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ON THE NUMERICAL SOLUTION OF FISHER'S EQUATION BY ITERATIVE OPERATOR-SPLITTING METHOD

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ABSTRACT. In this paper, iterative operator-splitting method is used to solve the Fisher's equation numerically. We compare the results to noniterative splitting one. This method is based on splitting the problem into sub-equations, each sub-equation combined with iterative scheme is solved via suitable integrators. To obtain stability criteria for the proposed method, we perform Von Neumann analysis. The numerical results obtained by iterative splitting method and the noniterative splitting method are compared with the exact solutions that was given by Bastani [3] and Wang [17]. We showed that the iterative splitting method is as good as strang splitting method.

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1. INTRODUCTION

Fisher's equation is a parabolic type of partial differential equations which model the population growth in mathematical ecology. Various methods have been developed to solve this equation numerically, for example finite difference method [2, 13], recursion scheme [15], sinc collocation method [1], operator-splitting methods [7, 4], see [14] for other methods. Geiser et. al. in [7] have shown that the traditional operator-splitting apart from its benefits has several drawbacks: 1) For noncommuting operators, there may be a very large constant in the local splitting error, requiring the use of an unrealistically small splitting time-step, 2) Within a full splitting step in one subinterval, the inner values are not an approximation to the solution of the original problem, 3) Splitting the original problem into the different subproblem with one operator (i.e., neglecting the other components) is physically correct, see for example the Strang splitting, but the method is physically questionable when we aim to get consistent approximation after each inner step because we lose the exact starting conditions. In this paper we use the iterative operator splitting method, to avoid those mentioned situations.

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A similar method has been applied to KdV equation see Gücüyenen et.al [11]. They showed the method gives the least numerical error in comparison with other noniterative splitting method.

This paper is organized as follows, in Section 2 outline of the iterative splitting method is presented. Stability analysis of the method which based on Fisher equation is derived in Section 3. The application of the method to Fisher's equation is presented at section 4 along with the numerical implementation and discussion at section 5.

2. OUTLINE OF THE METHOD

Let $\mathbf{X}$ be a Banach space, $A, B : \mathbf{X} \to \mathbf{X}$ be bounded linear operators. Consider the abstract Cauchy problem in $\mathbf{X}$

$$\begin{aligned} c'(t) &= (A+B)c(t), \quad t \in \mathbb{R}^+ \cup \{0\}, \\ c(0) &= c_0 \in \mathbf{X} \end{aligned} \tag{2.1}$$

where c_0 is initial condition.

Let $[0,T]$ be time interval with arbitrary but fixed $T > 0$ and $0 = t^0 < t^1 < t^2 < \cdots < t^N = T$ be a partition of $[0,T]$. The time step $\Delta t_n = t^{n+1} - t^n$ for $n = 0, 1, \ldots, N-1$. Let us now consider time interval $[t^n, t^{n+1}]$. The iterative splitting method solves the following problems consecutively for $i = 1, 3, \ldots, 2m+1$

$$c_i'(t) = Ac_i(t) + Bc_{i-1}(t), \tag{2.2}$$

$$c_{i+1}'(t) = Ac_i(t) + Bc_{i+1}(t), \quad t \in [t^n, t^{n+1}] \tag{2.3}$$

Here $c_0(t) \equiv 0$ is the initial guess, $c_1(t^0) = c_0$ is initial condition. For the subsequent time interval as well as next iteration, the initial guess is $c_{i-1}(t) = c_{2m}(t)$. For example if $i = 3$ then $m = 1$, hence $c_2(t)$ is our initial guess, which has been determined from the previous computation.

The scheme (2.2) and (2.3) is an iterative method, in which each step includes both operators A and B. Hence, in these equations, there is no actual separation of the different physical processes. We would like to apply this spitting method to Fisher's equation which is nonlinear. Thus the method is not directly applicable. As is shown in Geiser and Noack [9], this problem can be solved by introducing a variation in the method to handle nonlinearity of a certain type. In section 4, we will modify the Fisher's equation to fit the method proposed in Geiser and Noack [9].

3. STABILITY ANALYSIS OF ITERATIVE SPLITTING METHOD ON FISHER EQUATION

In this section, we will investigate the stability analysis of iterative splitting method applied to Fisher's equation via Von Neumann approach. Suppose $u(x,t)$ is a smooth real valued function with variable $t \in \mathbb{R}^+ \cup \{0\}$ and $x \in [a,b] \subset \mathbb{R}$. The initial-value problem of Fisher's equation in nondimensional form, is as follows:

$$\begin{array}{rcl} u_t & = & u_{xx} + u(1-u) \quad t \in (0,T], \quad x \in (a,b) \\ u(x,0) & = & u_0(x) \quad x \in [a,b]. \end{array} \tag{3.1}$$

where u_0 is continuous real valued function from $[a,b]$ to $\mathbb{R}$. Applying the iterative splitting schemes to eq. (3.1), we have the following scheme:

$$u_i' = (u_i)_{xx} + u_{i-1}(1-u_{i-1}) \tag{3.2}$$

$$u_{i+1}' = (u_i)_{xx} + u_i(1-u_{i+1}) \tag{3.3}$$

where $i = 1, 3, \ldots, 2m+1$. In this approach, it is not necessary to specify a spatial discretization technique. In the algorithm (3.2) and (3.3) we deal with nonlinear term which comes from the reaction part of $f(u) = u(1-u)$.

Let $u = u^*$ be a steady state of eq. (3.1). By taking the first two terms of Taylor series around u^*, we approximate $f(u)$ with linear form. Hence the equation (3.1) becomes

$$u_t = u_{xx} + (1-2u^*)(u-u^*). \tag{3.4}$$

Eq. (3.1) has two steady states that is $u_1^* = 0$ and $u_2^* = 1$ as unstable and stable equilibrium respectively (see [2, 15]). Putting $u^* = 1$ to eq.(3.4) and appying itterative splitting method we have the following scheme

$$u_i' = L_1 u_i + L_2 u_{i-1} + 1, \tag{3.5}$$

$$u_{i+1}' = L_1 u_i + L_2 u_{i+1} + 1, \tag{3.6}$$

where $L_1 = \frac{\partial^2}{\partial x^2}$, $L_2 = -1$ and $i = 1, 3, \ldots, 2m+1$. Combining scheme (3.5) and (3.6) with second order midpoint rule, we have

$$\begin{pmatrix} u_i^{n+1} \\ u_{i+1}^{n+1} \end{pmatrix} = \begin{pmatrix} u_i^n \\ u_i^{n+1} \end{pmatrix} + \Delta t \begin{pmatrix} L_1 \frac{u_i^n + u_i^{n+1}}{2} + L_2 \frac{u_{i-1}^n + u_{i-1}^{n+1}}{2} + 1 \\ L_1 \frac{u_i^n + u_i^{n+1}}{2} + L_2 \frac{u_{i+1}^n + u_{i+1}^{n+1}}{2} + 1 \end{pmatrix}. \tag{3.7}$$

By taking Fourier transform according to the formula

$$\tilde{u}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\omega x} u(x) \mathrm{d}x. \tag{3.8}$$

Eq. (3.7) can be put in the following matrix form

$$\begin{pmatrix} \tilde{u}_i^{n+1} \\ \tilde{u}_{i+1}^{n+1} \end{pmatrix} = \begin{pmatrix} \frac{1-\frac{\Delta t}{2}\omega^2}{1+\frac{\Delta t}{2}\omega^2} & 0 \\ \frac{(1-\frac{\Delta t}{2}\omega^2)^2}{(1+\frac{\Delta t}{2}\omega^2)(1+\frac{\Delta t}{2})} & \frac{1-\frac{\Delta t}{2}}{1+\frac{\Delta t}{2}} \end{pmatrix} \begin{pmatrix} \tilde{u}_i^n \\ \tilde{u}_{i+1}^n \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}. \tag{3.9}$$

The eq. (3.9) can be written in the following form

$$\tilde{\mathbf{u}}^{n+1} = \tilde{A}\tilde{\mathbf{u}}^n + \mathbf{k}. \tag{3.10}$$

The eigenvalues of $\tilde{A}$ are $\lambda_1 = \frac{1-\frac{\Delta t}{2}\omega^2}{1+\frac{\Delta t}{2}\omega^2}$, $\lambda_2 = \frac{1-\frac{\Delta t}{2}}{1+\frac{\Delta t}{2}}$ and for stability criterion eigenvalues must be $|\lambda_i| \leq 1, \quad i = 1, 2$. However this holds for any choice of Δt and ω, hence the method is unconditionally stable. Note that this approach is advantageous since the nonlinear problem can now be analyzed as a linear problem. For the consistency and convergence of the method see Geiser [7] and Gücüyenen [12].

4. NUMERICAL IMPLEMENTATION

Following Geiser and Noack [9] to handle the nonlinearity term, we define

$$Au = \frac{\partial^2}{\partial x^2}u + u \quad \text{and} \quad Bu = -u.$$

Then the Fisher's equation can be written as

$$u_t = Au + uBu. \tag{4.1}$$

Now let us consider the initial value problem (4.1). Let $0 = t^0 < t^1 < \cdots < t^N = T$ and $a = x^0 < x^1 < x^2 < \cdots < x^M = b$ be partition of time interval $[0, T]$ and spatial interval $[a, b]$ respectively. The iterative splitting method for (4.1) is

$$u_i' = Au_i + u_{i-1}Bu_{i-1}, \tag{4.2}$$

$$u_{i+1}' = Au_i + u_iBu_{i+1} \tag{4.3}$$

where $i = 1, 3, \ldots, 2m + 1$.

We approximate u_{xx} with centered differences scheme as follows

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{(x^m,t)} = \frac{1}{\Delta x^2}(u_{m+1} - 2u_m + u_{m-1}) + O(\Delta x^2) \tag{4.4}$$

where $\Delta x = x^{m+1} - x^m$ as spatial step and $m = 0, 1, 2, \ldots, M-1$. Applying midpoint rule at each subinterval $[t^n, t^{n+1}]$, $n = 0, 1, \ldots, M$ to the equations (4.2) and (4.3) we have the following scheme:

$$\begin{aligned} u_i^{n+1} = {} & \big(I - \frac{\Delta t}{2}A\big)^{-1}\Big(\big(I + \frac{\Delta t}{2}A\big)u_i^n + \\ & + \frac{\Delta t}{4}(u_{i-1}^{n+1} + u_{i-1}^n)B(u_{i-1}^{n+1} + u_{i-1}^n)\Big) \end{aligned} \tag{4.5}$$

$$\begin{aligned} u_{i+1}^{n+1} = {} & \big(I + \frac{\Delta t}{4}B(u_i^{n+1} + u_i^n)\big)^{-1}\big(\frac{\Delta t}{2}A(u_i^{n+1} + u_i^n) + \\ & + \big(I - \frac{\Delta t}{4}(u_i^{n+1} + u_i^n)B\big)u_{i+1}^n\big) \end{aligned} \tag{4.6}$$

where Δt is the time step and iteration starts from $i = 1$, initial guess $u_0(t^n) = u_0(t^{n+1}) = 0$, initial conditions $u_1(t^0) = u_0$ and $u_2(t^n) = u_0$.

Example 1: We now consider the Fisher's equation $u_t = u_{xx} + u(1 - u)$ subject to the initial condition $\psi(x) = \frac{1}{(1+e^{\sqrt{\frac{1}{6}}x})^2}$. Bastani in [3] proposed the exact solution

	Δx	error L_2	error L_∞	CPU time (in seconds)
Iterative splitting	$\frac{1}{4}$	2.0995×10^{-5}	4.1191×10^{-5}	111.8148
	$\frac{1}{8}$	9.7247×10^{-5}	2.1235×10^{-4}	111.8725
	$\frac{1}{16}$	2.5832×10^{-4}	5.3737×10^{-4}	111.9074
Strang splitting	$\frac{1}{4}$	6.1883×10^{-5}	1.1199×10^{-4}	112.2426
	$\frac{1}{8}$	1.4782×10^{-4}	2.8065×10^{-4}	111.0621
	$\frac{1}{16}$	3.112×10^{-4}	6.0349×10^{-4}	111.0920
Crank-Nicholson	$\frac{1}{4}$	6.3402×10^{-5}	1.9012×10^{-4}	110.9589
	$\frac{1}{8}$	1.4904×10^{-4}	9.3095×10^{-4}	111.1694
	$\frac{1}{16}$	3.1241×10^{-4}	4.1×10^{-3}	111.1574

TABLE 1. Comparison of errors at $T = 5$ with various Δx and $\Delta t = 0.005$ for example 1.

as

$$u(x,t) = \frac{1}{(1+e^{\sqrt{\frac{1}{6}}x-\frac{5}{6}t})^2}. \tag{4.7}$$

We will compare this result with numerical solutions constructed by iterative splitting method with $i = 1$, Strang splitting method and Finite difference method with Crank-Nicholson scheme for $x \in [0, 1]$. The error are measured in L_2-norm and L_∞-norm. We use MATLAB to run the code installed in computer with intel core i5 processor and 4GB of RAM. The results are presented in table 1.

The results show that the iterative operator-splitting method with $i = 1$ is as good as the Strang splitting method indicated by the same order of error at 10^{-4} while the Crank-Nicholson gives larger error at 10^{-3} for $\Delta x = \frac{1}{16}$.

Example 2: Consider the Fisher's equation $u_t = u_{xx} + u(1-u)$ subject to initial condition:

$$u(x,0) = \left[\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{x}{2\sqrt{6}}\right)\right]^2, \tag{4.8}$$

and the boundary conditions:

$$\lim_{x\to-\infty} u(x,t) = 1 \quad \text{and} \quad \lim_{x\to\infty} u(x,t) = 0. \tag{4.9}$$

The exact soution of this problem is presented in [17] as:

$$u(x,t) = \left[\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{x}{2\sqrt{6}} - \frac{5t}{12}\right)\right]^2. \tag{4.10}$$

We use the same setting as example 1 for $x \in [-20, 20]$ and the results are presented in table 2.

The results show that the iterative operator-splitting method is as good as the Strang splitting method indicated by the same order of error at 10^{-6} for error L_∞

	Δx	error L_2	error L_∞	CPU time (in seconds)
Iterative splitting	$\frac{1}{4}$	2.1×10^{-3}	4.3349×10^{-6}	111.2393
	$\frac{1}{8}$	2.0×10^{-3}	4.4097×10^{-6}	114.3591
	$\frac{1}{16}$	2.0×10^{-3}	9.0714×10^{-6}	125.9973
Strang splitting	$\frac{1}{4}$	1.6×10^{-3}	2.4139×10^{-6}	110.9700
	$\frac{1}{8}$	1.7×10^{-3}	4.1471×10^{-6}	112.0813
	$\frac{1}{16}$	1.8×10^{-3}	9.1337×10^{-6}	120.0134
Crank-Nicholson	$\frac{1}{4}$	4.6×10^{-3}	5.73×10^{-2}	111.1888
	$\frac{1}{8}$	4.8×10^{-3}	1.2×10^{-1}	111.8068
	$\frac{1}{16}$	4.8×10^{-3}	2.4×10^{-1}	118.2812

TABLE 2. Comparison of errors at $T = 5$ with various Δx and $\Delta t = 0.005$ for example 2.

and at 10^{-3} for error L_2, while the Crank-Nicholson gives larger error at 10^{-1} for $\Delta x = \frac{1}{16}$.

5. DISCUSSION

The results show that the iterative splitting method is as good as the Strang spitting method, based on the order of the difference with the exact solution for Fisher's equation. However, by splitting the equation into the *diffusion* part and the *reaction* part, we suspect that the error might be generated more severely in compare with the iterative splitting method since for the latter, there is no actual splitting of the operator. For future investigation, we are aiming on comparing the performance of the Strang splitting method againts the iterative splitting method in dealing with strong nonlinearity.

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