Neural, Parallel, and Scientific Computations 25 (2017) 313-324

WAVELET PACKETS WITH THEIR FOURIER PROPERTIES ON LOCAL FIELDS OF PRIME CHARACTERISTIC

M. YOUNUS BHAT

Department of Mathematics, National Institute of Technology
Srinagar-190006, Jammu and Kashmir, India.
gyounusg@gmail.com

ABSTRACT. Wavelet packets have greater decorrelation properties than standard wavelets in that they induce a finer partitioning of the frequency domain of the process generating the data. This allows our procedure to be applied to a wide class of processes. The concept of wavelet packets on local field of positive characteristic was considered by Behera and Jahan by proving a version of splitting lemma for this setup. In this paper, we investigate the properties of wavelet packets by means of the Fourier transform using a prime element $\mathfrak{p}$ of a local field K of prime characteristic.

Keywords: Wavelet; wavelet packet; Fourier transform; local field.

AMS (MOS) Subject Classification. Primary 42C40; Secondary 42C15, 43A70, 11S85

1. INTRODUCTION

Multiresolution analysis (MRA) is an important mathematical tool since it provides a natural framework for understanding and constructing discrete wavelet systems. A multiresolution analysis is an increasing family of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ such that $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and which satisfies $f \in V_j$ if and only if $f(2\cdot) \in V_{j+1}$. Furthermore, there exists an element $\varphi \in V_0$ such that the collection of integer translates of function φ, $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ represents a complete orthonormal system for V_0. The function φ is called the *scaling function* or the *father wavelet*. The concept of multiresolution analysis has been extended in various ways in recent years. These concepts are generalized to $L^2(\mathbb{R}^d)$, to lattices different from $\mathbb{Z}^d$, allowing the subspaces of multiresolution analysis to be generated by Riesz basis instead of orthonormal basis, admitting a finite number of scaling functions, replacing the dilation factor 2 by an integer $M \geq 2$ or by an expansive matrix $A \in GL_d(\mathbb{R})$ as long as $A \subset A\mathbb{Z}^d$. For more about wavelets and their applications, we refer the monograph [7].

In recent years there has been a considerable interest in the problem of constructing wavelet bases on various groups, namely, Cantor dyadic groups [12], locally compact Abelian groups [8], p-adic fields [11] and Vilenkin groups [14]. Wavelets

Received December 31, 2016

have also been redefined based on an algebraic basis and exploitation of the group symmetries for the two scale equations by A. Ludu and etal[13]. Here the algebraic properties of Fourier series are obtained and it has been shown that wavelets of certain types obey exactly the q-deformation of the algebraic structure of the Fourier series. Recently, R. L. Benedetto and J. J. Benedetto [2] developed a wavelet theory for local fields and related groups. They did not develop the multiresolution analysis (MRA) approach, their method is based on the theory of wavelet sets and only allows the construction of wavelet functions whose Fourier transforms are characteristic functions of some sets. Since local fields are essentially of two types: zero and prime characteristic (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). Examples of local fields of characteristic zero include the p-adic field $\mathbb{Q}_p$ where as local fields of prime characteristic are the Cantor dyadic group and the Vilenkin p-groups. Even though the structures and metrics of local fields of zero and prime characteristics are similar, but their wavelet and multiresolution analysis theory are quite different. The concept of multiresolution analysis on a local field K of prime characteristic was introduced by Jiang et al.[10]. They pointed out a method for constructing orthogonal wavelets on local field K with a constant generating sequence. Subsequently, tight wavelet frames on local fields of prime characteristic were constructed by Shah and Debnath [23] using extension principles. More results in this direction can also be found in [16, 17, 18, 19, 20, 21, 22] and the references therein.

It is well known that the classical orthonormal wavelet bases have poor frequency localization. For example, if the wavelet ψ is band limited, then the measure of the supp of $(\psi_{j,k})^{\wedge}$ is 2^j-times that of supp $\hat{\psi}$. To overcome this disadvantage, Coifman et al.[5] introduced the notion of orthogonal univariate wavelet packets. Well known Daubechies orthogonal wavelets are a special of wavelet packets. Chui and Li [4] generalized the concept of orthogonal wavelet packets to the case of non-orthogonal wavelet packets so that they can be employed to the spline wavelets and so on. Shen [24] generalized the notion of univariate orthogonal wavelet packets to the case of multivariate wavelet packets. The construction of wavelet packets and wavelet frame packets on local fields of prime characteristic were recently reported by Behera and Jahan in [1]. They proved lemma on the so-called splitting trick and several theorems concerning the Fourier transform of the wavelet packets and the construction of wavelet packets to show that their translates form an orthonormal basis of $L^2(K)$. Other notable generalizations are the vector-valued wavelet packets [22], nonuninform wavelet packets [20] and Tight framelet packets [18]. Several authors like Daubechies [6] have studied Fourier transform of Wavelets and scaling functions. Where as Coifman et al. [5], Wickerhauser [26] and Hernandez and Weises[9] have obtained several results on Fourier transform of Wavelet packets.

Motivated and inspired by the concept of wavelet packets on local fields of prime characteristic, we investigate their properties by means of Fourier transform. This paper is organized as follows. In Section 2, we discuss some preliminary facts about local fields of prime characteristic and also some results and some which are required in the subsequent sections on local fields of prime characteristic. In Section 3, we introduce the notion of wavelet packets on local field K and examine their properties by means of the Fourier transform.

2. PRELIMINARIES ON LOCAL FIELDS

Let K be a field and a topological space. Then K is called a *local field* if both K^+ and K^* are locally compact Abelian groups, where K^+ and K^* denote the additive and multiplicative groups of K, respectively. If K is any field and is endowed with the discrete topology, then K is a local field. Further, if K is connected, then K is either $\mathbb{R}$ or $\mathbb{C}$. If K is not connected, then it is totally disconnected. Hence by a local field, we mean a field K which is locally compact, non-discrete and totally disconnected. The p-adic fields are examples of local fields. More details are referred to [15, 25]. In the rest of this paper, we use the symbols $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

Let K be a local field. Let dx be the Haar measure on the locally compact Abelian group K^+. If $\alpha \in K$ and $\alpha \neq 0$, then $d(\alpha x)$ is also a Haar measure. Let $d(\alpha x) = |\alpha| dx$. We call $|\alpha|$ the *absolute value* of α. Moreover, the map $x \to |x|$ has the following properties: (a) $|x| = 0$ if and only if $x = 0$; (b) $|xy| = |x||y|$ for all $x, y \in K$; and (c) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$. Property (c) is called the *ultrametric inequality.* The set $\mathfrak{D} = \{x \in K : |x| \leq 1\}$ is called the *ring of integers* in K. Define $\mathfrak{B} = \{x \in K : |x| < 1\}$. The set $\mathfrak{B}$ is called the *prime ideal* in K. The prime ideal in K is the unique maximal ideal in $\mathfrak{D}$ and hence as result $\mathfrak{B}$ is both principal and prime. Since the local field K is totally disconnected, so there exist an element of $\mathfrak{B}$ of maximal absolute value. Let $\mathfrak{p}$ be a fixed element of maximum absolute value in $\mathfrak{B}$. Such an element is called a *prime element* of K. Therefore, for such an ideal $\mathfrak{B}$ in $\mathfrak{D}$, we have $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$. As it was proved in [25] the set $\mathfrak{D}$ is compact and open. Hence, $\mathfrak{B}$ is compact and open. Therefore, the residue space $\mathfrak{D}/\mathfrak{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^k$ for some prime p and $k \in \mathbb{N}$.

Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in K : |x| = 1\}$. Then, it can be proved that $\mathfrak{D}^*$ is a group of units in K^* and if $x \neq 0$, then we may write $x = \mathfrak{p}^k x', x' \in \mathfrak{D}^*$. For a proof of this fact we refer to [15]. Moreover, each $\mathfrak{B}^k = \mathfrak{p}^k\mathfrak{D} = \left\{x \in K : |x| < q^{-k}\right\}$ is a compact subgroup of K^+ and usually known as the *fractional ideals* of K^+. Let $\mathcal{U} = \{a_i\}_{i=0}^{q-1}$ be any fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$, then every element $x \in K$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell \mathfrak{p}^\ell$ with $c_\ell \in \mathcal{U}$. Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but is non-trivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$ so if $y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(yx), x \in K$. Suppose that χ_u is any character on K^+, then clearly the restriction $\chi_u|\mathfrak{D}$ is also a character on $\mathfrak{D}$. Therefore, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in K^+, then, as it was proved in [25], the set $\left\{\chi_{u(n)} : n \in \mathbb{N}_0\right\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

The Fourier transform $\hat{f}$ of a function $f \in L^1(K) \cap L^2(K)$ is defined by

$$\hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}dx. \tag{2.1}$$

It is noted that

$$\hat{f}(\xi) = \int_K f(x)\,\overline{\chi_\xi(x)}dx = \int_K f(x)\chi(-\xi x)dx.$$

Furthermore, the properties of Fourier transform on local field K are much similar to those of on the real line. In particular Fourier transform is unitary on $L^2(K)$.

To define the Fourier transform of a function in $L^2(K)$, we consider the function $\mathbf{1_k}$. For $k \in \mathbb{Z}$, let $\mathbf{1_k}$ be the characteristic function of $\mathfrak{B}^k$.

Definition 2.1. For $f \in L^2(K)$, let $f_k = f\mathbf{1}_{-k}$, then

$$\hat{f}(\xi) = \lim_{k\to\infty} \hat{f}_k(\xi) = \lim_{k\to\infty} \int_{|x|\le q^k} f(x)\,\overline{\chi_\xi(x)}dx,$$

where the limit is taken in $L^2(K)$.

Theorem 2.2. [25] *The Fourier transform is unitary on $L^2(K)$.*

Theorem 2.3. [25] *Let $\{u(n) : n \in \mathbb{N}_0\}$ be a complete list of (distinct) coset representation of $\mathfrak{D}$ in K^+. Then $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$ is a list of (distinct) characters on $\mathfrak{D}$. Moreover, it is a complete orthonormal system on $\mathfrak{D}$.*

Given such a list of characters $\{\chi_{u(n)}\}_{n=0}^{\infty}$, we define the Fourier coefficients of a function $f \in L^1(\mathfrak{D})$ as

$$\hat{f}\big(u(n)\big) = \int_{\mathfrak{D}} f(x)\overline{\chi_{u(n)}(x)}dx.$$

The series $\sum\limits_{n=0}^{\infty} \hat{f}(u(n))\chi_{u(n)}(x)$ is called the Fourier series of f . From the standard L^2-theory for compact Abelian groups we conclude that the Fourier series of f converges to f in $L^2(\mathfrak{D})$ and Parseval's identity holds:

$$\int_{\mathfrak{D}} \left|f(x)\right|^2 dx = \sum_{n=0}^{\infty} \left|\hat{f}\big(u(n)\big)\right|^2.$$

Moreover, if $f \in L^1(\mathfrak{D})$ and $\hat{f}(u(n)) = 0$ for all $n \in \mathbb{N}_0$ then $f = 0$ a.e.

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q)$ where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{c-1}\} \subset \mathfrak{D}^*$ such that span $\{\zeta_j\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \le n < q,\ n = a_0 + a_1 p + \cdots + a_{c-1} p^{c-1},\ 0 \le a_k < p, \text{ and } k = 0, 1, \dots, c-1,$$

we define

$$u(n) = (a_0 + a_1\zeta_1 + \cdots + a_{c-1}\zeta_{c-1})\,\mathfrak{p}^{-1}. \tag{2.2}$$

Also, for $n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s,\ n \in \mathbb{N}_0,\ 0 \le b_k < q, k = 0, 1, 2, \dots, s$, we set

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_s)\mathfrak{p}^{-s}. \tag{2.3}$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m)+u(n)$. But, if $r, k \in \mathbb{N}_0$ and $0 \le s < q^k$, then $u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s)$. Further, it is also easy to verify that $u(n) = 0$ if and only if $n = 0$ and $\{u(\ell) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$. Hereafter we use the notation $\chi_n = \chi_{u(n)},\ n \ge 0$.

Theorem 2.4. [25] *For $n \in \mathbb{N}_0$, let $u(n)$ be as defined above. Then*

(a) *$u(n) = 0$ if and only if $n = 0$. If $k \ge 1$, then $|u(n)| = q^k$ if and only if $q^{k-1} \le n < q^k$.*
(b) *$\{u(n) : n \in \mathbb{N}_0\} = \{-u(n) : n \in \mathbb{N}_0\}$.*
(c) *$\{u(k) + u(\ell) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$.*

In the above Theorem, formulae (b) and (c) are satisfied for the fields of characteristic t but are not satisfied for fields of characteristic zero.

Let the local field K be of characteristic $p > 0$ and $\zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\zeta_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p), & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1, \dots, c-1 \text{ or } j \ne 1. \end{cases} \tag{2.4}$$

Let us recall the definition of an MRA on local fields of prime characteristic ([10]).

Definition 2.5. Let K be a local field of prime characteristic $t > 0$ and $\mathfrak{p}$ be a prime element of K. A multiresolution analysis(MRA) of $L^2(K)$ is a sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(K)$ satisfying the following properties:

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
(b) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(K)$;
(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
(d) $f(x) \in V_j$ if and only if $f(\mathfrak{p}^{-1}x) \in V_{j+1}$ for all $j \in \mathbb{Z}, x \in K$;
(e) there is a function $\varphi \in V_0$, called the *scaling function*, such that $\{\varphi(x - u(k)) : k \in \mathbb{N}_0\}$ forms an orthonormal basis for V_0.

Since $\varphi \in V_0 \subset V_1$ and $\{\varphi_{1,k} : k \in \mathbb{N}_0\}$ is an orthonormal basis of V_1, we have

$$\varphi(x) = \sqrt{q} \sum_{k \in \mathbb{N}_0} a_k \, \varphi\big(\mathfrak{p}^{-1}x - u(k)\big), \tag{2.5}$$

where $a_k = \langle \varphi, \varphi_{1,k} \rangle$ and $\{a_k\}_{k\in\mathbb{N}_0} \in \ell^2(\mathbb{N}_0)$. Taking Fourier transform of equation (2.5), we get

$$\begin{aligned} \hat{\varphi}(\xi) &= \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} a_k \, \overline{\chi_k(\xi)} \hat{\varphi}(\xi) \\ &= m_0(\mathfrak{p}\xi) \, \hat{\varphi}(\mathfrak{p}\xi), \end{aligned} \tag{2.6}$$

where $m_0(\xi) = \dfrac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} a_k \, \overline{\chi_k(\xi)}$ is an integral periodic function over $L^2(\mathfrak{D})$.

Replacing ξ by $\mathfrak{p}\xi$ in equation (2.6), we have

$$\hat{\varphi}(\mathfrak{p}\xi) = m_0(\mathfrak{p}^2\xi) \, \hat{\varphi}(\mathfrak{p}^2\xi)$$

and hence

$$\hat{\varphi}(\xi) = m_0(\mathfrak{p}^2\xi) \, m_0(\mathfrak{p}^2\xi) \, \hat{\varphi}(\mathfrak{p}^2\xi)$$

Iterating above equation, we get

$$\hat{\varphi}(\xi) = \left[\prod_{n=1}^{j} m_0(\mathfrak{p}^n\xi) \right] \hat{\varphi}(\mathfrak{p}^j\xi). \tag{2.7}$$

Allowing $j \to \infty$ and using $\hat{\varphi}(0) = 1$, in (2.7), we get

$$\hat{\varphi}(\xi) = \prod_{n=1}^{\infty} m_0(\mathfrak{p}^n\xi). \tag{2.8}$$

It then follows from equation (2.6) that $m_0(0) = 1$.

Let W_j, $j \in \mathbb{Z}$ be the direct complementary subspace of V_j in V_{j+1}. Assume that there exists $q-1$ functions $\{\psi^1, \psi^2, \ldots, \psi^{q-1}\}$ in $L^2(K)$ such that their translates and dilations form an orthonormal bases of W_j, i.e.,

$$W_j = \overline{\operatorname{span}} \left\{ q^{j/2} \, \psi^\ell\big(\mathfrak{p}^{-j}x - u(k)\big), \; k \in \mathbb{N}_0, \; 1 \le \ell \le q-1, \; j \in \mathbb{Z} \right\}. \tag{2.9}$$

Since $\psi^\ell \in W_0 \subset V_1$, there exist a sequence $\{a_k^\ell\} \in l^2(\mathbb{N}_0)$ such that

$$\psi^\ell(x) = \sqrt{q} \sum_{k \in \mathbb{N}_0} a_k^\ell \, \varphi\big(\mathfrak{p}^{-1}x - u(k)\big), \; 1 \le \ell \le q-1. \tag{2.10}$$

Equation (2.10) can be written in the frequency domain as

$$\begin{aligned} \hat{\psi}^\ell(\xi) &= \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} a_k^\ell \, \overline{\chi_k(\xi)} \hat{\varphi}(\xi) \\ &= m_\ell(\mathfrak{p}\xi) \, \hat{\varphi}(\mathfrak{p}\xi), \end{aligned} \tag{2.11}$$

where $m_\ell(\xi) = \dfrac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} a_k^\ell \, \overline{\chi_k(\xi)}, \; 1 \le \ell \le q-1.$

Remark 2.6. Let $\{\psi^1, \psi^2, \ldots, \psi^{q-1}\} \subset L^2(K)$, be the wavelets constructed from the scaling function φ such that $|\hat{\varphi}|$ and $m_0(\xi)$ are continuous. Then $\hat{\psi}^\ell(\mathfrak{p}^{-1}t) = 0 \quad \forall\, t \in \mathbb{N}_0$.

Lemma 2.7. [1] *Let $\varphi \in L^2(K)$ be a scaling function. Then $\{\varphi(x - u(k))\}_{k\in\mathbb{N}_0}$ is an orthonormal basis of V_0 if and only if* $\sum_{k\in\mathbb{N}_0} \left|\hat{\varphi}(\xi + u(k))\right|^2 = 1 \quad$ *for a.e.* $\xi \in K$.

Lemma 2.8. [1] *Let $\{\psi^1, \psi^2, \ldots, \psi^{q-1}\} \subset L^2(K)$ be the wavelets constructed from the scaling function φ. Then* $|\hat{\varphi}(\xi)|^2 = \sum_{\ell=1}^{q-1} \sum_{j=1}^{\infty} \left|\hat{\psi}_\ell(\mathfrak{p}^{-j}\xi)\right|^2 \quad$ *for a.e.* $\xi \in K$.

For $n \geq 0$, the basic wavelet packets associated with a scaling function $\varphi(x)$ on a local field K of prime characteristic are defined recursively by

$$\omega_n(x) = \omega_{qr+\ell}(x) = \sqrt{q} \sum_{k\in\mathbb{N}_0} a_k^\ell\, \omega_r(\mathfrak{p}^{-1}x - u(k)), \;\; 0 \leq \ell \leq q-1. \tag{2.12}$$

where $r \in \mathbb{N}_0$ is the unique element such that $n = qr + \ell$, $0 \leq \ell \leq q-1$ holds(see [1]). For $n = 0$, we have $\omega_0 = \varphi$, the scaling function and for $1 \leq n \leq q-1$, we have the basic wavelets $\omega_n = \psi_n$. Taking $r = 0$ in (2.12), we obtain

$$\omega_n(x) = \sqrt{q} \sum_{k\in\mathbb{N}_0} a_k^\ell\, \omega_0(\mathfrak{p}^{-1}x - u(k)), \;\; 0 \leq \ell \leq q-1. \tag{2.13}$$

Taking Fourier transform of equation (2.13) and using identity (2.8), we obtain

$$\hat{\omega}_n(\xi) = m_\ell(\mathfrak{p}\xi) \prod_{j=1}^{\infty} m_0(\mathfrak{p}^j\xi).$$

Using the expansion $n = \sum_{j=1}^{\infty} \mu_j q^{j-1}$, it can be verified that

$$\hat{\omega}_n(\xi) = \prod_{j=1}^{\infty} m_{\mu_j}(\mathfrak{p}^j\xi). \tag{2.14}$$

3. **FOURIER TRANSFORM OF WAVELET PACKETS**

In this section, we study some properties of the Fourier transform wavelet packets on local fields of prime characteristic, constructed in the Section 2.

Theorem 3.1. *If $\{\omega_n : n \in \mathbb{N}_0\} \subset L^2(K)$ are the basic wavelet packets associated with the scaling function $\varphi = \omega_0$, then $\hat{\omega}_n(0) = 0$ for all $n \in \mathbb{N}_0$.*

Proof. It is obvious that for $n > 0$, the right-hand side of identity (2.14) contains at least one term of the type $m_\ell(\mathfrak{p}^j\xi)$, $1 \leq \ell \leq L$, $j \in \mathbb{Z}$. Therefore, using the definition of $m_\ell(x)$ and the fact that $m_\ell(0) = 1$, result follows. □

Theorem 3.2. *If $\{\omega_n : n \in \mathbb{N}_0\} \subset L^2(K)$ are the basic wavelet packets associated with the scaling function $\varphi = \omega_0$, then $\hat{\omega}_n(\mathfrak{p}^{-1}mt) = 0$ for all $t \in \mathbb{N}_0$, where $m = q^j$ for $j \in \mathbb{Z}$ and $|\hat{\varphi}|$ and $|m_0|$ are continuous.*

Proof. From the definition of wavelet packets, we have

$$\omega_{qn}(x) = \sqrt{q} \sum_{k \in \mathbb{N}_0} a_k \omega_{qn}\big(\mathfrak{p}^{-1}x - u(k)\big). \tag{3.1}$$

Equation (3.1) can be reformulated in the frequency domain as

$$\hat{\omega}_{qn}(\xi) = m_0(\mathfrak{p}\xi)\hat{\omega}_n(\mathfrak{p}\xi). \tag{3.2}$$

On taking $m = q^j$ and $\xi = qmt = q^{j+1}t$, we obtain

$$\hat{\omega}_n(\mathfrak{p}^{-1}mt) = \hat{\omega}_{q^j}(\mathfrak{p}^{-j-1}t). \tag{3.3}$$

Using Remark 2.6 and equation (3.2), the identity (3.3) can be rewritten as

$$\begin{aligned} \hat{\omega}_n(\mathfrak{p}^{-1}mt) &= m_0(\mathfrak{p}^{-j-1}t)\hat{\omega}_{q^j-\ell}(\mathfrak{p}^{-j-1}t) \\ &= m_0(\mathfrak{p}^{-j-1}t)m_0(\mathfrak{p}^{-j}t)xsm_0(\mathfrak{p}^{-1}t)\hat{\omega}_\ell(\mathfrak{p}^{-1}t) \\ &= \left\{\prod_{r=1}^{j+1} m_0(\mathfrak{p}^{-r}t)\right\}\hat{\psi}_\ell(\mathfrak{p}^{-1}t) \\ &= 0. \end{aligned}$$

This completes the proof. □

Theorem 3.3. *If $\{\omega_n : n \in \mathbb{N}_0\} \subset L^2(K)$ are the basic wavelet packets associated with the scaling function $\varphi = \omega_0$, then*

$$|\hat{\omega}_n(\xi)|^2 = \sum_{s=0}^{q^r-1} \left|\hat{\omega}_{q^r n+s}(\mathfrak{p}^{-r}\xi)\right|^2. \tag{3.4}$$

Proof. Using identities (2.5)-(2.11) and Fourier transform of wavelet packets, we have

$$\begin{aligned} |\hat{\omega}_{qn}(\mathfrak{p}^{-1}\xi)|^2 + \sum_{\ell=1}^{q-1} |\hat{\omega}_{qn+\ell}(\mathfrak{p}^{-1}\xi)|^2 &= |m_0(\xi)\hat{\omega}_n(\xi)|^2 + \sum_{\ell=1}^{q-1} |m_\ell(\xi)\hat{\omega}_n(\xi)|^2 \\ &= |\hat{\omega}_n(\xi)|^2 \left(\sum_{\ell=0}^{q-1} |m_\ell(\xi)|^2\right). \end{aligned} \tag{3.5}$$

Under the assumption that the matrix $M(\xi) = \big[m_\ell\big(\mathfrak{p}\xi + \mathfrak{p}u(k)\big)\big]_{\ell,k=0}^{q-1}$ formed by the integral periodic functions m_ℓ, $1 \le \ell \le q-1$ is unitary for a.e. $\xi \in \mathfrak{D}$, the sum inside the braces in the above system will be equal to 1. Hence, the equation (3.5) reduces to

$$|\hat{\omega}_n(\xi)|^2 = |\hat{\omega}_{qn}(\mathfrak{p}^{-1}\xi)|^2 + \sum_{\ell=1}^{q-1} |\hat{\omega}_{qn+\ell}(\mathfrak{p}^{-1}\xi)|^2.$$

On iterating the above system and using Lemma 2.8, we get the result. □

Remark 3.4. For $n = 0$, equation (3.4) reduces to

$$|\hat{\varphi}(\xi)|^2 = |\hat{\omega}_0(\xi)|^2 = \sum_{r=0}^{q^r-1} \left|\hat{\omega}_s(\mathfrak{p}^{-r}\xi)\right|^2. \tag{3.6}$$

In the following theorem, we provide the characterization of wavelet packets in terms of Fourier transform.

Theorem 3.5. *If* $\{\omega_n : n \in \mathbb{N}_0\} \subset L^2(K)$ *are the basic wavelet packets associated with the scaling function* $\varphi = \omega_0$*, then for* $r \in \mathbb{N}_0$,

$$\sum_{k\in\mathbb{N}_0}\sum_{s=0}^{q^r-1}\left|\hat{\omega}_s\big(\mathfrak{p}^{-r}\big(\xi+u(k)\big)\big)\right|^2 = 1, \quad \textit{for a.e.}\, \xi \in K.$$

Proof. From equation (3.6), we have

$$\left|\hat{\varphi}\big(\xi+u(k)\big)\right|^2 = \left|\hat{\omega}_0\big(\xi+u(k)\big)\right|^2. \tag{3.7}$$

Therefore, using Theorem 3.3, the right hand side of the identity (3.7) can be rewritten as

$$\left|\hat{\varphi}\big(\xi+u(k)\big)\right|^2 = \sum_{s=0}^{q^r-1}\left|\hat{\omega}_s\big(\mathfrak{p}^{-r}\big(\xi+u(k)\big)\big)\right|^2.$$

Hence, using Lemma 2.7, we have

$$\sum_{k\in\mathbb{N}_0}\sum_{s=0}^{q^r-1}\left|\hat{\omega}_s\big(\mathfrak{p}^{-r}\big(\xi+u(k)\big)\big)\right|^2 = \sum_{k\in\mathbb{N}_0}\left|\hat{\varphi}\big(\xi+u(k)\big)\right|^2 = 1.$$

This completes the proof. □

As a consequence of Theorem 3.5, we obtain a new characterization of wavelet packets that have been constructed in Section 2

Theorem 3.6. *If* $\{\omega_n : n \in \mathbb{N}_0\} \subset L^2(K)$ *are the basic wavelet packets associated with the scaling function* $\varphi = \omega_0$*, then for* $r \in \mathbb{N}_0$,

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\sum_{s=0}^{q^r-1}\left|\hat{\omega}_s\big(\mathfrak{p}^{-j-r}\big(\xi+u(k)\big)\big)\right|^2 = 1, \quad \textit{for a.e.}\, \xi \in K. \tag{3.8}$$

Proof. Using Theorem 3.3, we have

$$|\hat{\omega}_n(\xi)|^2 = |\hat{\omega}_{qn}(\mathfrak{p}^{-1}\xi)|^2 + \sum_{\ell=1}^{q-1}|\hat{\omega}_{qn+\ell}(\mathfrak{p}^{-1}\xi)|^2.$$

A simple iteration of this result yields

$$|\hat{\omega}_n(\xi)|^2 = \sum_{s=0}^{q^r-1}\left|\hat{\omega}_{q^rn+s}(\mathfrak{p}^{-r}\xi)\right|^2.$$

Hence

$$|\hat{\omega}_n(\xi)|^2 = \sum_{s=0}^{q^r-1}\left|\hat{\omega}_{q^rn+s}(\mathfrak{p}^{-r}\xi)\right|^2 = \sum_{n=q^r}^{q^{r+1}-1}\left|\hat{\omega}_n(\mathfrak{p}^{-r}\xi)\right|^2. \tag{3.9}$$

Since $\omega_\ell = \psi_\ell$, $0 \le \ell \le q-1$, Lemma's 2.7 and 2.8 give

$$\sum_{k\in\mathbb{N}_0} \left|\hat{\varphi}\big(\xi + u(k)\big)\right|^2 = \sum_{\ell=1}^{q-1}\sum_{j=1}^{\infty}\sum_{k\in\mathbb{N}_0} \left|\hat{\omega}_\ell\big(\mathfrak{p}^{-j}\big(\xi + u(k)\big)\big)\right|^2.$$

Hence

$$\sum_{\ell=1}^{q-1}\sum_{j=1}^{\infty}\sum_{k\in\mathbb{N}_0} \left|\hat{\omega}_\ell\big(\mathfrak{p}^{-j}\big(\xi + u(k)\big)\big)\right|^2 = 1, \text{for a.e. } \xi \in K.$$

As the wavelet space W_j can not be decomposed greater then j times, it follows from equations (3.8) and (3.9) that r ranges from 1 to j. This completes the proof. □

The following theorem, the main result of this section shows the existence of a maximal decomposition property of wavelet packets.

Theorem 3.7. *If $\{\omega_n : n \in \mathbb{N}_0\} \subset L^2(K)$ are the basic wavelet packets associated with the scaling function $\varphi = \omega_0$, then the expression*

$$D_\omega(\xi) = \sum_{j=1}^{\infty}\sum_{k\in\mathbb{N}_0}\sum_{n=q^r}^{q^{r+1}-1} \left|\hat{\omega}_n\big(\mathfrak{p}^{-j-r}\big(\xi + u(k)\big)\big)\right|^2.$$

is well defined for a.e. $\xi \in K$. Moreover $\int_{\mathfrak{D}} D_\omega(\xi)d\xi = 1$.

Proof. For the proof of the theorem, it is sufficient to prove only the second part of the theorem. So, we have

$$\begin{aligned}
\int_{\mathfrak{D}} D_\omega(\xi)d\xi &= \int_{\mathfrak{D}} \sum_{j=1}^{\infty}\sum_{k\in\mathbb{N}_0}\sum_{n=q^r}^{q^{r+1}-1} \left|\hat{\omega}_n\big(\mathfrak{p}^{-j-r}\big(\xi + u(k)\big)\big)\right|^2 d\xi \\
&= \sum_{j=1}^{\infty}\sum_{k\in\mathbb{N}_0}\sum_{n=q^r}^{q^{r+1}-1} \int_{\mathfrak{D}} \left|\hat{\omega}_n\big(\mathfrak{p}^{-j-r}\big(\xi + u(k)\big)\big)\right|^2 d\xi \\
&= \sum_{j=1}^{\infty}\sum_{k\in\mathbb{N}_0}\sum_{n=q^r}^{q^{r+1}-1} \int_{k+\mathfrak{D}} \left|\hat{\omega}_n\big(\mathfrak{p}^{-j-r}(\xi)\big)\right|^2 d\xi \\
&= \sum_{j=1}^{\infty}\sum_{n=q^r}^{q^{r+1}-1} q^{-j-r} \int_K \left|\hat{\omega}_n(\xi)\right|^2 d\xi \\
&= \sum_{j=1}^{\infty} q^{-j-r} \sum_{n=q^r}^{q^{r+1}-1} \|\hat{\omega}_n\|_2^2 \\
&= \sum_{j=1}^{\infty} q^{-j-r} q^r \|\hat{\omega}_n\|_2^2 \\
&= \sum_{j=1}^{\infty} q^{-j} \|\hat{\omega}_n\|_2^2 \; = \; 1.
\end{aligned}$$

This completes the proof. □

4. CONCLUSION

The Fourier transform due its deep significance has subsequently been recognized by mathematicians and physicists. Many applications, including the analysis of stationary signals and real-time signal processing, make an effective use of the Fourier transform in time and frequency domains. In the present paper, we have study fourier transform of wavelet packets on local fields. Results 3.5 and 3.6 give the characterization of wavelet packets on local fields. Theorem 3.7 shows the existence of a maximal decomposition property of wavelet packets. These results provide new constructed wavelet packages in terms of Fourier transform. They also provide a way for obtaining new characterization of wavelet packets in terms of low pass and high pass filters.

ACKNOWLEDGMENTS

We are deeply indebted to the referees for their valuable suggestions which greatly improved the presentation of this paper.

REFERENCES

[1] B. Behera and Q. Jahan, Wavelet packets and wavelet frame packets on local fields of positive characteristic, *J. Math. Anal. Appl.*, 395:1–14, 2012 .

[2] J. J. Benedetto and R.L. Benedetto, A wavelet theory for local fields and related groups, *J. Geom. Anal.*, 14:423–456, 2004 .

[3] M. Y. Bhat and B.A. Khandy, On the Fourier transform of wavelet packets on local fields of positive characteristic, *Int. J. Inn. Res. Dev.*, 14:423–456, 2004.

[4] C. K. Chui and C. Li, Non-orthogonal wavelet packets, *SIAM J. Math. Anal.*, 24:712–738, 1993.

[5] R. R. Coifman, Y. Meyer, S. Quake and M.V. Wickerhauser, *Signal processing and compression with wavelet packets,* Technical Report, Yale University, 1990.

[6] I. Daubechies, *Ten Lectures on Wavelets*, Regional Conf. Appl. Math. , Philadelphia, SIAM, 1992.

[7] L. Debnath and F. A. Shah, *Wavelet Transforms and Their Applications*, Birkhäuser, New York, 2015.

[8] Yu. A. Farkov, Orthogonal wavelets with compact support on locally compact Abelian groups, *Izv. Math.*, 69:623–650, 2005.

[9] E. Hernandez and G. Weiss, *A First Course on Wavelets*, CRC Press, 1996.

[10] H. K. Jiang, D.F. Li and N. Jin, Multiresolution analysis on local fields, *J. Math. Anal. Appl.*, 294:523–532, 2004.

[11] A. Yu. Khrennikov, V. M. Shelkovich and M. Skopina, p-Adic refinable functions and MRA-based wavelets, *J. Approx. Theory.*, 161:226–238, 2009.

[12] W. C. Lang, Orthogonal wavelets on the Cantor dyadic group, *SIAM J. Math. Anal.*, 27:305–312, 1996.

[13] A. Ludu, M. Greiner and J. P. Draayer, Wavelets and Quantum algebras, *SIAM J. Math. phys.*, 39(4):2346–2361, 1998.

[14] S. F. Lukomskii, Step refinable functions and orthogonal MRA on Vilenkin groups, *J. Fourier Anal. Appl.*, 20:42–65, 2014.

[15] D. Ramakrishnan and R. J. Valenza, *Fourier Analysis on Number Fields.* : Graduate Texts in Mathematics, vol. 186. Springer, New York 1999.

[16] F. A. Shah and M. Y. Bhat, Vector-valued nonuniform multiresolution analysis on local fields, *Int. J. Wavelets Multiresolut. Inf. Process.*, 13(4), 2015 .

[17] F. A. Shah and M. Y. Bhat, Semi-orthogonal wavelet frames on local fields, *Analysis*, 36(3):173–181, 2016.

[18] F. A. Shah and M. Y. Bhat, Tight framelet packets on local fields of positive characteristic, *J. Class. Anal.*, 6(1):85–101, 2015.

[19] F. A. Shah and M. Y. Bhat, On framelet kernels of M-band wavelet frames, *Gulf J. Math.*, 3(4):59–66, 2015.

[20] F. A. Shah and M. Y. Bhat, Nonuniform wavelet packets on local fields of positive characteristic, *Filomat*, 31(6):1491–1505, 2016.

[21] F. A. Shah and M. Y. Bhat, Biorthogonal wavelet packets on local fields of positive characteristic, *Turkish J. Math.*, 40:292–309, 2016.

[22] F. A. Shah and M. Y. Bhat, Vector-valued wavelet packets on local fields of positive characteristic, *NewZealand J. Math.*, 46:9–20, 2016.

[23] F. A. Shah and L. Debnath, Tight wavelet frames on local fields, *Analysis*, 33:293–307, 2013.

[24] Z. Shen, Non-tensor product wavelet packets in $L^2(\mathbb{R}^s)$, *SIAM J. Math. Anal.*, 26:1061–1074, 1995.

[25] M. H. Taibleson, *Fourier Analysis on Local Fields.* Princeton University Press, Princeton 1975.

[26] M. V. Wickerhauser, *Adapted Wavelet Analysis from Theory to Software*, Wellesley(USA), A. K. Peters Ltd.