) if $f^{(j)}(x) = 0$, *for* $j = 1, \ldots, N-1$, *we have*

$$\|A_n(f,x) - f(x)\| \leq \frac{(0.0868)}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D^{\alpha}_{x-}f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1\left(D^{\alpha}_{*x}f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} \right.$$

(96) $$\left. + \frac{2}{\pi^2(n^{1-\beta}-2)} \left(\left\|D^{\alpha}_{x-}f\right\|_{\infty,[a,x]} (x-a)^{\alpha} + \left\|D^{\alpha}_{*x}f\right\|_{\infty,[x,b]} (b-x)^{\alpha} \right) \right\},$$

iii)

$$\|A_n(f,x) - f(x)\| \leq (0.0868)$$

$$\times \left\{ \sum_{j=1}^{N-1} \frac{\left\|f^{(j)}(x)\right\|}{j!} \left\{ \frac{1}{n^{\beta j}} + (b-a)^j \frac{2}{\pi^2(n^{1-\beta}-2)} \right\} \right.$$

$$+ \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D^{\alpha}_{x-}f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1\left(D^{\alpha}_{*x}f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} \right.$$

(97) $$\left.\left. + \frac{2}{\pi^2(n^{1-\beta}-2)} \left(\left\|D^{\alpha}_{x-}f\right\|_{\infty,[a,x]} (x-a)^{\alpha} + \left\|D^{\alpha}_{*x}f\right\|_{\infty,[x,b]} (b-x)^{\alpha} \right) \right\} \right\},$$

$\forall\, x \in [a,b]$, *and*

iv)

$$\|A_n f - f\|_{\infty} \leq (0.0868)$$

$$\times \left\{ \sum_{j=1}^{N-1} \frac{\left\|f^{(j)}\right\|_{\infty}}{j!} \left\{ \frac{1}{n^{\beta j}} + (b-a)^j \frac{2}{\pi^2(n^{1-\beta}-2)} \right\} \right.$$

$$+ \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1\left(D^{\alpha}_{x-}f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1\left(D^{\alpha}_{*x}f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} \right.$$

(98)

$$\left.\left. + \frac{2}{\pi^2(n^{1-\beta}-2)} (b-a)^{\alpha} \left(\sup_{x\in[a,b]} \left\|D^{\alpha}_{x-}f\right\|_{\infty,[a,x]} + \sup_{x\in[a,b]} \left\|D^{\alpha}_{*x}f\right\|_{\infty,[x,b]} \right) \right\} \right\}.$$

Above, when $N = 1$ *the sum* $\sum_{j=1}^{N-1} \cdot = 0$.

As we see here we obtain X*-valued fractionally type pointwise and uniform convergence with rates of* $A_n \to I$ *the unit operator, as* $n \to \infty$.

Proof. Let $x \in [a,b]$. We have that $D^{\alpha}_{x-}f(x) = D^{\alpha}_{*x}f(x) = 0$.

From Theorem 18, we get by the left Caputo fractional Taylor formula that

(99) $$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{k}{n} - x\right)^j$$

$$+ \frac{1}{\Gamma(\alpha)} \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} \left(D^{\alpha}_{*x}f(J) - D^{\alpha}_{*x}f(x)\right) dJ,$$

for all $x \leq \frac{k}{n} \leq b$.

Also from Theorem 19, using the right Caputo fractional Taylor formula we get

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^j + \frac{1}{\Gamma(\alpha)} \int_{\frac{k}{n}}^{x} \left(J-\frac{k}{n}\right)^{\alpha-1} \left(D_{x-}^{\alpha} f(J) - D_{x-}^{\alpha} f(x)\right) dJ, \tag{100}$$

for all $a \leq \frac{k}{n} \leq x$.

Hence we have

$$f\left(\frac{k}{n}\right)\psi(nx-k) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!}\psi(nx-k)\left(\frac{k}{n}-x\right)^j + \frac{\psi(nx-k)}{\Gamma(\alpha)} \int_{x}^{\frac{k}{n}} \left(\frac{k}{n}-J\right)^{\alpha-1} \left(D_{*x}^{\alpha} f(J) - D_{*x}^{\alpha} f(x)\right) dJ, \tag{101}$$

for all $x \leq \frac{k}{n} \leq b$, iff $\lceil nx \rceil \leq k \leq \lfloor nb \rfloor$, and

$$f\left(\frac{k}{n}\right)\psi(nx-k) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!}\psi(nx-k)\left(\frac{k}{n}-x\right)^j + \frac{\psi(nx-k)}{\Gamma(\alpha)} \int_{\frac{k}{n}}^{x} \left(J-\frac{k}{n}\right)^{\alpha-1} \left(D_{x-}^{\alpha} f(J) - D_{x-}^{\alpha} f(x)\right) dJ, \tag{102}$$

for all $a \leq \frac{k}{n} \leq x$, iff $\lceil na \rceil \leq k \leq \lfloor nx \rfloor$.

Therefore it holds

$$\sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right)\psi(nx-k) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} \psi(nx-k)\left(\frac{k}{n}-x\right)^j + \frac{1}{\Gamma(\alpha)} \sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} \psi(nx-k) \int_{x}^{\frac{k}{n}} \left(\frac{k}{n}-J\right)^{\alpha-1} \left(D_{*x}^{\alpha} f(J) - D_{*x}^{\alpha} f(x)\right) dJ, \tag{103}$$

and

$$\sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} f\left(\frac{k}{n}\right)\psi(nx-k) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \psi(nx-k)\left(\frac{k}{n}-x\right)^j + \frac{1}{\Gamma(\alpha)} \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \psi(nx-k) \int_{\frac{k}{n}}^{x} \left(J-\frac{k}{n}\right)^{\alpha-1} \left(D_{x-}^{\alpha} f(J) - D_{x-}^{\alpha} f(x)\right) dJ. \tag{104}$$

Adding the last two equalities (103) and (104) obtain

(105)

$$A_n^*(f,x) = \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)\psi(nx-k)$$
$$= \sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)\left(\frac{k}{n}-x\right)^j$$
$$+\frac{1}{\Gamma(\alpha)}\left\{\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\psi(nx-k)\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J)-D_{x-}^{\alpha}f(x)\right)dJ\right.$$
$$\left.+\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\psi(nx-k)\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right)dJ\right\}.$$

So we have derived

$$A_n^*(f,x)-f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)\right)=\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}A_n^*\left((\cdot-x)^j\right)+u_n(x), \tag{106}$$

where

$$u_n(x) := \frac{1}{\Gamma(\alpha)}\left\{\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\psi(nx-k)\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J)-D_{x-}^{\alpha}f(x)\right)dJ\right.$$
$$\left.+\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\psi(nx-k)\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right)dJ\right\}. \tag{107}$$

We set

(108)

$$u_{1n}(x) := \frac{1}{\Gamma(\alpha)}\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\psi(nx-k)\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J)-D_{x-}^{\alpha}f(x)\right)dJ,$$

and

$$u_{2n} := \frac{1}{\Gamma(\alpha)}\sum_{k=\lceil nx\rceil+1}^{\lfloor nb\rfloor}\psi(nx-k)\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right)dJ, \tag{109}$$

i.e.

$$u_n(x) = u_{1n}(x)+u_{2n}(x). \tag{110}$$

We assume $b-a>\frac{1}{n^\beta}$, $0<\beta<1$, which is always the case for large enough $n\in\mathbb{N}$, that is when $n>\left\lceil (b-a)^{-\frac{1}{\beta}}\right\rceil$. It is always true that either $\left|\frac{k}{n}-x\right|\le\frac{1}{n^\beta}$ or $\left|\frac{k}{n}-x\right|>\frac{1}{n^\beta}$.

For $k = \lceil na \rceil, \dots, \lfloor nx \rfloor$, we consider

$$\gamma_{1k} := \left\| \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\alpha-1} \left(D_{x-}^{\alpha} f(J) - D_{x-}^{\alpha} f(x) \right) dJ \right\| \tag{111}$$

$$= \left\| \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\alpha-1} D_{x-}^{\alpha} f(J)\, dJ \right\| \leq \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\alpha-1} \left\| D_{x-}^{\alpha} f(J) \right\| dJ$$

$$\leq \left\| D_{x-}^{\alpha} f(J) \right\|_{\infty,[a,x]} \frac{\left(x - \frac{k}{n} \right)^{\alpha}}{\alpha} \leq \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} \frac{(x-a)^{\alpha}}{\alpha}. \tag{112}$$

That is

$$\gamma_{1k} \leq \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} \frac{(x-a)^{\alpha}}{\alpha}, \tag{113}$$

for $k = \lceil na \rceil, \dots, \lfloor nx \rfloor$.

Also we have in case of $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}}$ that

$$\gamma_{1k} \leq \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\alpha-1} \left\| D_{x-}^{\alpha} f(J) - D_{x-}^{\alpha} f(x) \right\| dJ$$

$$\leq \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\alpha-1} \omega_1 \left(D_{x-}^{\alpha} f, |J - x| \right)_{[a,x]} dJ \tag{114}$$

$$\leq \omega_1 \left(D_{x-}^{\alpha} f, \left| x - \frac{k}{n} \right| \right)_{[a,x]} \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\alpha-1} dJ$$

$$\leq \omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[a,x]} \frac{\left(x - \frac{k}{n} \right)^{\alpha}}{\alpha} \leq \omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[a,x]} \frac{1}{\alpha n^{\alpha\beta}}.$$

That is when $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}}$, then

$$\gamma_{1k} \leq \frac{\omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[a,x]}}{\alpha n^{\alpha\beta}}. \tag{115}$$

Consequently we obtain

(116)

$$\left\| u_{1n}(x) \right\| \leq \frac{1}{\Gamma(\alpha)} \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \psi(nx - k)\, \gamma_{1k}$$

$$= \frac{1}{\Gamma(\alpha)} \left\{ \sum_{\left\{ \begin{array}{l} k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}} \end{array} \right.}^{\lfloor nx \rfloor} \psi(nx - k)\, \gamma_{1k} + \sum_{\left\{ \begin{array}{l} k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^{\beta}} \end{array} \right.}^{\lfloor nx \rfloor} \psi(nx - k)\, \gamma_{1k} \right\}$$

$$\leq \frac{1}{\Gamma(\alpha)} \left\{ \left(\sum_{\left\{\begin{array}{l} k=\lceil na \rceil \\ : \left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\beta}} \end{array}\right.}^{\lfloor nx \rfloor} \psi(nx-k) \right) \frac{\omega_1\left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[a,x]}}{\alpha n^{\alpha\beta}} \right.$$

(117)

$$\left. + \left(\sum_{\left\{\begin{array}{l} k=\lceil na \rceil \\ : \left|\frac{k}{n}-x\right| > \frac{1}{n^{\beta}} \end{array}\right.}^{\lfloor nx \rfloor} \psi(nx-k) \right) \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} \frac{(x-a)^{\alpha}}{\alpha} \right\}$$

$$\leq \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\omega_1\left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[a,x]}}{n^{\alpha\beta}} \right.$$

$$\left. + \left(\sum_{\left\{\begin{array}{l} k=-\infty \\ : |nx-k| > n^{1-\beta} \end{array}\right.}^{\infty} \psi(nx-k) \right) \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} (x-a)^{\alpha} \right\}$$

$$\leq \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\omega_1\left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[a,x]}}{n^{\alpha\beta}} + \frac{2\left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} (x-a)^{\alpha}}{\pi^2\left(n^{1-\beta}-2\right)} \right\}.$$

So we have proved that

$$\|u_{1n}(x)\| \leq \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\omega_1\left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[a,x]}}{n^{\alpha\beta}} + \frac{2}{\pi^2\left(n^{1-\beta}-2\right)} \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} (x-a)^{\alpha} \right\}. \tag{118}$$

Next when $k = \lfloor nx \rfloor + 1, \ldots, \lfloor nb \rfloor$ we consider

$$\gamma_{2k} := \left\| \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} \left(D_{*x}^{\alpha} f(J) - D_{*x}^{\alpha} f(x)\right) dJ \right\| \tag{119}$$

$$\leq \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} \left\| D_{*x}^{\alpha} f(J) - D_{*x}^{\alpha} f(x) \right\| dJ$$

$$= \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} \left\| D_{*x}^{\alpha} f(J) \right\| dJ$$

$$\leq \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} \frac{\left(\frac{k}{n}-x\right)^{\alpha}}{\alpha} \leq \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} \frac{(b-x)^{\alpha}}{\alpha}. \tag{120}$$

Therefore when $k = \lfloor nx \rfloor + 1, \ldots, \lfloor nb \rfloor$ we get that

$$\gamma_{2k} \leq \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} \frac{(b-x)^{\alpha}}{\alpha}. \tag{121}$$

In case of $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}}$ we have

$$\begin{aligned} \gamma_{2k} &\leq \int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - J \right)^{\alpha-1} \omega_1 \left(D_{*x}^{\alpha} f, |J - x| \right)_{[x,b]} dJ \\ &\leq \omega_1 \left(D_{*x}^{\alpha} f, \left| \frac{k}{n} - x \right| \right)_{[x,b]} \int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - J \right)^{\alpha-1} dJ \\ &\leq \omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x,b]} \frac{\left(\frac{k}{n} - x \right)^{\alpha}}{\alpha} \leq \omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x,b]} \frac{1}{\alpha n^{\alpha\beta}}. \end{aligned} \tag{122}$$

So when $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}}$ we derived that

$$\gamma_{2k} \leq \frac{\omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x,b]}}{\alpha n^{\alpha\beta}}. \tag{123}$$

Similarly we have that

$$\begin{aligned} \left\| u_{2n}(x) \right\| &\leq \frac{1}{\Gamma(\alpha)} \left(\sum_{k=\lfloor nx \rfloor + 1}^{\lfloor nb \rfloor} \psi(nx-k) \gamma_{2k} \right) \\ &= \frac{1}{\Gamma(\alpha)} \left\{ \sum_{\substack{k = \lfloor nx \rfloor + 1 \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}}}}^{\lfloor nb \rfloor} \psi(nx-k) \gamma_{2k} + \sum_{\substack{k = \lfloor nx \rfloor + 1 \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^{\beta}}}}^{\lfloor nb \rfloor} \psi(nx-k) \gamma_{2k} \right\} \\ &\leq \frac{1}{\Gamma(\alpha)} \left\{ \left(\sum_{\substack{k = \lfloor nx \rfloor + 1 \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}}}}^{\lfloor nb \rfloor} \psi(nx-k) \right) \frac{\omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x,b]}}{\alpha n^{\alpha\beta}} \right. \\ &\quad \left. + \left(\sum_{\substack{k = \lfloor nx \rfloor + 1 \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^{\beta}}}}^{\lfloor nb \rfloor} \psi(nx-k) \right) \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} \frac{(b-x)^{\alpha}}{\alpha} \right\} \\ &\leq \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x,b]}}{n^{\alpha\beta}} \right. \end{aligned} \tag{124}$$

(125)

$$+\left(\sum_{\left\{\begin{array}{l}k=-\infty\\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\beta}}\end{array}\right.}^{\infty}\psi\left(nx-k\right)\right)\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\left(b-x\right)^{\alpha}\right\}$$

$$\leq\frac{1}{\Gamma\left(\alpha+1\right)}\left\{\frac{\omega_{1}\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}}{n^{\alpha\beta}}+\frac{2}{\pi^{2}\left(n^{1-\beta}-2\right)}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\left(b-x\right)^{\alpha}\right\}.$$

So we have proved that

$$\left\|u_{2n}\left(x\right)\right\|\leq\frac{1}{\Gamma\left(\alpha+1\right)}\left\{\frac{\omega_{1}\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}}{n^{\alpha\beta}}+\frac{2}{\pi^{2}\left(n^{1-\beta}-2\right)}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\left(b-x\right)^{\alpha}\right\}. \tag{126}$$

Therefore

$$\left\|u_{n}\left(x\right)\right\|\leq\left\|u_{1n}\left(x\right)\right\|+\left\|u_{2n}\left(x\right)\right\|$$

$$\leq\frac{1}{\Gamma\left(\alpha+1\right)}\left\{\frac{\omega_{1}\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_{1}\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}}{n^{\alpha\beta}}+\frac{2}{\pi^{2}\left(n^{1-\beta}-2\right)}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}\left(x-a\right)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\left(b-x\right)^{\alpha}\right)\right\}. \tag{127}$$

From the proof of Theorem 13 we get that

$$\left|A_{n}^{*}\left(\left(\cdot-x\right)^{j}\right)\left(x\right)\right|\leq\frac{1}{n^{\beta j}}+\left(b-a\right)^{j}\frac{2}{\pi^{2}\left(n^{1-\beta}-2\right)}, \tag{128}$$

for $j=1,\ldots,N-1$, $\forall\ x\in[a,b]$.

Putting things together, we have established

(129)

$$\left\|A_{n}^{*}\left(f,x\right)-f\left(x\right)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi\left(nx-k\right)\right)\right\|\leq\sum_{j=1}^{N-1}\frac{\left\|f^{(j)}\left(x\right)\right\|}{j!}\left[\frac{1}{n^{\beta j}}+\left(b-a\right)^{j}\frac{2}{\pi^{2}\left(n^{1-\alpha}-2\right)}\right]+\frac{1}{\Gamma\left(\alpha+1\right)}\left\{\frac{\omega_{1}\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_{1}\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}}{n^{\alpha\beta}}\right.$$

(130)

$$\left.+\frac{2}{\pi^{2}\left(n^{1-\beta}-2\right)}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}\left(x-a\right)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\left(b-x\right)^{\alpha}\right)\right\}=:K_{n}\left(x\right).$$

As a result we derive

$$\|A_n(f,x) - f(x)\| \leq 0.0868 K_n(x), \quad \forall\, x \in [a,b]. \tag{131}$$

We further have that

(132)

$$\|K_n\|_\infty \leq \sum_{j=1}^{N-1} \frac{\|f^{(j)}\|_\infty}{j!} \left[\frac{1}{n^{\beta j}} + (b-a)^j \frac{2}{\pi^2 (n^{1-\alpha} - 2)} \right]$$
$$+ \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left\{ \sup_{x\in[a,b]} \left(\omega_1 \left(D^\alpha_{x-} f, \frac{1}{n^\beta} \right)_{[a,x]} \right) + \sup_{x\in[a,b]} \left(\omega_1 \left(D^\alpha_{*x} f, \frac{1}{n^\beta} \right)_{[x,b]} \right) \right\}}{n^{\alpha\beta}} \right.$$
$$+ \frac{2}{\pi^2 (n^{1-\beta} - 2)} (b-a)^\alpha$$
$$\left. \times \left\{ \left(\sup_{x\in[a,b]} \left(\left\| D^\alpha_{x-} f \right\| \right) + \sup_{x\in[a,b]} \left(\|D^\alpha_{*x} f\|_{\infty,[x,b]} \right) \right) \right\} \right\} =: E_n.$$

Hence it holds

$$\|A_n f - f\|_\infty \leq 0.0868 E_n. \tag{133}$$

We observe the following:

We have

$$\left(D^\alpha_{x-} f \right)(y) = \frac{(-1)^N}{\Gamma(N-\alpha)} \int_y^x (J-y)^{N-\alpha-1} f^{(N)}(J)\, dJ, \quad \forall\, y \in [a,x] \tag{134}$$

and

$$\left\| \left(D^\alpha_{x-} f \right)(y) \right\| \leq \frac{1}{\Gamma(N-\alpha)} \left(\int_y^x (J-y)^{N-\alpha-1} dJ \right) \left\| f^{(N)} \right\|_\infty$$
$$= \frac{1}{\Gamma(N-\alpha)} \frac{(x-y)^{N-\alpha}}{(N-\alpha)} \left\| f^{(N)} \right\|_\infty = \frac{(x-y)^{N-\alpha}}{\Gamma(N-\alpha+1)} \left\| f^{(N)} \right\|_\infty$$
$$\leq \frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)} \left\| f^{(N)} \right\|_\infty. \tag{135}$$

That is

$$\left\| D^\alpha_{x-} f \right\|_{\infty,[a,x]} \leq \frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)} \left\| f^{(N)} \right\|_\infty, \tag{136}$$

and

$$\sup_{x\in[a,b]} \left\| D^\alpha_{x-} f \right\|_{\infty,[a,x]} \leq \frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)} \left\| f^{(N)} \right\|_\infty. \tag{137}$$

Similarly we have

$$\left(D^\alpha_{*x} f \right)(y) = \frac{1}{\Gamma(N-\alpha)} \int_x^y (y-t)^{N-\alpha-1} f^{(N)}(t)\, dt, \quad \forall\, y \in [x,b]. \tag{138}$$

Thus we get

$$\left\| \left(D_{*x}^{\alpha} f\right)(y)\right\| \leq \frac{1}{\Gamma(N-\alpha)} \left(\int_x^y (y-t)^{N-\alpha-1} dt \right) \left\| f^{(N)} \right\|_{\infty} \tag{139}$$

$$\leq \frac{1}{\Gamma(N-\alpha)} \frac{(y-x)^{N-\alpha}}{(N-\alpha)} \left\| f^{(N)} \right\|_{\infty} \leq \frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)} \left\| f^{(N)} \right\|_{\infty}.$$

Hence

$$\left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} \leq \frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)} \left\| f^{(N)} \right\|_{\infty}, \tag{140}$$

and

$$\sup_{x\in[a,b]} \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} \leq \frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)} \left\| f^{(N)} \right\|_{\infty}. \tag{141}$$

From (93) and (94) we get

$$\sup_{x\in[a,b]} \omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[a,x]} \leq \frac{2\left\| f^{(N)} \right\|_{\infty}}{\Gamma(N-\alpha+1)} (b-a)^{N-\alpha}, \tag{142}$$

and

$$\sup_{x\in[a,b]} \omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x,b]} \leq \frac{2\left\| f^{(N)} \right\|_{\infty}}{\Gamma(N-\alpha+1)} (b-a)^{N-\alpha}. \tag{143}$$

That is $E_n < \infty$.

We finally notice that

$$A_n(f,x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n\left((\cdot - x)^j \right)(x) - f(x)$$

$$= \frac{A_n^*(f,x)}{\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \psi(nx-k)\right)} - \frac{1}{\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \psi(nx-k)\right)}$$

$$\times \left(\sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n^*\left((\cdot - x)^j \right)(x) \right) - f(x)$$

$$= \frac{1}{\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \psi(nx-k)\right)} \left[A_n^*(f,x) - \left(\sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n^*\left((\cdot - x)^j \right)(x) \right) \right. \tag{144}$$

$$\left. - \left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \psi(nx-k) \right) f(x) \right].$$

Therefore we get

$$\left\| A_n(f,x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n\left((\cdot - x)^j \right)(x) - f(x) \right\| \leq (0.0868) \cdot$$

$$\left\| A_n^* (f, x) - \left(\sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n^* \left((\cdot - x)^j \right)(x) \right) - \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi (nx - k) \right) f(x) \right\|, \tag{145}$$

$\forall\ x \in [a, b]$.

The proof of the theorem is now finished. □

Next we apply Theorem 29 for $N = 1$.

Theorem 30. *Let* $0 < \alpha, \beta < 1$, $f \in C^1([a,b], X)$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. *Then*

i)

$$\|A_n(f,x) - f(x)\| \leq \frac{(0.0868)}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[a,x]} + \omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x,b]} \right)}{n^{\alpha\beta}} \right.$$

$$\left. + \frac{2}{\pi^2 (n^{1-\beta} - 2)} \left(\left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} (x-a)^{\alpha} + \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} (b-x)^{\alpha} \right) \right\}, \tag{146}$$

and

ii)

$$\|A_n f - f\|_{\infty} \leq \frac{(0.0868)}{\Gamma(\alpha+1)} \left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x,b]} \right)}{n^{\alpha\beta}} \right.$$

$$\left. + \frac{2(b-a)^{\alpha}}{\pi^2 (n^{1-\beta} - 2)} \left(\sup_{x\in[a,b]} \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} + \sup_{x\in[a,b]} \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} \right) \right\}. \tag{147}$$

When $\alpha = \frac{1}{2}$ we derive

Corollary 31. *Let* $0 < \beta < 1$, $f \in C^1([a,b], X)$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. *Then*

i)

$$\|A_n(f,x) - f(x)\| \leq \frac{0.1736}{\sqrt{\pi}} \left\{ \frac{\left(\omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^{\beta}} \right)_{[a,x]} + \omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^{\beta}} \right)_{[x,b]} \right)}{n^{\frac{\beta}{2}}} \right.$$

$$\left. + \frac{2}{\pi^2 (n^{1-\beta} - 2)} \left(\left\| D_{x-}^{\frac{1}{2}} f \right\|_{\infty,[a,x]} \sqrt{(x-a)} + \left\| D_{*x}^{\frac{1}{2}} f \right\|_{\infty,[x,b]} \sqrt{(b-x)} \right) \right\}, \tag{148}$$

and

ii)

$$\|A_n f - f\|_\infty \leq \frac{0.1736}{\sqrt{\pi}} \left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^\beta} \right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\frac{\beta}{2}}} \right.$$

$$(149) \qquad \left. + \frac{2\sqrt{(b-a)}}{\pi^2 (n^{1-\beta} - 2)} \left(\sup_{x\in[a,b]} \left\| D_{x-}^{\frac{1}{2}} f \right\|_{\infty,[a,x]} + \sup_{x\in[a,b]} \left\| D_{*x}^{\frac{1}{2}} f \right\|_{\infty,[x,b]} \right) \right\} < \infty.$$

We finish with

Remark 32. Some convergence analysis follows:

Let $0 < \beta < 1$, $f \in C^1([a,b], X)$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. We elaborate on (149). Assume that

$$(150) \qquad \omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^\beta} \right)_{[a,x]} \leq \frac{K_1}{n^\beta},$$

and

$$(151) \qquad \omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^\beta} \right)_{[x,b]} \leq \frac{K_2}{n^\beta},$$

$\forall\, x \in [a,b]$, $\forall\, n \in \mathbb{N}$, where $K_1, K_2 > 0$.

Then it holds

$$\frac{\left[\sup_{x\in[a,b]} \omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^\beta} \right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^\beta} \right)_{[x,b]} \right]}{n^{\frac{\beta}{2}}}$$

$$(152) \qquad \leq \frac{\frac{(K_1+K_2)}{n^\beta}}{n^{\frac{\beta}{2}}} = \frac{(K_1+K_2)}{n^{\frac{3\beta}{2}}} = \frac{K}{n^{\frac{3\beta}{2}}},$$

where $K := K_1 + K_2 > 0$.

The other summand of the right hand side of (149), for large enough n, converges to zero at the speed $\frac{1}{n^{1-\beta}}$, so it is about $\frac{L}{N^{1-\beta}}$, where $L > 0$ is a constant.

Then, for large enough $n \in \mathbb{N}$, by (149), (152) and the above comment, we obtain that

$$(153) \qquad \|A_n f - f\|_\infty \leq \frac{M}{\min\left(n^{\frac{3\beta}{2}}, n^{1-\beta} \right)},$$

where $M > 0$.

Clearly we have two cases:

i)

$$(154) \qquad \|A_n f - f\|_\infty \leq \frac{M}{n^{1-\beta}}, \quad \text{when } \frac{2}{5} \leq \beta < 1,$$

with speed of convergence $\frac{1}{n^{1-\beta}}$, and ii)

$$\|A_n f - f\|_\infty \leq \frac{M}{n^{\frac{3\beta}{2}}}, \quad \text{when } 0 < \beta < \frac{2}{5}, \tag{155}$$

with speed of convergence $\frac{1}{n^{\frac{3\beta}{2}}}$.

In Theorem 10, for $f \in C([a,b], X)$ and for large enough $n \in \mathbb{N}$, when $0 < \beta \leq \frac{1}{2}$, the speed is $\frac{1}{n^\beta}$. So when $0 < \beta < \frac{2}{5} (< \frac{1}{2})$, we get by (155) that $\|A_n f - f\|_\infty$ converges must faster to zero. The last comes because we assumed differentiability of f.

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