Neural, Parallel, and Scientific Computations 25 (2017) 435-455

STABILITY ANALYSIS OF MAY'S TWO PREY AND TWO PREDATOR MODEL

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ABSTRACT. This paper deals dynamical study of Mays Prey-Predator Model in the case of two preys and two predators. The local stability analysis of the equilibrium points are studied. The study is further carried out simulating the behavior exhibited by the interaction two preys and two predators.

AMS (MOS) Subject Classification. 92D25.

1. INTRODUCTION

The Mathematician of 20th century claim that most of the research activities in late twenties and twenty first century will be in the field of mathematical biology and ecology. The volume of knowledge of mathematics in biology and ecology will be far more than the existing literature in mathematics. The field of bio-mathematical research is fast growing day by day and its applications and usefulness are related to the mutual existence of flora and fauna and ecological balance of the nature. Most of these problems and their mathematical models are described by the interaction between different species of animals, micro-organisms and plants in various forms. The prey-predator model in different forms are mostly used models for such ecological problems. The main objective of these models are to describe the dynamical behavior of interacting populations. The natural balance and their stability are described by such models.

The population modeling drew the attention of the biologist and ecologist in 20th century as human civilization faced the pressure on limited sustenance food and resources and imbalance in ecological system due to human population growth. According to Pulley[5] the European biologist Remond Pearl in 1921 started the modeling study in collaborations with physicist Alfred Lotka (1880-1949). Lotka, fascinated by the molecular dynamics in certain chemical reaction has already published an article with the title "analytical note on certain rhythmic relation in organic system". He made

Received December 31, 2016

and study of the biological system and its dynamics of the species inside it. He had considered the herbivore feeding on plants as predator and prey and published a model in 1920. During the same period an Italian Mathematician Vito Volterra (1860-1940) independently published a model in 1926 considering the population dynamics of two species first as prey and second as predator, in order to analyze the cyclic variations observed in the Shark and Food fish populations in the Adriatic sea. After 1926, the above model developed independently is recognized among the researchers as Lotka-Volterra model.

1.1. **LOTKA-VOLTERRA MODEL.** Let $H(t)$ and $P(t)$ denote the population of Prey and predator species at time t. In the absence of predators, the prey population would grow at natural way, that is proportional to the population of the prey, with

$$\frac{dH(t)}{dt} = a_1 H, \quad a_1 > 0$$

where a_1 is per capita rate or intrinsic rate of increase.
In the absence of prey, the predator population would decline at a natural way, with

$$\frac{dP(t)}{dt} = -b_1 P, \quad b_1 > 0$$

where b_1 is death rate.
When both predator and prey are present, the presence of both is beneficial to growth predator species and decline in the prey species.consequently the consumption of prey by predators results in an interaction rate of decline $-\alpha_1 HP$ $(\alpha_1 > 0)$ in the prey population H , where α_1 measures the attack rate of predators on their prey, and an interaction rate of growth $\beta_1 HP (\beta_1 > 0)$ in predator population P, where β_1 measure of conversion efficiency (the rate at which the predator converts prey biomass in to new predator offspring).
When we combine the natural and interaction rates $a_1 H$ and $-\alpha_1 HP$ for the prey population H, as well as the natural and interaction rates $-b_1 P$ and $\beta_1 HP$ for the predator population, we get the predator-prey system

$$\begin{aligned} \frac{dH}{dt} &= H(a_1 - \alpha_1 P), \quad a_1, \alpha_1 > 0 \\ \frac{dP}{dt} &= P(-b_1 + \beta_1 H), \quad b_1, \beta_1 > 0 \end{aligned} \tag{1.1}$$

The equation (1.1) along with initial conditions

$$H(0) = H_0, \quad P(0) = P_0 \tag{1.2}$$

are known as Lokta-Volterra equations.

2. MAY'S GENERAL MODEL

In 1971 Robert M May [6] proposed multi species prey-predator model under Lotka-Volterra assumptions. That is, he made the assumptions of interaction between preys and predators having no interactions on the same species.

Let $H_i(t)$ and $P_i(t), i = 1, 2, 3, \cdots, n$ be the population of n-prey and n-predator species (or of host and parasite Species) at time t. There is an interaction between preys and predators only, then the May's general prey-predator model [6] is given by

$$\frac{dH_i}{dt} = H_i(t)\Big\{a_i - \sum_{j=1}^{n} \alpha_{ij} P_j(t)\Big\}, \tag{2.1}$$

$$\frac{dP_i}{dt} = P_i(t)\Big\{-b_i + \sum_{j=1}^{n} \beta_{ij} H_j(t)\Big\}, \tag{2.2}$$

$i = 1, 2, 3, \cdots, n$ with a_i are natural birth rate for prey, b_i are natural death rate for predator, α_{ij} are attack rate of predator j on prey i and β_{ij} are conversion efficiency of predator i into its offspring by attacking prey j. Also all a_i, b_i, α_{ij}, $\beta_{ij} > 0$. But there is a complexity in dynamical study of interactions among multi-species prey-predator.

After May's model, various researchers [1, 2, 3, 4, 7, 8] studied the dynamical behavior of prey-predator interactions in various aspects. But we focused on the interaction of two prey and two predator having no interactions between the same species.

2.1. MAY'S TWO PREY AND TWO PREDATOR MODEL.

When $n = 1$, the model reduce to Lotka-Volterra prey-predator model. For the case of two prey and two predator, we use $n = 2$ then the model equation becomes

$$\frac{dH_1}{dt} = H_1(t)\{a_1 - \alpha_{11}P_1(t) - \alpha_{12}P_2(t)\} \tag{2.3}$$

$$\frac{dH_2}{dt} = H_2(t)\{a_2 - \alpha_{21}P_1(t) - \alpha_{22}P_2(t)\} \tag{2.4}$$

$$\frac{dP_1}{dt} = P_1(t)\{-b_1 + \beta_{11}H_1(t) + \beta_{12}H_2(t)\} \tag{2.5}$$

$$\frac{dP_2}{dt} = P_2(t)\{-b_2 + \beta_{21}H_1(t) + \beta_{22}H_2(t)\} \tag{2.6}$$

2.2. EQUILIBRIUM POSITIONS.

The equilibrium positions means time independent solution, so solutions of the system of equations (2.3) - (2.6) when time derivative of state variable is set as zero, is called equilibrium positions. Let $\bar{H}_1, \bar{H}_2, \bar{P}_1, \bar{P}_2$

be the equilibrium position of prey-predator. Then

$$\begin{aligned}
\bar{H}_1\{a_1 - \alpha_{11}\bar{P}_1 - \alpha_{12}\bar{P}_2\} &= 0\\
\bar{H}_2\{a_2 - \alpha_{21}\bar{P}_1 - \alpha_{22}\bar{P}_2\} &= 0\\
\bar{P}_1\{-b_1 + \beta_{11}\bar{H}_1 + \beta_{12}\bar{H}_2\} &= 0\\
\bar{P}_2\{-b_2 + \beta_{21}\bar{H}_1 + \beta_{22}\bar{H}_2\} &= 0
\end{aligned}$$

Now above four equations imply

$$\bar{H}_1 = 0 \quad \text{or} \quad a_1 - \alpha_{11}\bar{P}_1 - \alpha_{12}\bar{P}_2 = 0 \tag{2.7}$$

$$\bar{H}_2 = 0 \quad \text{or} \quad a_2 - \alpha_{21}\bar{P}_1 - \alpha_{22}\bar{P}_2 = 0 \tag{2.8}$$

$$\bar{P}_1 = 0 \quad \text{or} \quad -b_1 + \beta_{11}\bar{H}_1 + \beta_{12}\bar{H}_2 = 0 \tag{2.9}$$

$$\bar{P}_2 = 0 \quad \text{or} \quad -b_2 + \beta_{21}\bar{H}_1 + \beta_{22}\bar{H}_2 = 0 \tag{2.10}$$

From the above four equations, we obtain the following six equilibrium positions.

(a)

$$\bar{H}_1 = 0, \quad \bar{H}_2 = 0, \quad \bar{P}_1 = 0, \quad \bar{P}_2 = 0,$$

The first possible equilibrium is

$$1EP = \big(\bar{H}_1, \bar{H}_2, \bar{P}_1, \bar{P}_2\big) = \big(0, 0, 0, 0\big) \tag{2.11}$$

(b) If $\bar{H}_1 = 0, \bar{P}_1 = 0$, then from equations (2.8) and (2.10), we obtain $\bar{P}_2 = \dfrac{a_2}{\alpha_{22}}, \bar{H}_2 = \dfrac{b_2}{\beta_{22}}$.
The second possible equilibrium is

$$2EP = \left(\bar{H}_1, \bar{H}_2, \bar{P}_1, \bar{P}_2\right) = \left(0, \frac{b_2}{\beta_{22}}, 0, \frac{a_2}{\alpha_{22}}\right) \tag{2.12}$$

(c) If $\bar{H}_2 = 0, \bar{P}_2 = 0$, then from equations (2.7) and (2.9), we obtain $\bar{P}_1 = \dfrac{a_1}{\alpha_{11}}, \bar{H}_1 = \dfrac{b_1}{\beta_{11}}$.
The third possible equilibrium is

$$3EP = \left(\bar{H}_1, \bar{H}_2, \bar{P}_1, \bar{P}_2\right) = \left(\frac{b_1}{\beta_{11}}, 0, \frac{a_1}{\alpha_{11}}, 0\right) \tag{2.13}$$

(d) If $\bar{H}_1 = 0, \bar{P}_2 = 0$, then from equations (2.8) and (2.9), we obtain $\bar{P}_1 = \dfrac{a_2}{\alpha_{21}}, \bar{H}_2 = \dfrac{b_1}{\beta_{12}}$.
The fourth possible equilibrium is

$$4EP = \left(\bar{H}_1, \bar{H}_2, \bar{P}_1, \bar{P}_2\right) = \left(0, \frac{b_1}{\beta_{12}}, \frac{a_1}{\alpha_{21}}, 0\right) \tag{2.14}$$

(e) If $\bar{H}_2 = 0, \bar{P}_1 = 0$, then from equations (2.7) and (2.10), we obtain $\bar{P}_2 = \frac{a_1}{\alpha_{12}}, \bar{H}_1 = \frac{b_2}{\beta_{21}}$.

The fifth possible equilibrium is

$$5EP = \left(\bar{H}_1, \bar{H}_2, \bar{P}_1, \bar{P}_2\right) = \left(\frac{b_2}{\beta_{21}}, 0, 0, \frac{a_1}{\alpha_{12}}\right) \tag{2.15}$$

(f) Now from equations (2.7) and (2.8)

$$\begin{aligned} \alpha_{11}\bar{P}_1 + \alpha_{12}\bar{P}_2 - a_1 &= 0 \\ \alpha_{21}\bar{P}_1 + \alpha_{22}\bar{P}_2 - a_2 &= 0 \end{aligned}$$

Using by cross-multiplication method.

$$\frac{\bar{P}_1}{a_1\alpha_{22} - a_2\alpha_{12}} = \frac{\bar{P}_2}{a_2\alpha_{11} - a_1\alpha_{21}} = \frac{1}{\alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12}}$$

These results,

$$\bar{P}_1 = \frac{a_1\alpha_{22} - a_2\alpha_{12}}{\alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12}}, \qquad \bar{P}_2 = \frac{a_2\alpha_{11} - a_1\alpha_{21}}{\alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12}}$$

Also from equations (2.9) and (2.10)

$$\begin{aligned} \beta_{11}\bar{H}_1 + \beta_{12}\bar{H}_2 - b_1 &= 0 \\ \beta_{21}\bar{H}_1 + \beta_{22}\bar{H}_2 - b_2 &= 0 \end{aligned}$$

using cross-multiplication method

$$\frac{\bar{H}_1}{b_1\beta_{22} - b_2\beta_{12}} = \frac{\bar{H}_2}{b_2\beta_{11} - b_1\beta_{21}} = \frac{1}{\beta_{11}\beta_{22} - \beta_{21}\beta_{12}}$$

these results,

$$\bar{H}_1 = \frac{b_1\beta_{22} - b_2\beta_{12}}{\beta_{11}\beta_{22} - \beta_{21}\beta_{12}}, \qquad \bar{H}_2 = \frac{b_2\beta_{11} - b_1\beta_{21}}{\beta_{11}\beta_{22} - \beta_{21}\beta_{12}}$$

Thus the sixth possible equilibrium is

$$6EP = \begin{pmatrix} \bar{H}_1 \\ \bar{H}_2 \\ \bar{P}_1 \\ \bar{P}_2 \end{pmatrix} = \begin{pmatrix} \frac{b_1\beta_{22} - b_2\beta_{12}}{\beta_{11}\beta_{22} - \beta_{21}\beta_{12}} \\ \frac{b_2\beta_{11} - b_1\beta_{21}}{\beta_{11}\beta_{22} - \beta_{21}\beta_{12}} \\ \frac{a_1\alpha_{22} - a_2\alpha_{12}}{\alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12}} \\ \frac{a_2\alpha_{11} - a_1\alpha_{21}}{\alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12}} \end{pmatrix} \tag{2.16}$$

which is only a non zero equilibrium position.

Let us introduce the new six parameters A_1, A_2, A_3, B_1, B_2 and B_3 defined by

$$\begin{aligned} A_1 &= \frac{a_1}{\alpha_{12}} - \frac{a_2}{\alpha_{22}} \\ A_2 &= \frac{a_2}{\alpha_{21}} - \frac{a_1}{\alpha_{11}} \\ A_3 &= \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} \\ B_1 &= \frac{b_1}{\beta_{12}} - \frac{b_2}{\beta_{22}}, \\ B_2 &= \frac{b_2}{\beta_{21}} - \frac{b_1}{\beta_{11}}, \\ B_3 &= \beta_{11}\beta_{22} - \beta_{12}\beta_{21} \end{aligned}$$

to reduce the non-zero equilibrium position into simplified form.

Then equation (2.16) becomes

$$\bar{H}_1 = \frac{\beta_{12}\beta_{22}\left(\frac{b_1}{\beta_{12}} - \frac{b_2}{\beta_{22}}\right)}{\beta_{11}\beta_{22} - \beta_{12}\beta_{21}} = \beta_{12}\beta_{22}\frac{B_1}{B_3}$$

$$\bar{H}_2 = \frac{\beta_{11}\beta_{21}\left(\frac{b_2}{\beta_{21}} - \frac{b_1}{\beta_{11}}\right)}{\beta_{11}\beta_{22} - \beta_{12}\beta_{21}} = \beta_{11}\beta_{21}\frac{B_2}{B_3}$$

$$\bar{P}_1 = \frac{\alpha_{12}\alpha_{22}\left(\frac{a_1}{\alpha_{12}} - \frac{a_2}{\alpha_{22}}\right)}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} = \alpha_{12}\alpha_{22}\frac{A_1}{A_3}$$

$$\bar{P}_2 = \frac{\alpha_{11}\alpha_{21}\left(\frac{a_2}{\alpha_{21}} - \frac{a_1}{\alpha_{11}}\right)}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} = \alpha_{11}\alpha_{21}\frac{A_2}{A_3}$$

Therefore

$$6EP = \begin{pmatrix} \bar{H}_1 \\ \bar{H}_2 \\ \bar{P}_1 \\ \bar{P}_2 \end{pmatrix} = \begin{pmatrix} \beta_{12}\beta_{22}\frac{B_1}{B_3} \\ \beta_{11}\beta_{21}\frac{B_2}{B_3} \\ \alpha_{12}\alpha_{22}\frac{A_1}{A_3} \\ \alpha_{11}\alpha_{21}\frac{A_2}{A_3} \end{pmatrix} \tag{2.17}$$

We assume that each of $A_1, A_2, A_3, B_1, B_2, B_3$ is either positive or negative so that there are apparently 64 cases. We now further illustrate the above cases under the following conditions:

Condition 1:

If $A_1 > 0$, $A_2 > 0$ then

$$\frac{a_1}{\alpha_{12}} - \frac{a_2}{\alpha_{22}} > 0, \qquad \frac{a_2}{\alpha_{21}} - \frac{a_1}{\alpha_{11}} > 0$$

which imply

$$\frac{a_1}{a_2} > \frac{\alpha_{12}}{\alpha_{22}}, \qquad \frac{a_1}{a_2} < \frac{\alpha_{11}}{\alpha_{21}}$$

The above two inequalities become

$$\frac{\alpha_{11}}{\alpha_{21}} > \frac{a_1}{a_2} > \frac{\alpha_{12}}{\alpha_{22}}$$

which results,

$$\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} > 0$$

It means

$$A_3 > 0$$

This implies that we can neglect the case for (A_1, A_2, A_3) having sign pattern $(+, +, -)$.

Condition 2:

If $\quad A_1 < 0, \qquad A_2 < 0$

$$\frac{a_1}{\alpha_{12}} - \frac{a_2}{\alpha_{22}} < 0, \qquad \frac{a_2}{\alpha_{21}} - \frac{a_1}{\alpha_{11}} < 0$$
$$\frac{a_1}{a_2} < \frac{\alpha_{12}}{\alpha_{22}}, \qquad \frac{a_1}{a_2} > \frac{\alpha_{11}}{\alpha_{21}}$$

The above two inequalities become

$$\frac{\alpha_{11}}{\alpha_{21}} < \frac{a_1}{a_2} < \frac{\alpha_{12}}{\alpha_{22}}$$

which results,

$$\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} < 0$$

It means

$$A_3 < 0$$

This implies that we can neglect the case for (A_1, A_2, A_3) having sign pattern $(-, -, +)$.

Condition 3:

If $\quad B_1 > 0, \qquad B_2 > 0$

$$\frac{b_1}{\beta_{12}} - \frac{b_2}{\beta_{22}} > 0, \qquad \frac{b_2}{\beta_{21}} - \frac{b_1}{\beta_{11}} > 0$$
$$\frac{b_1}{b_2} > \frac{\beta_{12}}{\beta_{22}}, \qquad \frac{b_1}{b_2} < \frac{\beta_{11}}{\beta_{21}}$$

The above two inequality become

$$\frac{\beta_{11}}{\beta_{21}} > \frac{b_1}{b_2} > \frac{\beta_{12}}{\beta_{22}}$$

which results,

$$\beta_{11}\beta_{22} - \beta_{12}\beta_{21} > 0$$

It means

$$B_3 > 0$$

This implies that we can neglect the case for (B_1, B_2, B_3) having sign pattern $(+, +, -)$.

Condition 4:

If $\quad B_1 < 0, \quad B_2 < 0$

$$\frac{b_1}{\beta_{12}} - \frac{b_2}{\beta_{22}} < 0, \qquad \frac{b_2}{\beta_{21}} - \frac{b_1}{\beta_{11}} < 0$$

$$\frac{b_1}{b_2} < \frac{\beta_{12}}{\beta_{22}}, \qquad \frac{b_1}{b_2} > \frac{\beta_{11}}{\beta_{21}}$$

The above two inequalities become

$$\frac{\beta_{11}}{\beta_{21}} < \frac{b_1}{b_2} < \frac{\beta_{12}}{\beta_{22}}$$

which results,

$$\beta_{11}\beta_{22} - \beta_{12}\beta_{21} < 0$$

It means

$$B_3 < 0$$

This implies that we can neglect the case for (B_1, B_2, B_3) having sign pattern $(+, +, -)$.

The above four conditions shows that if A_1, A_2 are positive then A_3 must be positive. Also, if A_1, A_2 are negative then A_3 must be negative. This implies (A_1, A_2, A_3) can not have the signs

$$(+, +, -), \quad (-, -, +)$$

Therefore (A_1, A_2, A_3) can have the signs

$$(+, +, +), \quad (-, -, -), \quad (+, -, +), \quad (+, -, -), \quad (-, +, +), \quad (- + -)$$

Similarly if B_1, B_2 are positive then B_3 must be positive. Also if B_1, B_2 are negative then B_3 must be negative.This implies (B_1, B_2, B_3) can not have the signs

$$(+, +, -), \quad (-, -, +)$$

Therefore (B_1, B_2, B_3) can have the signs

$$(+, +, +), \quad (-, -, -), \quad (+, -, +), \quad (+, -, -), \quad (-, +, +), \quad (-, +, -)$$

Thus instead of 64 cases there are only 36 cases of signs of $(A_1, A_2, A_3)$and(B_1, B_2, B_3).

2.3. **STABILITY ANALYSIS OF EQUILIBRIUM POSITIONS.** The coefficient matrix of prey-predator model equations (2.3) – (2.6) is

$$A = \begin{bmatrix} C & 0 & -\alpha_{11}H_1 & -\alpha_{12}H_1 \\ 0 & D & -\alpha_{21}H_2 & -\alpha_{22}H_2 \\ \beta_{11}P_1 & \beta_{12}P_1 & E & 0 \\ \beta_{21}P_2 & \beta_{22}P_2 & 0 & F \end{bmatrix}$$

where

$$\begin{aligned} C &= a_1 - \alpha_{11}P_1 - \alpha_{12}P_2 \\ D &= a_2 - \alpha_{21}P_1 - \alpha_{22}P_2 \\ E &= -b_1 + \beta_{11}H_1 + \beta_{12}H_2 \\ F &= -b_2 + \beta_{21}H_1 + \beta_{22}H_2 \end{aligned}$$

The characteristic equation coefficients matrix A is $|A - \lambda I| = 0$ where I is a 4×4 identity matrix and λ is the eigenvalues of coefficient matrix A. The determinants $|A - \lambda I|$ is the Jacobian of $\bar{H}_1, \bar{H}_2, \bar{P}_1, \bar{P}_2$ which we denote by $J(\bar{H}_1, \bar{H}_2, \bar{P}_1, \bar{P}_2)$. That is

$$J(\bar{H}_1, \bar{H}_2, \bar{P}_1, \bar{P}_2) = \begin{vmatrix} \bar{C} & 0 & -\alpha_{11}\bar{H}_1 & -\alpha_{12}\bar{H}_1 \\ 0 & \bar{D} & -\alpha_{21}\bar{H}_2 & -\alpha_{22}\bar{H}_2 \\ \beta_{11}\bar{P}_1 & \beta_{12}\bar{P}_1 & \bar{E} & 0 \\ \beta_{21}\bar{P}_2 & \beta_{22}\bar{P}_2 & 0 & \bar{F} \end{vmatrix} = 0 \tag{2.18}$$

where

$$\begin{aligned} \bar{C} &= a_1 - \alpha_{11}\bar{P}_1 - \alpha_{12}\bar{P}_2 - \lambda \\ \bar{D} &= a_2 - \alpha_{21}\bar{P}_1 - \alpha_{22}\bar{P}_2 - \lambda \\ \bar{E} &= -b_1 + \beta_{11}\bar{H}_1 + \beta_{12}\bar{H}_2 - \lambda \\ \bar{F} &= -b_2 + \beta_{21}\bar{H}_1 + \beta_{22}\bar{H}_2 - \lambda \end{aligned}$$

Now we analyze the different equilibrium positions.

Case 1:

For the first equilibrium position $1EP = \big(\bar{H}_1, \bar{H}_2, \bar{P}_1, \bar{P}_2\big) = \big(0, 0, 0, 0\big)$

$$J(0,0,0,0) = \begin{vmatrix} a_1 - \lambda & 0 & 0 & 0 \\ 0 & a_2 - \lambda & 0 & 0 \\ 0 & 0 & -b_1 - \lambda & 0 \\ 0 & 0 & 0 & -b_2 - \lambda \end{vmatrix} = 0$$

$$(a_1 - \lambda)(a_2 - \lambda)(-b_1 - \lambda)(-b_2 - \lambda) = 0 \tag{2.19}$$

This equation gives $\lambda_1 = a_1, \lambda_2 = a_2, \lambda_3 = -b_1, \lambda_4 = -b_2$ which are real and distinct. Therefore the equilibrium position $1EP = \big(0, 0, 0, 0\big)$ is always unstable.

Case 2:

For the second equilibrium position $2EP = \left(\bar{H}_1, \bar{H}_2, \bar{P}_1, \bar{P}_2\right) = \left(0, \frac{b_2}{\beta_{22}}, 0, \frac{a_2}{\alpha_{22}}\right)$

Here,

$$J\left(0, \frac{b_2}{\beta_{22}}, 0, \frac{a_2}{\alpha_{22}}\right) = 0$$

This implies

$$\begin{vmatrix} a_1 - \alpha_{12}\frac{a_2}{\alpha_{22}} - \lambda & 0 & 0 & 0 \\ 0 & a_2 - a_2 - \lambda & -\alpha_{21}\frac{b_2}{\beta_{22}} & -\alpha_{22}\frac{b_2}{\beta_{22}} \\ 0 & 0 & -b_1 + \beta_{12}\frac{b_2}{\beta_{22}} - \lambda & 0 \\ \beta_{21}\frac{a_2}{\alpha_{22}} & \beta_{22}\frac{a_2}{\alpha_{22}} & 0 & -b_2 + b_2 - \lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} \alpha_{12}A_1 - \lambda & 0 & 0 & 0 \\ 0 & -\lambda & -\alpha_{21}\frac{b_2}{\beta_{22}} & -\alpha_{22}\frac{b_2}{\beta_{22}} \\ 0 & 0 & -\beta_{12}B_1 - \lambda & 0 \\ \beta_{21}\frac{a_2}{\alpha_{22}} & \beta_2\frac{a_2}{\alpha_{22}} & 0 & -\lambda \end{vmatrix} = 0$$

On expanding we get

$$(\alpha_{12}A_1 - \lambda)(\beta_{12}B_1 + \lambda)(\lambda^2 + a_2b_2) = 0 \tag{2.20}$$

which yields

$$\lambda_1 = \alpha_{12}A_1, \ \ \lambda_2 = -\beta_{12}B_1, \ \ \lambda_3 = \sqrt{a_2b_2}\ i, \ \ \lambda_4 = -\sqrt{a_2b_2}\ i$$

If $A_1 < 0, \ B_1 > 0$ then λ_1, λ_2 are negative real roots and λ_3 and λ_4 are imaginary. Therefore the second equilibrium position $2EP = \left(\bar{H}_1, \bar{H}_2, \bar{P}_1, \bar{P}_2\right) = \left(0, \frac{b_2}{\beta_{22}}, 0, \frac{a_2}{\alpha_{22}}\right)$

(i) is neutral if $A_1 < 0, \ B_1 > 0$.

(ii) otherwise it is unstable.

Case 3:

For the third equilibrium position $3EP = \left(\bar{H}_1, \bar{H}_2, \bar{P}_1, \bar{P}_2\right) = \left(\frac{b_1}{\beta_{11}}, 0, \frac{a_1}{\alpha_{11}}, 0\right)$

$$J\left(\frac{b_1}{\beta_{11}}, 0, \frac{a_1}{\alpha_{11}}, 0\right) = 0$$

$$\begin{vmatrix} -\lambda & 0 & -\alpha_{11}\frac{b_1}{\beta_{11}} & -\alpha_{12}\frac{b_1}{\beta_{11}} \\ 0 & a_2 - \alpha_{21}\frac{a_1}{\alpha_{11}} - \lambda & 0 & 0 \\ \beta_{11}\frac{a_1}{\alpha_{11}} & \beta_{12}\frac{a_1}{\alpha_{11}} & -b_1 + b_1 - \lambda & 0 \\ 0 & 0 & 0 & -b_2 - \beta_{21}\frac{b_1}{\beta_{11}} - \lambda \end{vmatrix} = 0$$

$$(-\beta_{21}B_2 - \lambda)(\alpha_{21}A_2 - \lambda)(\lambda^2 + a_1b_1) = 0 \tag{2.21}$$

On solving

$$\lambda_1 = \alpha_{21}A_2,\ \ \lambda_2 = -\beta_{12}B_2,\ \ \lambda_3 = \sqrt{a_1 b_1}\ i,\ \ \lambda_4 = -\sqrt{a_1 b_1}\ i$$

If $A_2 < 0,\ \ B_2 > 0$ then λ_1, λ_2 are negative real roots and λ_3 and λ_4 are imaginary. Therefore the equilibrium position $3EP = \left(\bar{H}_1, \bar{H}_2, \bar{P}_1, \bar{P}_2\right) = \left(\frac{b_1}{\beta_{11}}, 0, \frac{a_1}{\alpha_{11}}, 0\right)$

(i) is neutral if $A_2 < 0,\ \ B_2 > 0$.
(ii) otherwise it is unstable.

Case 4:
For the fourth equilibrium position $4EP = \left(\bar{H}_1, \bar{H}_2, \bar{P}_1, \bar{P}_2\right) = \left(0, \frac{b_1}{\beta_{12}}, \frac{a_2}{\alpha_{21}}, 0\right)$
Here,

$$J\left(0, \frac{b_1}{\beta_{12}}, \frac{a_2}{\alpha_{21}}, 0\right) = 0$$

$$\begin{vmatrix} a_1 - \alpha_{11}\frac{a_2}{\alpha_{21}} - \lambda & 0 & 0 & 0 \\ 0 & a_2 - a_2 - \lambda & -\alpha_{21}\frac{b_1}{\beta_{12}} & -\alpha_{22}\frac{b_1}{\beta_{12}} \\ \beta_{11}\frac{a_2}{\alpha_{21}} & \beta_{12}\frac{a_2}{\alpha_{21}} & -b_1 + b_1 - \lambda & 0 \\ 0 & 0 & 0 & -b_2 + \beta_{22}\frac{b_1}{\beta_{12}} - \lambda \end{vmatrix} = 0$$

$$(\alpha_{11}A_2 + \lambda)(\beta_{22}B_1 - \lambda)(\lambda^2 + a_2 b_1) = 0 \tag{2.22}$$

On solving

$$\lambda_1 = -\alpha_{11}A_2,\ \ \lambda_2 = \beta_{22}B_1,\ \ \lambda_3 = \sqrt{a_2 b_1}\ i,\ \ \lambda_4 = -\sqrt{a_2 b_1}\ i$$

If $A_2 > 0,\ \ B_1 < 0$ then λ_1, λ_2 are negative real roots and λ_3 and λ_4 are imaginary. Therefore the equilibrium position $4EP = \left(\bar{H}_1, \bar{H}_2, \bar{P}_1, \bar{P}_2\right) = \left(0, \frac{b_1}{\beta_{12}}, \frac{a_2}{\alpha_{21}}, 0\right)$

(i) is neutral if $A_2 > 0,\ \ B_1 < 0$.
(ii) otherwise it is unstable.

Case 5:
For the fifth equilibrium position $5EP = \left(\bar{H}_1, \bar{H}_2, \bar{P}_1, \bar{P}_2\right) = \left(\frac{b_2}{\beta_{21}}, 0, 0, \frac{a_1}{\alpha_{12}}\right)$

Here,

$$J\left(\frac{b_2}{\beta_{21}}, 0, 0, \frac{a_1}{\alpha_{12}}\right) = 0$$

$$\begin{vmatrix} a_1 - a_1 - \lambda & 0 & -\alpha_{11}\frac{b_2}{\beta_{21}} & -\alpha_{12}\frac{b_2}{\beta_{21}} \\ 0 & a_2 - \frac{\alpha_{22}a_1}{\alpha_{12}} & 0 & 0 \\ 0 & 0 & -b_1 + \frac{\beta_{11}b_2}{\beta 21} - \lambda & 0 \\ \beta_{21}\frac{a_1}{\alpha_{12}} & \beta_{22}\frac{a_1}{\alpha_{12}} & 0 & -b_2 + b_2 - \lambda \end{vmatrix} = 0$$

$$(\alpha_{22}A_1 + \lambda)(\beta_{11}B_2 - \lambda)(\lambda^2 + a_1 b_2) = 0 \tag{2.23}$$

On solving,

$$\lambda_1 = -\alpha_{22}A_1, \;\; \lambda_2 = \beta_{11}B_2, \;\; \lambda_3 = \sqrt{a_1 b_2}\; i, \;\; \lambda_4 = -\sqrt{a_1 b_2}\; i$$

If $A_1 > 0, \;\; B_2 < 0$ then λ_1, λ_2 are negative real roots and λ_3 and λ_4 are imaginary. Therefore the equilibrium position $5EP = \left(\bar{H}_1, \bar{H}_2, \bar{P}_1, \bar{P}_2\right) = \left(\frac{b_2}{\beta_{21}}, 0, 0, \frac{a_1}{\alpha_{12}}\right)$

(i) is neutral if $A_1 > 0, \;\; B_2 < 0$.
(ii) otherwise it is unstable.

Case 6:
For the sixth equilibrium position

$$6EP = \begin{pmatrix} \bar{H}_1 \\ \bar{H}_2 \\ \bar{P}_1 \\ \bar{P}_2 \end{pmatrix} = \begin{pmatrix} \beta_{12}\beta_{22}\frac{B_1}{B_3} \\ \beta_{11}\beta_{21}\frac{B_2}{B_3} \\ \alpha_{12}\alpha_{22}\frac{A_1}{A_3} \\ \alpha_{11}\alpha_{21}\frac{A_2}{A_3} \end{pmatrix},$$

In this case,

$$J(\bar{H}_1, \bar{H}_2, \bar{P}_1, \bar{P}_2) = \begin{vmatrix} -\lambda & 0 & -\alpha_{11}\bar{H}_1 & -\alpha_{12}\bar{H}_1 \\ 0 & -\lambda & -\alpha_{21}\bar{H}_2 & -\alpha_{22}\bar{H}_2 \\ \beta_{11}\bar{P}_1 & \beta_{12}\bar{P}_1 & -\lambda & 0 \\ \beta_{21}\bar{P}_2 & \beta_{22}\bar{P}_2 & 0 & -\lambda \end{vmatrix} = 0$$

On expanding we get

$$\begin{aligned} -\lambda[-\lambda^3 - \alpha_{22}\bar{H}_2\beta_{22}\bar{P}_2\lambda - \alpha_{21}\bar{H}_2\beta_{12}\bar{P}_1] &\;\; - \\ \alpha_{11}\bar{H}_1[-\alpha_{22}\bar{H}_2\beta_{11}\bar{P}_1\beta_{22}\bar{P}_2 + \alpha_{22}\bar{H}_2\beta_{12}\bar{P}_1\beta_{21}\bar{P}_2 - \beta_{11}\bar{P}_1\lambda^2] &\;\; + \\ \alpha_{12}\bar{H}_1[\beta_{21}\bar{P}_2\lambda^2 - \alpha_{21}\bar{H}_2\beta_{11}\bar{P}_1\beta_{22}\bar{P}_2 + \alpha_{21}\bar{H}_2\beta_{12}\bar{P}_1\beta_{21}\bar{P}_2] &= 0 \end{aligned}$$

$$
\begin{aligned}
\lambda^4 + \lambda^2(\alpha_{11}\bar{P}_1\beta_{11}\bar{H}_1 + \alpha_{12}\bar{P}_2\beta_{21}\bar{H}_1 + \alpha_{21}\bar{P}_1\beta_{12}\bar{H}_2 + \alpha_{22}\bar{P}_2\beta_{22}\bar{H}_2) &\quad + \\
\alpha_{11}\bar{P}_1\alpha_{22}\bar{P}_2\beta_{11}\bar{H}_1\beta_{22}\bar{H}_2 - \alpha_{11}\bar{P}_1\alpha_{22}\bar{P}_2\beta_{21}\bar{H}_1\beta_{12}\bar{H}_2 &\quad - \\
\alpha_{12}\bar{P}_2\alpha_{21}\bar{P}_1\beta_{11}\bar{H}_1\beta_{22}\bar{H}_2 + \alpha_{21}\bar{P}_1\alpha_{12}\bar{P}_2\beta_{21}\bar{H}_1\beta_{12}\bar{H}_2 &\quad = \quad 0
\end{aligned}
$$

$$
\begin{aligned}
\lambda^4 + \lambda^2(\alpha_{11}\bar{P}_1\beta_{11}\bar{H}_1 + \alpha_{12}\bar{P}_2\beta_{21}\bar{H}_1 + \alpha_{21}\bar{P}_1\beta_{12}\bar{H}_2 + \alpha_{22}\bar{P}_2\beta_{22}\bar{H}_2) &\quad + \\
(\beta_{11}\bar{H}_1\beta_{22}\bar{H}_2 - \beta_{21}\bar{H}_1\beta_{12}\bar{H}_2)(\alpha_{11}\bar{P}_1\alpha_{22}\bar{P}_2 - \alpha_{12}\bar{P}_2\alpha_{21}\bar{P}_1) &\quad = \quad 0
\end{aligned}
$$

Substituting $\begin{pmatrix} \bar{H}_1 \\ \bar{H}_2 \\ \bar{P}_1 \\ \bar{P}_2 \end{pmatrix} = \begin{pmatrix} \beta_{12}\beta_{22}\frac{B_1}{B_3} \\ \beta_{11}\beta_{21}\frac{B_2}{B_3} \\ \alpha_{12}\alpha_{22}\frac{A_1}{A_3} \\ \alpha_{11}\alpha_{21}\frac{A_2}{A_3} \end{pmatrix}$ in above equation, we get

$$
\begin{aligned}
\lambda^4 + \lambda^2(\alpha_{11}\alpha_{12}\alpha_{22}\frac{A_1}{A_3}\beta_{11}\beta_{12}\beta_{22}\frac{B_1}{B_3} + \alpha_{11}\alpha_{12}\alpha_{21}\frac{A_2}{A_3}\beta_{21}\beta_{12}\beta_{22}\frac{B_1}{B_3} &\quad + \\
\alpha_{11}\alpha_{21}\alpha_{22}\frac{A_2}{A_3}\beta_{11}\beta_{21}\beta_{22}\frac{B_2}{B_3} + \alpha_{12}\alpha_{21}\alpha_{22}\frac{A_1}{A_3}\beta_{11}\beta 12\beta 21\frac{B_2}{B_3}) + (\beta_{11}\beta_{22} &\quad - \\
\beta_{12}\beta_{21})\beta_{12}\beta_{22}\frac{B_1}{B_3}\beta_{11}\beta_{21}\frac{B_2}{B_3}(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})\alpha_{12}\alpha_{22}\frac{A_1}{A_3}\alpha_{11}\alpha_{21}\frac{A_2}{A_3} &\quad = \quad 0
\end{aligned}
$$

$$
\begin{aligned}
A_3B_3\lambda^4 + \lambda^2(\alpha_{11}\alpha_{12}\alpha_{22}\beta_{11}\beta_{12}\beta_{22}A_1B_1 + \alpha_{11}\alpha_{12}\alpha_{21}\beta_{12}\beta_{21}\beta_{22}A_2B_1 &\quad + \\
\alpha_{11}\alpha_{21}\alpha_{22}\beta_{11}\beta_{21}\beta_{22}A_2B_2 + \alpha_{12}\alpha_{21}\alpha_{22}\beta_{11}\beta_{12}\beta_{21}A_1B_2) &\quad + \\
\alpha_{11}\alpha_{12}\alpha_{21}\alpha_{22}\beta_{11}\beta_{12}\beta_{21}\beta_{22}A_1A_2B_1B_2 &\quad = \quad 0
\end{aligned} \tag{2.24}
$$

If $A_1 > 0, A_2 > 0 \implies A_3 > 0$ and if $B_1 > 0, B_2 > 0 \implies B_3 > 0$. In this case all the coefficient of polynomial equation (2.24) are positive and bi-quadratic form. So all the roots are complex conjugate.Thus equilibrium position

$$
6EP = \begin{pmatrix} \bar{H}_1 \\ \bar{H}_2 \\ \bar{P}_1 \\ \bar{P}_2 \end{pmatrix} = \begin{pmatrix} \beta_{12}\beta_{22}\frac{B_1}{B_3} \\ \beta_{11}\beta_{21}\frac{B_2}{B_3} \\ \alpha_{12}\alpha_{22}\frac{A_1}{A_3} \\ \alpha_{11}\alpha_{21}\frac{A_2}{A_3} \end{pmatrix}
$$

(i) is neutral if $A_3 > 0, B_3 > 0$ or $A_3 < 0, B_3 < 0$.
(ii) is unstable if $A_3 > 0, B_3 < 0$ or $A_3 < 0, B_3 > 0$.

The above discussion is summarized as the Table 1. The numbers 1 to 6 in the table corresponds the equilibrium positions $1EP$ to $6EP$ respectively, where red colored number denotes the neutral equilibrium position and other denote the unstable equilibrium positions.

At the intersection of rows and columns, there appears only one neutral equilibrium except at the intersection of the row-columns positions $(1, 6)$ and $(6, 1)$. At that positions there appears two neutral equilibriums.

Thus, out of 36 cases, as presented in Table 1, we observe the following facts.

TABLE 1. Stability analysis with sign of A_1, A_2, A_3, B_1, B_2 and B_3.

	B_1, B_2, B_3					
A_1, A_2, A_3	$+,+,+$	$+,-,+$	$-,+,+$	$+,-,-$	$-,+,-$	$-,-,-$
$+,+,+$	1,2,3,4,5,**6**	1,2,3,4,**5**,6	1,2,3,**4**,5,6	1,2,3,4,**5**,6	1,2,3,**4**,5,6	1,2,3,**4**,**5**,6
$+,-,+$	1,2,**3**,4,5,6	1,2,3,4,**5**,6	1,2,**3**,4,5,6	1,2,3,4,**5**,6	1,2,**3**,4,5,6	1,2,3,4,**5**,6
$-,++$	1,**2**,3,4,5,6	1,**2**,3,4,5,6	1,2,3,**4**,5,6	1,**2**,3,4,5,6	1,2,3,**4**,5,6	1,2,3,**4**,5,6
$+,-,-$	1,2,**3**,4,5,6	1,2,3,4,**5**,6	1,2,**3**,4,5,6	1,2,3,4,**5**,6	1,2,**3**,4,5,6	1,2,3,4,**5**,6
$-,+,-$	1,**2**,3,4,5,6	1,**2**,3,4,5,6	1,2,3,**4**,5,6	1,**2**,3,4,5,6	1,2,3,**4**,5,6	1,2,3,**4**,5,6
$-,-,-$	1,**2**,**3**,4,5,6	1,**2**,3,4,5,6	1,2,**3**,4,5,6	1,**2**,3,4,5,6	1,2,**3**,4,5,6	1,2,3,4,5,**6**

(i) Equilibrium positions $2EP, 3EP, 4EP$ and $5EP$, are in neutral equilibrium in nine cases.

(ii) Equilibrium position $6EP$ is in neutral equilibrium in two cases.

(iii) There are two cases in which two equilibrium positions $2EP, 3EP$ or $4EP, 5EP$ can be in neutral equilibrium at the same time.

The facts observed then follows that there are 34 outcome cases which are independent of initial conditions and can be predicted only if the signs of (A_1, A_2, A_3), (B_1, B_2, B_3) are known. In two cases the outcome dependents on initial conditions besides sign of (A_1, A_2, A_3),(B_1, B_2, B_3). That is,

(a) **Independent on initial condition**

Condition 1:

$(+,+,+)$,$(+,+,+)$ or $(-,-,-)$,$(-,-,-)$ implies that all four species H_1, H_2, P_1, P_2 will survive and there will be conservative oscillation about the non zero equilibrium point.

Condition 2:

$(+,-,+)$,$(+,-,+)$ implies that the second prey species H_2 and first predator specie P_1 will be die out. Thus the first prey species H_1 and the second predator specie P_2 will be survive. There will be conservative oscillations about the prey population $\bar{H}_1 = \dfrac{b_2}{\beta_{12}}$ and predator populations $\bar{P}_2 = \dfrac{a_1}{\alpha_{12}}$. Similar behavior will be true for other 31 cases.

(b) **Dependent on initial condition**

Condition 3:

$(+,+,+)$,$(-,-,-)$ implies that either H_2, P_1 will die out and there will be conservative oscillation about $\bar{H}_1 = \dfrac{b_2}{\beta_{21}}$, $\bar{P}_2 = \dfrac{a_1}{\alpha_{12}}$ or H_1, P_2 will die out and there will be conservative oscillation about $\bar{H}_2 = \dfrac{b_1}{\beta_{12}}$, $\bar{P}_1 = \dfrac{a_1}{\alpha_{21}}$.

Condition 4:
$(-,-,-),(+,+,+)$ implies that either H_1, P_1 will die out and there will be conservative oscillation about $\bar{H}_2 = \dfrac{b_2}{\beta_{22}},\ \bar{P}_2 = \dfrac{a_2}{\alpha_{22}}$ or H_2, P_2 will die out and there will be conservative oscillation about $\bar{H}_1 = \dfrac{b_1}{\beta_{11}},\ \bar{P}_1 = \dfrac{a_1}{\alpha_{11}}$.

3. NUMERICAL RESULTS AND DISCUSSIONS

The model is analyzed numerically and graphically using Runge-Kutta fourth order method based on the parameter value from literatures published by researchers [6, 9]. The numerical and graphical results help to understand the qualitative behavior of each compartment of preys and predators. The conditions 1 and 3 described above are numerically and graphically presented below.

(a) **Independent on initial condition:**
Condition 1:
$a_1 = a_2 = 3,\ \alpha_{11} = \alpha_{22} = 2,\ \alpha_{12} = \alpha_{21} = 1$
$b_1 = 4, b_2 = 2, \beta_{11} = 3, \beta_{12} = \beta_{21} = \beta_{22} = 1$
Also,
$A_1 = \dfrac{a_1}{\alpha_{12}} - \dfrac{a_2}{\alpha_{22}} = \dfrac{3}{1} - \dfrac{3}{2} = 1.5 > 0$
$A_2 = \dfrac{a_2}{\alpha_{21}} - \dfrac{a_1}{\alpha_{11}} = \dfrac{3}{1} - \dfrac{3}{2} = 1.5 > 0$
$A_3 = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = 2 * 2 - 1 * 1 = 3 > 0$
$B_1 = \dfrac{b_1}{\beta_{12}} - \dfrac{b_2}{\beta_{22}} = \dfrac{4}{1} - \dfrac{2}{1} = 2 > 0$
$B_2 = \dfrac{b_2}{\beta_{21}} - \dfrac{b_1}{\beta_{11}} = \dfrac{2}{1} - \dfrac{4}{3} = \dfrac{2}{3} > 0$
$B_3 = \beta_{11}\beta_{22} - \beta_{12}\beta_{21} = 3 * 1 - 1 * 1 = 2 > 0$
Here the sign of (A_1, A_2, A_3) are $(+,+,+)$ and Sign of (B_1, B_2, B_3) are $(+,+,+)$.
The non zero equilibrium points are:
$\bar{H}_1 = \beta_{12}\,\beta_{22}\,\frac{B_1}{B_3} = 1 * 1 * \frac{2}{2} = 1$
$\bar{H}_2 = \beta_{11}\,\beta_{21}\,\frac{B_2}{B_3} = 1 * 3 * -\frac{2}{3} * \frac{1}{2} = 1$
$\bar{P}_1 = \alpha_{12}\,\alpha_{22}\,\frac{A_1}{A_3} = 1 * 2 * \frac{3}{2} * \frac{1}{3} = 1$
$\bar{P}_2 = \alpha_{11}\,\alpha_{21}\,\frac{A_2}{A_3} = 2 * 1 * \frac{3}{2} * \frac{1}{3} = 1$
Hence the nonzero equilibrium points $(\bar{H}_1, \bar{H}_2, \bar{P}_1, \bar{P}_2) = (1,1,1,1)$ exists and neutral. Now the characteristic equation is

$$\begin{vmatrix} -\lambda & 0 & -\alpha_{11}\bar{H}_1 & -\alpha_{12}\bar{H}_1 \\ 0 & -\lambda & -\alpha_{21}\bar{H}_2 & -\alpha_{22}\bar{H}_2 \\ \beta_{11}\bar{P}_1 & \beta_{12}\bar{P}_1 & -\lambda & 0 \\ \beta_{21}\bar{P}_2 & \beta_{22}\bar{P}_2 & 0 & -\lambda \end{vmatrix} = 0$$

$$\implies \begin{vmatrix} -\lambda & 0 & -2 & -1 \\ 0 & -\lambda & -1 & -2 \\ 3 & 1 & -\lambda & 0 \\ 1 & & 0 & -\lambda \end{vmatrix} = 0$$

$$\lambda^4 + 10\lambda^2 + 6 = 0$$

$$\lambda^2 = \frac{-10 \pm \sqrt{100-24}}{2} = -5 \pm \sqrt{19}$$

Case 1:

If we take initial condition of preys and predators as follows

$$H_1(0) = 1.25, H_2(0) = .75, P_1(0) = 1.25, P_2(0) = .75$$

Then the graphical results obtained are shown in Figures 1 to 3.

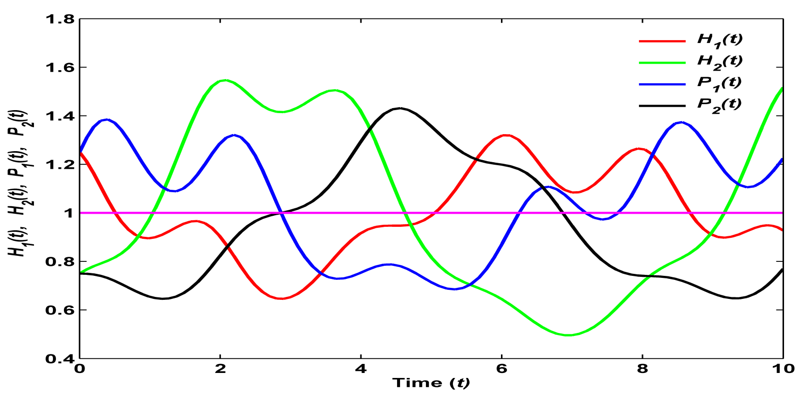

FIGURE 1. Conservation oscillations of $H_1(t)$, $H_2(t)$, $P_1(t)$ and $P_2(t)$ about the equilibrium position $(1, 1, 1, 1)$.

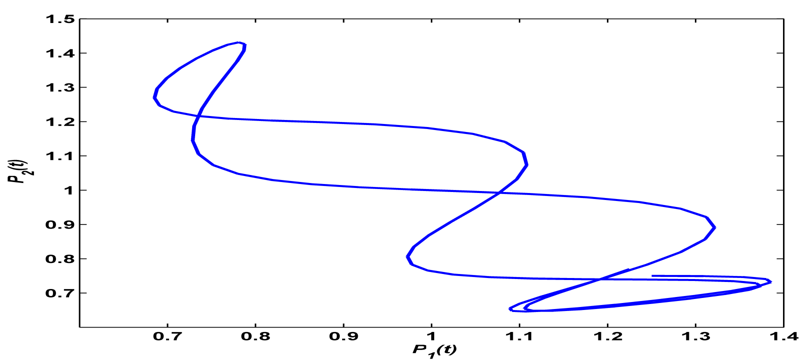

FIGURE 2. Projection of the trajectory on the P_1P_2-plane.

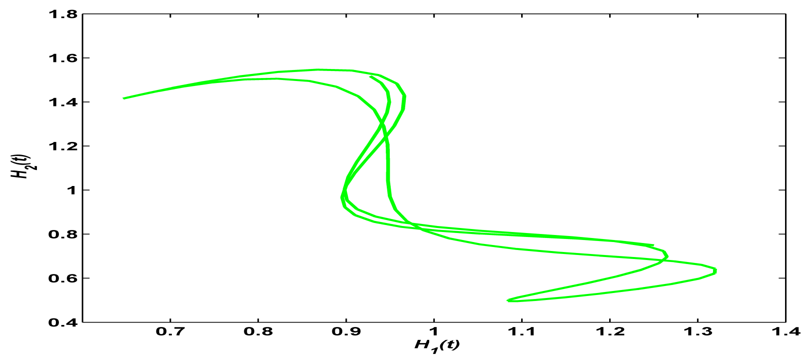

FIGURE 3. Projection of the trajectory on the H_1H_2-plane.

Case 2:

If we change initial condition of preys and predators as follows.

$$H_1(0) = 1.5, H_2(0) = 1.1, P_1(0) = 1.3, P_2(0) = .75$$

Then the graphical results obtained are shown in Figures 4 to 6.

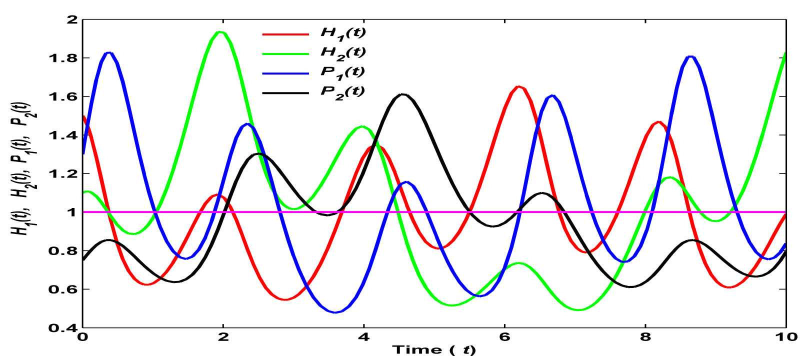

FIGURE 4. Conservation oscillations of $H_1(t)$, $H_2(t)$, $P_1(t)$ and $P_2(t)$ about the equilibrium position $(1, 1, 1, 1)$.

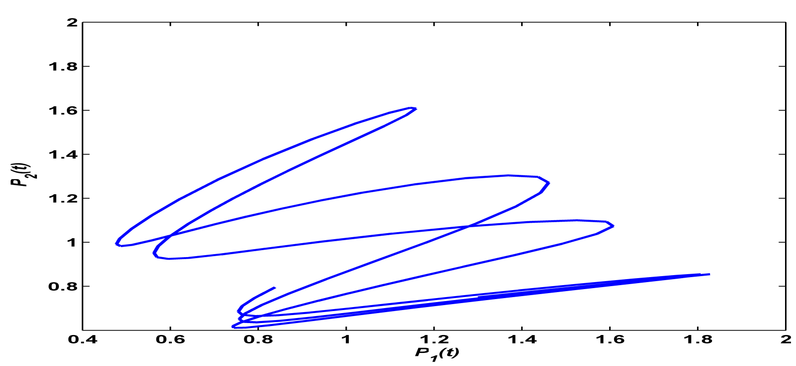

FIGURE 5. Projection of the trajectory on the P_1P_2-plane.

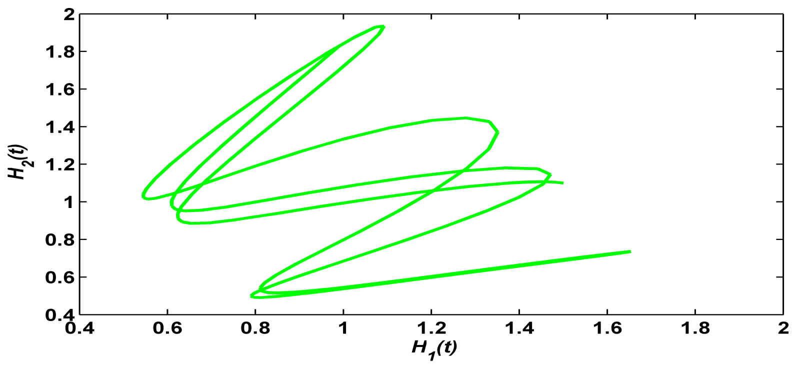

FIGURE 6. Projection of the trajectory on the H_1H_2-plane.

From graphical representations of Case 1 and Case 2 of **condition 1**, we observe the same behavior of population dynamics. So this condition is clearly independent of initial populations of preys and predators. That is, for any initial data value, all four species will survive and will oscillates about the nonzero equilibrium point.

(b) **Dependent on initial condition:**

Condition 3:

$a_1 = a_2 = 1,\ \alpha_{11} = \alpha_{22} = 1,\ \alpha_{12} = \alpha_{21} = 0.5$

$b_1 = 2, b_2 = 2, \beta_{11} = 1, \beta_{12} = \beta_{21} = 2, \beta_{22} = 1$

Also,

$$A_1 = \frac{a_1}{\alpha_{12}} - \frac{a_2}{\alpha_{22}} = \frac{1}{0.5} - \frac{1}{1} = 1 > 0$$
$$A_2 = \frac{a_2}{\alpha_{21}} - \frac{a_1}{\alpha_{11}} = \frac{1}{0.5} - \frac{1}{1} = 1 > 0$$
$$A_3 = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = 1 * 1 - 0.5 * 0.5 = 0.75 < 0$$
$$B_1 = \frac{b_1}{\beta_{12}} - \frac{b_2}{\beta_{22}} = \frac{2}{2} - \frac{2}{1} = -1 < 0$$
$$B_2 = \frac{b_2}{\beta_{21}} - \frac{b_1}{\beta_{11}} = \frac{2}{2} - \frac{2}{1} = -1 < 0$$
$$B_3 = \beta_{11}\beta_{22} - \beta_{12}\beta_{21} = 1 * 1 - 2 * 2 = -3 < 0$$

This implies $\big(A_1, A_2, A_3\big)$ having sign pattern $(+, +, +)$ and $\big(B_1, B_2, B_3\big)$ having sign pattern $(-, -, -)$.

The nonzero equilibrium position $(6EP)$ is

$$\bar{H}_1 = \beta_{12}\,\beta_{22}\,\frac{B_1}{B_3} = 2 * 1 * \frac{-1}{-3} = \frac{2}{3}$$
$$\bar{H}_2 = \beta_{11}\,\beta_{21}\,\frac{B_2}{B_3} = 1 * 2 * \frac{-1}{-3} = \frac{2}{3}$$
$$\bar{P}_1 = \alpha_{12}\alpha_{22}\frac{A_1}{A_3} = 0.5 * 1 * \frac{1}{0.75} = \frac{2}{3}$$
$$\bar{P}_2 = \alpha_{11}\alpha_{21}\frac{A_2}{A_3} = 1 * 0.5 * \frac{1}{0.75} = \frac{2}{3}$$

Hence, the nonzero equilibrium position $6EP = \big(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}\big)$ exists and unstable but the equilibrium positions $4EP = (0, 1, 2, 0)$ and $5EP = (1, 0, 0, 2)$ exits and neutral. Since $A_2 > 0$, $B_1 < 0$ and $A_1 > 0$, $B_2 < 0$.

Here sign of A_1, A_2, A_3 and $B_1, B_2, B_3$are $(+,+,+)$,$(-,-,-)$. Thus out come is dependent on initial condition which implies that either H_2, P_1 will die out and there will be conservative oscillation about $\bar{H}_1 = \dfrac{b_2}{\beta_{21}}$, $\bar{P}_2 = \dfrac{a_1}{\alpha_{12}}$ or H_1, P_2 will die out and there will be conservative oscillation about $\bar{H}_2 = \dfrac{b_1}{\beta_{12}}$ $\bar{P}_1 = \dfrac{a_1}{\alpha_{21}}$.

Case 1:
If we take initial condition of preys and predators as follows.

$$H_1(0) = 1.25, H_2(0) = .75, P_1(0) = 1.25, P_2(0) = .75$$

Then the graphical results obtained are shown in Figures 7 to 9

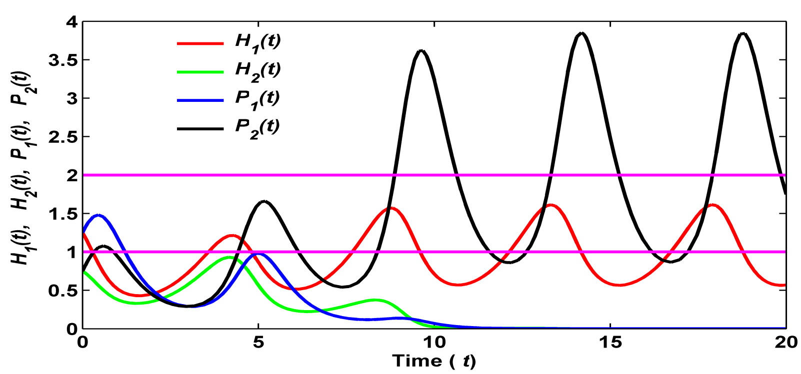

FIGURE 7. Conservation oscillations of $H_1(t)$ about the equilibrium point $\bar{H}_1 = 1$ and $P_2(t)$ about the equilibrium point $\bar{P}_2 = 2$. But $H_2(t)$ and $P_1(t)$ die out.

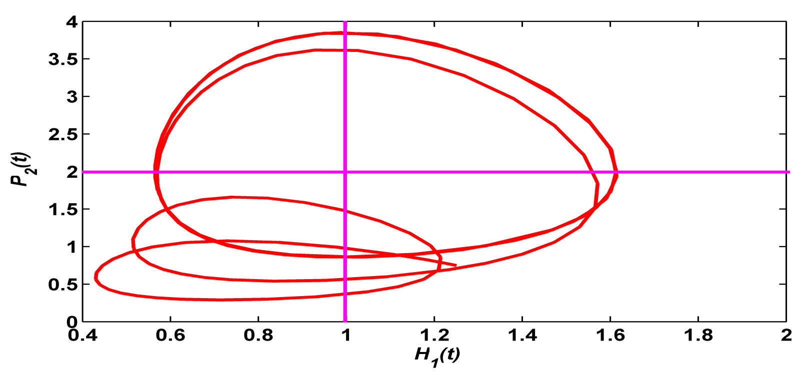

FIGURE 8. Projection of the trajectory on the H_1P_2-plane.

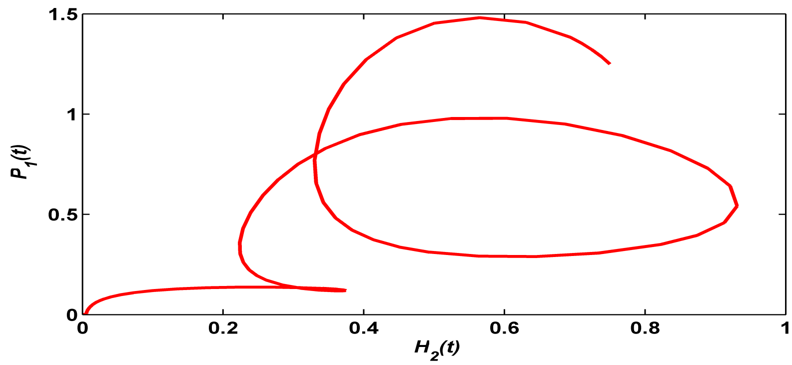

FIGURE 9. Projection of the trajectory on the H_2P_1-plane.

Case 2:
If we change initial condition of preys and predators as follows.

$$H_1(0) = 0.5, H_2(0) = 1.25, P_1(0) = 0.5, P_2(0) = 1.25$$

The graphical result shown in Figures 10 to 12.

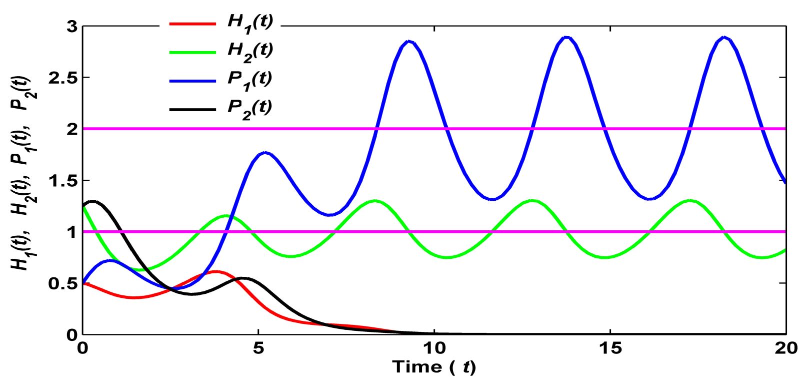

FIGURE 10. Conservation oscillations of $H_2(t)$ about the equilibrium point $\bar{H}_2 = 1$ and $P_1(t)$ about the equilibrium point $\bar{P}_1 = 2$. But $H_1(t)$ and $P_2(t)$ die out.

From graphical representations of Case 1 and Case 2 of **condition 3**, we observe the different behavior of population dynamics. So this condition is dependent of initial populations of preys and predators.

ACKNOWLEDGMENTS

The first author acknowledge to Cosmos College of Management and Technology, Lalitpur for providing financial support to carry out this study.

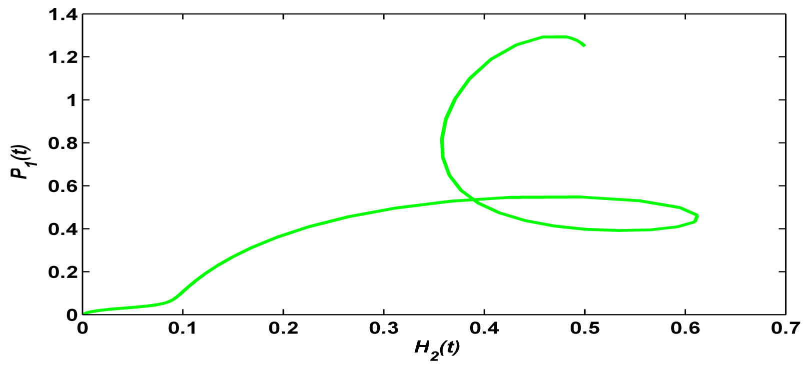

FIGURE 11. Projection of the trajectory on the H_1P_2−plane.

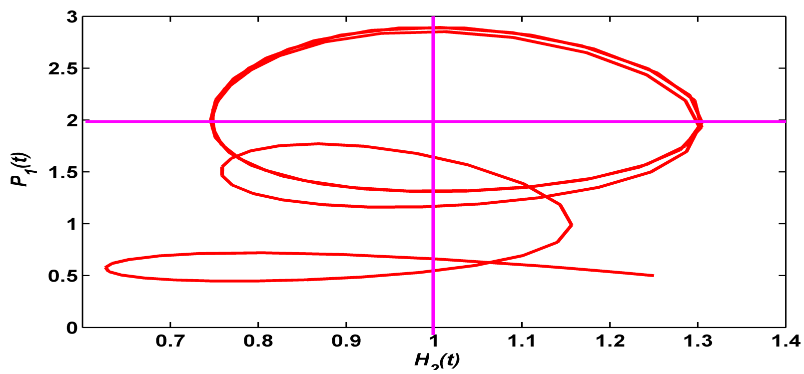

FIGURE 12. Projection of the trajectory on the H_2P_1−plane.

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