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EXPONENTIAL APPROXIMATION OF SOLUTIONS OF BIDIRECTIONAL NEURAL NETWORKS MODEL WITH POSITIVE DELAY

SYED ABBAS[1], ANIBAL CORONEL[2], MANUEL PINTO[3], SWATI TYAGI[4]

[1,4]School of Basic Sciences
Indian Institute of Technology Mandi
Mandi, H.P., 175001, INDIA

[2]Departamento de Ciencias Básicas
Facultad de Ciencias
Universidad del Bío-Bío, Campus Fernando May, Chillán, CHILE

[3]Departamento de Matemáticas
Facultad de Ciencias
Universidad de Chile, CHILE

[4]Department of Mathematics
Indian Institute of Technology Ropar
Punjab, INDIA

ABSTRACT: In this paper, a class of bidirectional neural network containing delay has been studied. Based on the theory of differential calculus and the generalized Gronwall inequality, the solution of the corresponding neural network model is approximated using discretization method and a sufficient condition is established for its stability. Some easily verifiable sufficient conditions are obtained ensuring that every solution of the discrete-time analogue converges exponentially to the unique solution. In the last section, two numerical example are also presented to validate the feasibility of the proposed results and illustrate the advantages of the discrete-time analogues in numerically simulating the continuous-time networks.

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1. INTRODUCTION

The term 'neural network' denotes a collection of interconnected, interacting neurons, which can be biological or artificial. A system of connected nodes constitutes an artificial neural network. Arranging the nodes in different configurations yields different artificial neural networks with various characteristic properties. Neural networks have been proved to be much promising practical tool for parallel computation. Several authors have studied the dynamics of neural networks [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] theoretically as well as computationally. Many authors have established the almost periodic solutions of cellular neural networks using discretization scheme [11, 12, 13, 14, 15]. Moreover, convergence of numerical methods for various delay differential equations has also gained attention in recent few years [16].

Two relevant concepts in the recent neural network theory are the bidirectional associative memories and the delay time. *First*, the bidirectional associative memories (BAM) have one input layer and one output layer. Information signals can travel in both directions; from input to output and back from output to input. The neurons in one layer are fully interconnected to the neurons in the second layer with no interconnection among neurons in the same layer. The weight from first layer to second layer is same as the weight from second layer to first layer. In [4] Kosko extended the Hopfield model by incorporating an additional layer for performing both recurrent autoassociations and heteroassociations on the stored memories. *Second*, we remark that the activation of neurons is given with a delay. Thus, the classical mathematical models based on ordinary differential equations must be modified in order to incorporate the delay behavior.

In this paper, we are concerned with the following bidirectional neural network model with delay:

$$\frac{dx_i(t)}{dt} = -a_i(t)x_i(t) + \sum_{j=1}^{m} b_{ij}(t) f_j(y_j(t-\sigma_{ij})) + I_i(t), \tag{1.1a}$$

$$\frac{dy_i(t)}{dt} = -c_i(t)y_i(t) + \sum_{j=1}^{m} d_{ij}(t) g_j(x_j(t-\tau_{ij})) + J_i(t), \tag{1.1b}$$

$$x_i(t) = \phi_i(t),\ t\in[-\tau_i,0]; \quad \tau_i = \max_j\{\tau_{ij}\}, \tag{1.1c}$$

$$y_i(t) = \psi_i(t),\ t\in[-\sigma_i,0]; \quad \sigma_i = \max_j\{\sigma_{ij}\}, \tag{1.1d}$$

for $i = 1, 2, \cdots, m$, and $\phi_i \in C([-\tau_i, 0], \mathbb{R}),\ \psi_i \in C([-\sigma_i, 0], \mathbb{R})$. System (1.1) consists of two sets of m neurons (or units) arranged in two layers, namely, $I-$layer and $J-$layer. On the $I-$layer, the set of m neurons having membrane potentials $x_i(.)$ receive external inputs $I_i(t)$; whereas on the $J-$layer, the other set of m neurons with membrane potentials $y_i(.)$ receive external inputs $J_i(t)$; b_{ij} and c_{ij} denote the synaptic connection weights; σ_{ij}, τ_{ij} denote the time delays in axonal transmission of signals and neural processing. $a_i(.)$ and $c_i(.)$ denote the rates with which the i^{th} neurons from the $I-$layer and the $J-$layer, respectively, reset their potentials to their resting states when disconnected from other neurons and external inputs; $f_j(.)$, $g_j(.)$ denote nonlinear activation functions.

On the other hand, we recall that the study of differential equations with piecewise constant argument (DEPCA) was initiated in [17, 18, 19] and later on, it has been widely investigated by several researchers by reducing to discrete equations (see for instance [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33]). Indeed, DEPCA are hybrid equations with characteristics of both continuous systems and discrete equations. The continuity of a solution in points combining two consecutive intervals implies the existence of iterative relations for the solution at such points. Thus the equations become similar in structure to those found in certain "sequentially continuous" models of the dynamic ones of the disease of the treaty by S. Busenberg and K.L. Cooke [17]. Other applications of DEPCA are discussed in [26, 34, 35]. Moreover, the delay differential equations with piecewise constant argument fortify several properties of the continuous dynamical systems generated by delay differential equations [36] and of the discrete dynamical systems generated by difference equations [37].

The novelty of this paper is to establish some approximating results for the solutions of delayed bidirectional neural network model (1.1) via the solutions of corresponding delay differential equations with piecewise constant arguments. A related work with this purpose is [23], where some relations have been obtained between the solutions of delay differential equations with continuous arguments and the solutions of some retarded delay DEPCA, which were used in computing the numerical solution of ordinary and delay differential equations. More precisely, we consider that the corresponding differential equations with piecewise constant argument is given by

$$\frac{dx_i^h(t)}{dt} = -a_i(t)x_i^h(t) + \sum_{j=1}^{m} b_{ij}(t)f_j(y_j^h(\gamma_h(t-\sigma_{ij}))) + I_i(t), \tag{1.2a}$$

$$\frac{dy_i^h(t)}{dt} = -c_i(t)y_i^h(t) + \sum_{j=1}^{m} d_{ij}(t)g_j(x_j^h(\gamma_h(t-\tau_{ij}))) + J_i(t), \tag{1.2b}$$

$$x_i^h(\ell h) = \phi_i(\ell h),\ \ell = -k, \cdots, 0, \tag{1.2c}$$

$$y_i^h(\ell h) = \psi_i(\ell h),\ \ell = -l, \cdots, 0, \tag{1.2d}$$

where $h \in \mathbb{R}^+$ and the step function γ_h is defined as follows

$$\gamma_h(t-r) = \left[\frac{t}{h} - \left[\frac{r}{h}\right]\right] h. \tag{1.2e}$$

The bracket $[\cdot]$ denotes the greatest integer function.

We introduce two main results given as Theorem 3.2 and Theorem 3.3. First, in Section 2, using variation of constants formula for delay differential equations, it has been shown that for the equations (1.2), the solutions $x_i^h(t)$, $y_i^h(t)$ and the sequences $x_i^h(nh)$ and $y_i^h(nh)$, $n \in \mathbb{Z}$, approximating the solutions $x_i(t)$ and $y_i(t)$, of system (1.1) are calculated, for major details see Theorem 2.2. Second, in Section 3, we establish an approximation result: if the zero solution of (3.4) and the coefficients satisfy some regularity assumptions (see (H1)-(H5)), then the difference of the solutions for (1.1) and (1.2) are bounded by a function depending of h and such that vanishes when $h \to 0$ uniformly on $\mathbb{R}^+$, see Theorem 3.3 and also converges exponentially to zero. Moreover, in Section 4, appropriate examples are given to illustrate the theory, depicting the approximated solution calculated via (1.2) versus actual solution along with the error estimation.

2. SOLUTION OF (??) AS DISCRETIZATION OF (??) ON $\mathbb{R}^+$

In this section we construct the solution of (1.2) in the sense of the following definition.

Definition 2.1. A solution of (1.2) is a family the $2m$ real functions $\{x_1^h, \dots, x_m^h, y_1^h, \dots, y_m^h\}$, such that for each $i = 1, 2, \cdots, m$, the following assertions

(i) The functions x_i^h and y_i^h are continuous on $\mathbb{R}^+$,

(ii) The derivatives $x_i^{h'}, y_i^{h'}$ exist at each point $t \in \mathbb{R}^+$ with possible exception of the points kh with $k \in \mathbb{N}$, where one sided derivatives with finite values exist,

(iii) The functions x_i^h and y_i^h satisfy the equations (1.2c)-(1.2d) on each interval $[kh, (k+1)h)$ with $k \in \mathbb{N}$,

are satisfied.

Now we assume that the the coefficients $a_i(t), b_{ij}(t), c_i(t), d_{ij}(t), I_i(t), J_i(t)$ are bounded continuous functions and the functions f_j, g_j are Lipschitz. These conditions are required for the existence of a unique solution of our problem.

Theorem 2.2. *The system (1.2) has a unique solution in the sense of the definition 2.1. Moreover, for each $i = 1, 2, \cdots, m$, and for $t \in [nh, (n+1)h)$ the functions x_i^h and y_i^h are in the form given by*

$$\begin{aligned}
x_i^h(t) &= \exp\left(-\int_0^t a_i(u)du\right)\phi_i(0) \\
&+ \sum_{\xi=0}^{n-1}\int_{\xi h}^{(\xi+1)h} \exp\left(-\int_s^t a_i(u)du\right)\left\{\sum_{j=1}^m b_{ij}(s) f_j(y_i^h(h(\xi - k_{ij})))) + I_i(s)\right\} ds \\
&+ \int_{nh}^t \exp\left(-\int_s^t a_i(u)du\right)\left\{\sum_{j=1}^m b_{ij}(s) f_j(y_i^h(h(n - k_{ij})))) + I_i(s)\right\} ds, \quad (2.1a)
\end{aligned}$$

$$
\begin{aligned}
y_i^h(t) &= \exp\left(-\int_0^t c_i(u)du\right)\psi_i(0) \\
&+\sum_{\xi=0}^{n-1}\int_{\xi h}^{(\xi+1)h}\exp\left(-\int_s^t c_i(u)du\right)\left\{\sum_{j=1}^m d_{ij}(s)g_j(x_i^h(h(\xi-l_{ij}))))+J_i(s)\right\}ds \\
&+\int_{nh}^t \exp\left(-\int_s^t c_i(u)du\right)\left\{\sum_{j=1}^m d_{ij}(s)g_j(x_i^h(h(n-l_{ij}))))+J_i(s)\right\}ds, \qquad (2.1b)
\end{aligned}
$$

where k_{ij} and l_{ij} are defined as follows

$$
k_{ij}=\left[\frac{\sigma_{ij}}{h}\right], \quad l_{ij}=\left[\frac{\tau_{ij}}{h}\right] \qquad (2.1c)
$$

and the sequence $\left(x_i^h(nh), y_i^h(nh)\right)$ satisfies the nonlinear difference equations

$$
\begin{aligned}
x_i^h((n+1)h) &= \exp\left(-\int_{nh}^{(n+1)h} a_i(u)du\right)x_i^h(nh) \\
&\quad+\int_{nh}^{(n+1)h}\exp\left(-\int_s^{(n+1)h} a_i(u)du\right) \\
&\quad\left\{\sum_{j=1}^m b_{ij}(s)f_j(y_j^h(h(n-k_{ij})))+I_i(s)\right\}ds, \qquad (2.2a) \\
y_i^h((n+1)h) &= \exp\left(-\int_{nh}^{(n+1)h} c_i(u)du\right)y_i^h(nh) \\
&\quad+\int_{nh}^{(n+1)h}\exp\left(-\int_s^{(n+1)h} c_i(u)du\right) \\
&\quad\left\{\sum_{j=1}^m d_{ij}(s)g_j(x_j^h(h(n-l_{ij})))+J_i(s)\right\}ds, \qquad (2.2b)
\end{aligned}
$$

with the initial conditions $x_i^h(nh)=\phi_i(nh)$ and $y_i^h(nh)=\psi_i(nh)$ for $n=-l_{ij},\dots,-1,0$ and $n=-k_{ij},\dots,-1,0$, respectively.

Proof. The proof is constructive and is based on variation of constants formula and the continuity property of a DEPCAG solution (see 2.1-(i)). Indeed, we first deduce the variation of constants formula on the intervals of the form $[kh,(k+1)h)$ with $k\in\mathbb{N}$, then by applying induction we extend the arguments to the interval $[0,t]$ and obtain (2.1).

Using the definition of γ_h given in (1.2e) and considering that $t\in[kh,(k+1)h)$ with $k\in\mathbb{N}$, we have that the equation (1.2) can be rewritten as follows

$$
\frac{dx_i^h(t)}{dt} = -a_i(t)x_i^h(t)+\sum_{j=1}^m b_{ij}(t)f_j(y_j^h(h(k-k_{ij})))+I_i(t), \qquad (2.3a)
$$

$$\begin{aligned}
\frac{dy_i^h(t)}{dt} &= -c_i(t)y_i^h(t) + \sum_{j=1}^{m} d_{ij}(t)g_j(x_j^h(h(k-l_{ij}))) + J_i(t), && \text{(2.3b)}\\
x_i^h(\ell h) &= \phi_i(\ell h), \quad \ell = -l_1, \cdots, 0, && \text{(2.3c)}\\
y_i^h(\ell h) &= \psi_i(\ell h), \quad \ell = -l_2, \cdots, 0, && \text{(2.3d)}
\end{aligned}$$

where k_{ij} and l_{ij} are defined on (2.1c). Then, by the variation of constants formula for $t \in [kh, (k+1)h)$, we can write the solution of (2.3) as follows

$$\begin{aligned}
x_i^h(t) &= \exp\left(-\int_{kh}^{t} a_i(u)du\right) x_i^h(kh)\\
&+ \int_{kh}^{t} \exp\left(-\int_{s}^{t} a_i(u)du\right)\left\{\sum_{j=1}^{m} b_{ij}(s)f_j(y_j^h(h(k-k_{ij}))) + I_i(s)\right\} ds, \qquad \text{(2.4a)}
\end{aligned}$$

$$\begin{aligned}
y_i^h(t) &= \exp\left(-\int_{kh}^{t} c_i(u)du\right) y_i^h(kh)\\
&+ \int_{kh}^{t} \exp\left(-\int_{s}^{t} c_i(u)du\right)\left\{\sum_{j=1}^{m} d_{ij}(s)g_j(x_j^h(h(k-l_{ij}))) + J_i(s)\right\} ds. \qquad \text{(2.4b)}
\end{aligned}$$

Thus, we have obtained the representation of the solution on intervals of the form $[kh, (k+1)h)$ with $k \in \mathbb{N}$. Now, by application the continuity property of a DEPCAG solution and application of (2.4) with $t = (k+1)h$ for $k = 0, 1, \ldots, [t/h]$ we can prove that the sequence $\left(x_i^h(nh), y_i^h(nh)\right)$ satisfies (2.2).

In order to prove (2.1) for $t \in \mathbb{R}^+$, we start by noticing that

$$[0, t] = \bigcup_{\xi=0}^{n-1} [\xi h, (\xi+1)h) \bigcup [nh, t] \quad \text{with} \quad n = \left[\frac{t}{h}\right]. \qquad \text{(2.5)}$$

Then, we apply (2.4) iteratively on $\xi = 0, \ldots, n$ and we use the continuity property of the functions x_i^h and y_i^h. Indeed, for $\xi = 0$ from (2.2), we obtain

$$\begin{aligned}
x_i^h(h) &= \exp\left(-\int_{0}^{h} a_i(u)du\right) x_i^h(0)\\
&\quad + \int_{0}^{h} \exp\left(-\int_{s}^{h} a_i(u)du\right)\left\{\sum_{j=1}^{m} b_{ij}(s)f_j(y_i^h(-hk_{ij}))) + I_i(s)\right\} ds,\\
y_i^h(h) &= \exp\left(-\int_{0}^{h} c_i(u)du\right) y_i^h(0)\\
&\quad + \int_{0}^{h} \exp\left(-\int_{s}^{h} c_i(u)du\right)\left\{\sum_{j=1}^{m} d_{ij}(s)g_j(x_i^h(-hl_{ij})) + J_i(s)\right\} ds.
\end{aligned}$$

Now, for $\xi = 1$, by a new application of (2.4), we have that

$$\begin{aligned}
x_i^h(2h) &= \exp\left(-\int_h^{2h} a_i(u)du\right) x_i^h(h) \\
&\quad + \int_h^{2h} \exp\left(-\int_s^{2h} a_i(u)du\right)\left\{\sum_{j=1}^m b_{ij}(s) f_j(y_i^h(h(1-k_{ij})))) + I_i(s)\right\} ds \\
&= \exp\left(-\int_h^{2h} a_i(u)du\right)\left[\exp\left(-\int_0^h a_i(u)du\right) x_i^h(0)\right. \\
&\quad \left. + \int_0^h \exp\left(-\int_s^h a_i(u)du\right)\left\{\sum_{j=1}^m b_{ij}(s) f_j(y_i^h(-hk_{ij}))) + I_i(s)\right\} ds\right] \\
&\quad + \int_h^{2h} \exp\left(-\int_s^{2h} a_i(u)du\right)\left\{\sum_{j=1}^m b_{ij}(s) f_j(y_i^h(h(1-k_{ij})))) + I_i(s)\right\} ds \\
&= \exp\left(-\int_0^{2h} a_i(u)du\right) x_i^h(0) \\
&\quad + \int_0^h \exp\left(-\int_s^{2h} a_i(u)du\right)\left\{\sum_{j=1}^m b_{ij}(s) f_j(y_i^h(-hk_{ij})))\right\} ds \\
&\quad + \int_h^{2h} \exp\left(-\int_s^{2h} a_i(u)du\right)\left\{\sum_{j=1}^m b_{ij}(s) f_j(y_i^h(h(1-k_{ij}))))\right\} ds \\
&\quad + \int_0^{2h} \exp\left(-\int_s^{2h} a_i(u)du\right) I_i(s) ds
\end{aligned}$$

and analogously we can deduce the following relation

$$\begin{aligned}
y_i^h(2h) &= \exp\left(-\int_0^{2h} c_i(u)du\right) y_i^h(0) \\
&\quad + \int_0^h \exp\left(-\int_s^{2h} c_i(u)du\right)\left\{\sum_{j=1}^m d_{ij}(s) g_j(x_i^h(-hl_{ij})))\right\} ds \\
&\quad + \int_h^{2h} \exp\left(-\int_s^{2h} c_i(u)du\right)\left\{\sum_{j=1}^m d_{ij}(s) g_j(x_i^h(h(1-l_{ij}))))\right\} ds \\
&\quad + \int_0^{2h} \exp\left(-\int_s^{2h} c_i(u)du\right) J_i(s) ds.
\end{aligned}$$

Then, based on the these calculations and proceeding similarly for $\xi = 2, 3, \ldots, n$ and using the fact that $x_i^h(0) = \phi_i(0)$ and $y_i^h(0) = \psi_i(0)$, we can deduce that

$$\begin{aligned} x_i^h(nh) &= \exp\left(-\int_0^{nh} a_i(u)du\right)\phi_i(0) \\ &+\sum_{\xi=0}^{n-1}\int_{\xi h}^{(\xi+1)h}\exp\left(-\int_s^{nh} a_i(u)du\right) \\ &\times\left\{\sum_{j=1}^{m} b_{ij}(s)f_j(y_i^h(h(\xi-k_{ij})))) + I_i(s)\right\}ds, \end{aligned} \tag{2.6a}$$

$$\begin{aligned} y_i^h(nh) &= \exp\left(-\int_0^{nh} c_i(u)du\right)\psi_i(0) \\ &+\sum_{\xi=0}^{n-1}\int_{\xi h}^{(\xi+1)h}\exp\left(-\int_s^{nh} c_i(u)du\right) \\ &\times\left\{\sum_{j=1}^{m} d_{ij}(s)g_j(x_i^h(h(\xi-l_{ij}))) + J_i(s)\right\}ds. \end{aligned} \tag{2.6b}$$

We note that the sequence from the initial condition, the above sequence is well defined. Now, an application of (2.4) on $[nh, t]$ implies that, the solution of system (1.2) can be written as

$$\begin{aligned} x_i^h(t) &= \exp\left(-\int_{nh}^{t} a_i(u)du\right)x_i^h(nh) \\ &+\int_{nh}^{t}\exp\left(-\int_s^{t} a_i(u)du\right)\left\{\sum_{j=1}^{m} b_{ij}(s)f_j(y_i^h(h(n-k_{ij})))) + I_i(s)\right\}ds, \end{aligned} \tag{2.7a}$$

$$\begin{aligned} y_i^h(t) &= \exp\left(-\int_{nh}^{t} c_i(u)du\right)y_i^h(nh) \\ &+\int_{nh}^{t}\exp\left(-\int_s^{t} c_i(u)du\right)\left\{\sum_{j=1}^{m} d_{ij}(s)g_j(x_i^h(h(n-l_{ij}))) + J_i(s)\right\}ds, \end{aligned} \tag{2.7b}$$

or equivalently, by replacing (2.6) in (2.4), we have that the solution of system (1.2) is given by (2.1). □

Remark 2.3. If the solutions of the system of non linear delay differential equations (1.2) and (4.1) are not unique, then there is no guarantee that the approximation method will be valid.

3. MAIN RESULTS

In this section, we prove that if the linear system of system (1.1) is exponentially stable, then the solution of the semilinear system of delay differential equations can be approximated by the solutions of (1.2) for large t(see Theorem 3.2). Moreover if the system (1.1) is exponentially stable, then the distance between the solution and the approximation tends to zero as $h \to 0$ uniformly on $\mathbb{R}^+$ and the approximated solution of (1.2) converges exponentially to actual solution.

3.1. ASSUMPTIONS

In order to establish the results, for each $i = 1, \dots, m$, we need to impose following restrictions,

(H1) The functions a_i and c_i are continuous on $\mathbb{R}^+$ and are bounded below by positive constants, i.e. there exists a positive constant α such that $\alpha \le a_i(t)$ and $\alpha \le c_i(t)$ for all $t \in \mathbb{R}^+$.

(H2) The functions f_i and g_i are Lipschitz, i.e. there exist the positive constants $L_f = \max_i\{L_{f_i}\}$ and $L_g = \max_i\{L_{g_i}\}$ such that the estimates

$$|f_i(y_1) - f_i(y_2)| \le L_f |y_1 - y_2| \quad \text{and} \quad |g_i(x_1) - g_i(x_2)| \le L_g |x_1 - x_2|,$$

hold for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

(H3) The functions I_i and J_i are continuous on $\mathbb{R}^+$.

(H4) The functions φ_i and ψ_i are continuous on $[-\theta, 0]$, where

$$\theta = \max_{1 \le i,j \le m} \Big\{ \tau_{ij}, \sigma_{ij} \Big\}. \tag{3.1}$$

(H5) The functions b_{ij} and c_{ij} are continuous on $\mathbb{R}^+$.

3.2. A PRELIMINARY RESULT

Let us consider θ given by (3.1) and the following notation

$$\begin{aligned}
&\mathbf{z} = (x_1, \dots, x_m, y_1, \dots, y_m), \quad \mathbf{\Phi} := (\phi_1, \dots, \phi_m, \psi_1, \dots \psi_m), \\
&\mathbf{f} = (f_1, \dots, f_m, g_1, \dots, g_m), \quad \mathbf{I} = (I_1, \dots, I_m, J_1, \dots, J_m), \\
&\mathbb{A} = \operatorname{diag}(a_1, \dots, a_m, c_1, \dots, c_m), \quad \mathbb{B} = [b_{ij}]_{m,m}, \\
&\mathbb{D} = [d_{ij}]_{m,m}, \quad \mathbb{E} = \begin{pmatrix} \mathbb{B} & 0 \\ 0 & \mathbb{D} \end{pmatrix},
\end{aligned} \tag{3.2}$$

$$\mathbf{F}(t,\mathbf{z}(t),\mathbf{z}(t-\theta)) = \mathbb{E}(t)\mathbf{f}(\mathbf{z}(t-\theta)) + \mathbf{I}(t).$$

Then, the system (1.1) can be rewritten as

$$\mathbf{z}'(t) = -\mathbb{A}(t)\mathbf{z}(t) + \mathbf{F}(t,\mathbf{z}(t),\mathbf{z}(t-\theta)) \text{ with } \mathbf{z}(t) = \hat{\mathbf{\Phi}}(t) \text{ for } t \in [-\theta, 0]. \tag{3.3}$$

If φ is a particular solution of (3.3), then $\mathbf{w} = \mathbf{z} - \varphi$ is a solution of the translated system

$$\mathbf{w}'(t) = -\mathbb{A}(t)\mathbf{w}(t) + \overline{\mathbf{F}}(t,\mathbf{w}(t),\mathbf{w}(t-\theta)) \text{ with } \mathbf{w}(t) = \mathbf{\Phi}(t) \text{ for } t \in [-\theta, 0], \tag{3.4}$$

where $\overline{\mathbf{F}}(t,\mathbf{w}(t),\mathbf{w}(t-\theta)) = \mathbb{E}(t)\Big[\mathbf{f}((\mathbf{w}+\varphi)(t-\theta)) - \mathbf{f}(\varphi(t-\theta))\Big]$, with L as Lipschitz constant for $\mathbf{f}(.)$. Now, the fact that φ exponentially stable is equivalent to the fact that $\mathbf{w} = 0$ exponentially stable.

Consider the following norms

$$\|\mathbf{z}\|_1 = \sum_{i=1}^{m}(|x_i| + |y_i|), \ \|\mathbf{z}\|_0 = \sup_{t\in[-\theta,0]} \|\mathbf{z}(t)\|_1, \ \|\mathbb{E}\|_\infty = \sup_{t\in\mathbb{R}_+} \sum_{i,j=1}^{m} (|b_{ij}(t)| + |d_{ij}(t)|). \tag{3.5}$$

Lemma 3.1. *If the linear system*

$$\mathbf{u}'(t) = -\mathbb{A}(t)\mathbf{u}(t) \tag{3.6}$$

is α-exponentially stable, i.e. for some $\alpha > 0$ the estimate $\|\mathbf{u}(t)\| \le \|\mathbf{u}(0)\|e^{-\alpha t}$ holds for $t > 0$, and $\mathbf{f}$ is a Lipschitz function with Lipschitz constant $L_\mathbf{f}$ such that

$$\alpha_0 := \alpha - e^{\alpha\theta}\|\mathbb{E}\|_\infty L_\mathbf{f} > 0, \tag{3.7}$$

then (3.4) *is α_0-exponentially stable.*

Proof. The solution of (3.4) is given by

$$\mathbf{w}(t) \quad = \quad \mathbf{\Psi}(t,0)\mathbf{\Phi}(0) + \int_0^t \mathbf{\Psi}(t,s)\mathbb{E}(s)\Big[\mathbf{f}((\mathbf{w}+\varphi)(s-\theta)) - \mathbf{f}(\varphi(s-\theta))\Big]ds,$$

where $\mathbf{\Psi}(t,s)$ is the fundamental matrix solution corresponding to system (3.4). Now, using the fact that $\|\mathbf{\Psi}(t,s)\| \le e^{-\alpha(t-s)}$ for $t \ge s$, and taking norm both sides, we obtain

$$\begin{aligned}\|\mathbf{w}(t)\| &= \|\mathbf{\Psi}(t,0)\mathbf{\Phi}(0)\| + \int_0^t \|\mathbf{\Psi}(t,s)\|\|\mathbb{E}(s)\|\|\mathbf{f}((\mathbf{w}+\varphi)(s-\theta)) - \mathbf{f}(\varphi(s-\theta))\|ds \\ &\le \|\mathbf{\Phi}(0)\|e^{-\alpha t} + \|\mathbb{E}\|_\infty L_\mathbf{f} \int_0^t \|\mathbf{\Psi}(t,s)\|\|\mathbf{w}(s-\theta)\|ds\end{aligned}$$

$$\leq \quad \|\mathbf{\Phi}(0)\|e^{-\alpha t} + \|\mathbb{E}\|_\infty L_{\mathbf{f}} \int_0^t e^{-\alpha(t-s)}\|\mathbf{w}(s-\theta)\|ds$$

or

$$m(t) \leq m(0) + \|\mathbb{E}\|_\infty L_{\mathbf{f}} e^{\alpha\theta} \int_0^t m(s)ds \quad \text{with} \quad m(t) = \sup_{s\in[-\theta,t]} e^{\alpha s}\|\mathbf{w}(s)\|.$$

Then, applying Gronwall integral lemma, we get

$$m(t) \leq m(0)\exp\Big(\|\mathbb{E}\|_\infty L_{\mathbf{f}} e^{\alpha\theta t}\Big).$$

Thus, we have

$$\|\mathbf{w}(t)\| \leq \left(\sup_{s\in[-\theta,0]} e^{\alpha s}\|\mathbf{\Phi}(s)\|\right) e^{-\alpha_0 t}, = \|e^{\alpha(\cdot)}\mathbf{\Phi}(\cdot)\|_0 e^{-\alpha_0 t}, \quad t \geq 0,$$

and we get the result that (3.4) is α_0-exponentially stable. □

Theorem 3.2. *Assume the hypotheses (H1)-(H5) hold and consider the notation* (3.2) *and* α_0 *a positive constant defined as follows* $\alpha_0 = \alpha - \|\mathbb{E}\|_\infty L e^{\alpha\theta}$ *for* $L = \max\{L_f, L_g\}$ *(see notation on (H2)). Denote by* $\mathbf{w} = (w_{x_1}, \ldots, w_{x_m}, w_{y_1}, \ldots, w_{y_m})$ *the solution of* (3.4) *with the initial condition* $\mathbf{\Phi}$*. Then, there exist* $M_1 = M_1(\mathbf{F}, \|\mathbf{\Phi}\|_0, h)$ > 0 *and* $M_2 = M_2(\mathbf{F}, \|\mathbf{\Phi}\|_0, h) > 0$ *(independent of* s *and* α*) defined by the following relations*

$$M_1 = \|\Phi\|_0 \exp\Big(\alpha_0(\theta+h)\Big) \times \left(\left[\sup_{s\geq 2\theta} \int_{\gamma_h(s-\theta)}^{s-\theta} \sum_{i=1}^m |a_i(\mu)|d\mu + \int_{\gamma_h(s-\theta)}^{s-\theta} \sum_{i,j=1}^m L_{f_j}|b_{ij}(\mu)|\exp(\alpha_0\theta)d\mu\right]\right), \tag{3.8}$$

$$M_2 = \|\Phi\|_0 \exp\Big(\alpha_0(\theta+h)\Big) \times \left(\left[\sup_{s\geq 2\theta} \int_{\gamma_h(s-\theta)}^{s-\theta} \sum_{i=1}^m |c_i(\mu)|d\mu + \int_{\gamma_h(s-\theta)}^{s-\theta} \sum_{i,j=1}^m L_{g_j}|d_{ij}(\mu)|\exp(\alpha_0\theta)d\mu\right]\right), \tag{3.9}$$

such that the following properties

$$|w_{x_i}(s-\theta) - w_{x_i}(\gamma_h(s-\theta))| \leq M_1 e^{-\alpha_0 s}, \quad s \geq 2\theta, \tag{3.10}$$

$$|w_{y_i}(s-\theta) - w_{y_i}(\gamma_h(s-\theta))| \leq M_2 e^{-\alpha_0 s}, \quad s \geq 2\theta, \tag{3.11}$$

$$\lim_{h\to 0} M_1(\mathbf{F}, \|\Phi\|_0, h) = \lim_{h\to 0} M_2(\mathbf{F}, \|\Phi\|_0, h) = 0, \tag{3.12}$$

hold. In particular, M_1 *and* M_2 *can be taken as*

$M_1 = \|\Phi\|_0\Big(\|\mathbf{a}\|_\infty + L\exp(\alpha_0\theta)\|\mathbb{B}\|_\infty\Big)h$ *and* $M_2 = \|\Phi\|_0\Big(\|\mathbf{c}\|_\infty + L\exp(\alpha_0\theta)\|\mathbb{D}\|_\infty\Big)h$,

whenever that the functions $a_i, b_i, c_{i,j}$ *and* $d_{i,j}$ *are bounded for all* $i, j \in \{1, \ldots, m\}$.

Proof. Let us prove (3.10). Considering the notation $\boldsymbol{\varphi} = \{\varphi_{x_1}, \dots, \varphi_{x_m}, \varphi_{y_1}, \dots, \varphi_{y_m}\}$ for a particular solution of (3.3). Then, by application of (3.4) and (H2), we can calculate the following estimate

$$
\begin{aligned}
&\Big| w_{x_i}(s-\theta) - w_{x_i}(\gamma_h(s-\theta)) \Big| \\
&\quad = \left| \int_{\gamma_h(s-\theta)}^{s-\theta} w'_{x_i}(\mu) d\mu \right| \\
&\quad = \Bigg| \int_{\gamma_h(s-\theta)}^{s-\theta} \Big\{ -(a_i(\mu) w_{x_i}(\mu) \\
&\quad + \sum_{j=1}^{m} b_{ij}(\mu) \Big[f_j((w_{y_j} + \varphi_{y_j})(\mu-\theta)) - f_j(\varphi_{y_j}(\mu-\theta)) \Big] \Big\} d\mu \Bigg| \\
&\quad \le \int_{\gamma_h(s-\theta)}^{s-\theta} |a_i(\mu)||w_{x_i}(\mu)| d\mu \\
&\qquad + \sum_{j=1}^{m} \int_{\gamma_h(s-\theta)}^{s-\theta} |b_{ij}(\mu)||f_j((w_{y_j} + \varphi_{y_j})(\mu-\theta)) - f_j(\varphi_{y_j}(\mu-\theta))| d\tau \\
&\quad \le \int_{\gamma_h(s-\theta)}^{s-\theta} |a_i(\mu)||w_{x_i}(\mu)| d\mu + \sum_{j=1}^{m} \int_{\gamma_h(s-\theta)}^{s-\theta} L_{f_j} |b_{ij}(\mu)||w_{y_j}(\mu-\theta)| d\mu.
\end{aligned} \tag{3.13}
$$

Now, by assuming (H2), we note that, we can apply the Lemma 3.1 to deduce that the solutions of the system (3.4) are α_0-exponentially stable. Hence, we have that

$$
|w_{x_i}(t)| \le \|\boldsymbol{\Phi}\|_0 e^{-\alpha_0 t} \quad \text{and} \quad |w_{y_i}(t)| \le \|\boldsymbol{\Phi}\|_0 e^{-\alpha_0 t}. \tag{3.14}
$$

Then, using (3.13) and (3.14), we obtain

$$
\begin{aligned}
&|w_{x_i}(s-\theta) - w_{x_i}(\gamma_h(s-\theta))| \\
&\le \int_{\gamma_h(s-\theta)}^{s-\theta} |a_i(\mu)| \|\boldsymbol{\Phi}\|_0 e^{-\alpha_0 \mu} d\mu + \sum_{j=1}^{m} \int_{\gamma_h(s-\theta)}^{s-\theta} L_{f_j} |b_{ij}(\mu)| \|\boldsymbol{\Phi}\|_0 e^{-\alpha_0(\mu-\theta)} d\mu \\
&\le \|\boldsymbol{\Phi}\|_0 \left(\int_{\gamma_h(s-\theta)}^{s-\theta} \sum_{i=1}^{m} |a_i(\mu)| e^{-\alpha_0 \mu} d\mu + \sum_{i,j=1}^{m} \int_{\gamma_h(s-\theta)}^{s-\theta} L_{f_j} |b_{ij}(\mu)| e^{-\alpha_0 \mu} e^{\alpha_0 \theta} d\mu \right) \\
&\le \|\boldsymbol{\Phi}\|_0 e^{-\alpha_0 \gamma_h(s-\theta)} \left(\int_{\gamma_h(s-\theta)}^{s-\theta} \sum_{i=1}^{m} |a_i(\mu)| d\mu + \sum_{i,j=1}^{m} \int_{\gamma_h(s-\theta)}^{s-\theta} L_{f_j} |b_{ij}(\mu)| e^{\alpha_0 \theta} d\mu \right) \\
&\le \|\boldsymbol{\Phi}\|_0 e^{-\alpha_0 s} e^{\alpha_0(\theta+h)} \left(\int_{\gamma_h(s-\theta)}^{s-\theta} \sum_{i=1}^{m} |a_i(\mu)| d\mu + \int_{\gamma_h(s-\theta)}^{s-\theta} \sum_{i,j=1}^{m} L_{f_j} |b_{ij}(\mu)| e^{\alpha_0 \theta} d\mu \right),
\end{aligned}
$$

since $h \le \theta$ and

$$
\mu \ge \gamma_h(s-\theta) = \left[\frac{s}{h}\right] h - \left[\frac{\theta}{h}\right] h = s - \theta - \left(\left\{\frac{s}{h}\right\} - \left\{\frac{\theta}{h}\right\} \right) h \ge s - \theta - h
$$

as $s \geq 2\theta$, where $\{\cdot\}$ denotes the fractional component. Thus, defining M_1 by (3.8), we deduce (3.10).

The proof of (3.11) with M_2 given by (3.9) is similar. $\square$

Now, the proof of (3.12) follows by the hypotheses (H1) and (H5) and the fact that the measure of the interval $[\gamma_h(s-\theta), s-\theta]$ is convergent to zero as $h \to 0$.

3.3. THE MAIN APPROXIMATION RESULT

Now we prove our next result, which says that the error in approximation tends to zero exponentially as t tends to infinity.

Theorem 3.3. *Assume the hypotheses and notation of Lemma 3.2. Consider that, for $i = 1, \dots, m$, the functions $w^h_{x_i}$ and $w^h_{y_i}$ are a solution of the following system*

$$\frac{dw^h_{x_i}(t)}{dt} = -a_i(t)w^h_{x_i}(t) + \sum_{j=1}^{m} b_{ij}(t)\Big[f_j\Big((w^h_{y_j} \circ \gamma_h + \varphi_{y_j})(t-\theta)\Big) - f_j(\varphi_{y_j}(t-\theta))\Big], \tag{3.15a}$$

$$\frac{dw^h_{y_i}(t)}{dt} = -c_i(t)w^h_{y_i}(t) + \sum_{j=1}^{m} d_{ij}(t)\Big[g_j\Big((w^h_{x_j} \circ \gamma_h + \varphi_{x_j})(t-\theta)\Big) - g_j(\varphi_{x_j}(t-\theta))\Big], \tag{3.15b}$$

$$w^h_{x_i}(\ell h) = \Phi_{x_i}(\ell h), \ \ell = -k, \cdots, 0, \tag{3.15c}$$

$$w^h_{y_i}(\ell h) = \Phi_{y_i}(\ell h), \ \ell = -l, \cdots, 0. \tag{3.15d}$$

Moreover consider α_0 defined by (3.7) with $L_{\mathbf{f}} = L := \max\Big\{L_f, L_g\Big\}$, where L_f and L_g are defied in (H2), and $\mathcal{C}$ defined as follows

$$\mathcal{C}(t) = \|\mathbf{w} - \mathbf{w}^h\|_0 + \|\mathbb{E}\|_\infty L(M_1 + M_2)t + \alpha_0^{-1}\|\mathbb{E}\|_\infty \, L\, \Omega(\mathbf{w}, h, \theta)(\exp(2\alpha_0\theta) - 1),$$

with M_1 and M_2 the positive constants defined on (3.8) and (3.9), and

$$\Omega(\mathbf{w}, h, \theta) = \sup_{[-\theta,\theta]} \|\mathbf{w} - \mathbf{w} \circ \gamma_h\|_0. \tag{3.16}$$

If the zero solution of (3.4) is α-exponentially stable and the parameters satisfy

$$\alpha_0 := \alpha - \|\mathbb{E}\|_\infty L e^{\alpha\theta} > 0 \quad \text{and} \quad L\exp(\alpha_0(\theta + h_0))\|\mathbb{E}\|_\infty < \alpha_0,$$

then for each $h \in (0, h_0)$ the estimate

$$|w_{x_i}(t) - w^h_{x_i}(t)| + |w_{y_i}(t) - w^h_{y_i}(t)| \leq \mathcal{C}e^{-\rho t} \quad \text{with} \quad \rho := (\alpha_0 - e^{\alpha_0(\theta+h)}\|\mathbb{E}\|_\infty L) \in \mathbb{R}^+,$$

holds.

Proof. There exists a constant $h_0 > 0$, such that $e^{\alpha_0(h_0+\theta)}\|E\|_\infty L < \alpha_0$. Let us consider the notation

$$\mathcal{E}_{w_{x_i},h}(t) = w_{x_i}(t) - w^h_{x_i}(t), \quad \mathcal{E}_{w_{y_i},h}(t) = w_{y_i}(t) - w^h_{y_i}(t). \tag{3.17}$$

Differentiating $\mathcal{E}_{w_{x_i},h}$ with respect to t, we get

$$\begin{aligned}
&\mathcal{E}'_{w_{x_i},h}(t) = w'_{x_i}(t) - {w^h_{x_i}}'(t)\\
=&- a_i(t)(w_{x_i}(t) - w^h_{x_i}(t))\\
&+\sum_{j=1}^m b_{ij}(t)\Big[f_j((w_{y_j}+\varphi_{y_j})(t-\theta)) - f_j(\varphi_{y_j}(t-\theta))\Big]\\
&-\sum_{j=1}^m b_{ij}(t)\Big[f_j\Big((w^h_{x_j}\circ\gamma_h+\varphi_{x_j})(t-\theta)\Big) - f_j(\varphi_{y_j}(t-\theta))\Big]\\
=&- a_i(t)\mathcal{E}_{w_{x_i},h}(t)\\
&+\sum_{j=1}^m b_{ij}(t)\Big[f_j\Big((w_{y_j}+\varphi_{y_j})(t-\theta)\Big) - f_j\Big((w^h_{y_j}\circ\gamma_h+\varphi_{y_j})(t-\theta)\Big)\Big].
\end{aligned} \tag{3.18}$$

Integrating equation (3.18) from 0 to t, we obtain

$$\begin{aligned}
\mathcal{E}_{w_{x_i},h}(t) \quad = \quad & \exp\Big(-\int_0^t a_i(u)du\Big)\mathcal{E}_{w_{x_i},h}(0)\\
&+\sum_{j=1}^m\int_0^t \exp\Big(-\int_s^t a_i(u)du\Big)b_{ij}(s)\\
&\times\Big[f_j\Big((w_{y_j}+\varphi_{y_j})(s-\theta)\Big) - f_j\Big((w^h_{y_j}\circ\gamma_h+\varphi_{y_j})(s-\theta)\Big)\Big]\,ds.
\end{aligned}$$

Further taking modulo and using the bound of functions a_i and the Lipschitz continuity of functions f_j (see (H1) and (H2)), we get

$$\begin{aligned}
&|\mathcal{E}_{w_{x_i},h}(t)| \le e^{-\alpha t}|\mathcal{E}_{w_{x_i},h}(0)|\\
&+\sum_{j=1}^m\int_0^t e^{-\alpha(t-s)}|b_{ij}(s)|L_f|w_{y_j}(s-\theta) - w^h_{y_j}(\gamma_h(s-\theta))|ds\\
=&e^{-\alpha t}|\mathcal{E}_{w_{x_i},h}(0)| + \sum_{j=1}^m\int_0^t e^{-\alpha(t-s)}|b_{ij}(s)|L_f|w_{y_j}(s-\theta) - w_{y_j}(\gamma_h(s-\theta))|ds\\
&+\sum_{j=1}^m\int_0^t e^{-\alpha(t-s)}|b_{ij}(s)|L_f|w_{y_j}(\gamma_h(s-\theta)) - w^h_{y_j}(\gamma_h(s-\theta))|ds\\
\le&e^{-\alpha t}|\mathcal{E}_{w_{x_i},h}(0)| + L_f\sum_{j=1}^m\int_0^{2\theta} e^{-\alpha(t-s)}|b_{ij}(s)||w_{y_j}(s-\theta) - w_{y_j}(\gamma_h(s-\theta))|ds\\
&+L_f\sum_{j=1}^m\int_{2\theta}^t e^{-\alpha(t-s)}|b_{ij}(s)||w_{y_j}(s-\theta) - w_{y_j}(\gamma_h(s-\theta))|ds
\end{aligned}$$

$$+ L_f \sum_{j=1}^{m} \int_0^t e^{-\alpha(t-s)} |b_{ij}(s)||\mathcal{E}_{w_{y_i},h}(\gamma_h(s-\theta))|ds. \tag{3.19}$$

Similarly using the bound of functions c_i and the Lipschitz continuity of functions g_j, we obtain

$$\begin{aligned} |\mathcal{E}_{w_{y_i},h}(t)| \leq & e^{-\alpha t}|\mathcal{E}_{w_{y_i},h}(0)| \\ & + L_g \sum_{j=1}^{m} \int_0^{2\theta} e^{-\alpha(t-s)}|d_{ij}(s)||w_{x_j}(s-\theta) - w_{x_j}(\gamma_h(s-\theta))|ds \\ & + L_g \sum_{j=1}^{m} \int_{2\theta}^{t} e^{-\alpha(t-s)}|d_{ij}(s)||w_{x_j}(s-\theta) - w_{x_j}(\gamma_h(s-\theta))|ds \\ & + L_g \sum_{j=1}^{m} \int_0^{t} e^{-\alpha(t-s)}|d_{ij}(s)||\mathcal{E}_{w_{x_i},h}(\gamma_h(s-\theta))|ds. \end{aligned} \tag{3.20}$$

Now, considering $L = \max\{L_f, L_g\}$, for $s \geq 2\theta$ applying the estimates (3.10) and (3.11) on Lemma 3.2, rearranging the terms of the inequalities (3.19) and (3.20) and using the fact that $\alpha_0 < \alpha$, we obtain

$$\begin{aligned} |\mathcal{E}_{w_{x_i},h}(t)| \leq & e^{-\alpha_0 t}|\mathcal{E}_{w_{x_i},h}(0)| \\ & + e^{-\alpha_0 t} L \sum_{j=1}^{m} \int_0^{2\theta} e^{\alpha_0 s}|b_{ij}(s)||w_{y_j}(s-\theta) - w_{y_j}(\gamma_h(s-\theta))|ds \\ & + L M_1 e^{-\alpha_0 t} \sum_{j=1}^{m} \int_{2\theta}^{t} |b_{ij}(s)|ds \\ & + L e^{-\alpha_0 t} \sum_{j=1}^{m} \int_0^{t} |b_{ij}(s)| e^{\alpha_0(s-\gamma_h(s-\theta))} e^{\alpha_0 \gamma_h(s-\theta)} \left|\mathcal{E}_{y_j,h}(\gamma_h(s-\theta))\right| ds \end{aligned} \tag{3.21}$$

$$\begin{aligned} |\mathcal{E}_{w_{y_i},h}(t)| \leq & e^{-\alpha_0 t}|\mathcal{E}_{w_{y_i},h}(0)| \\ & + L e^{-\alpha_0 t} \sum_{j=1}^{m} \int_0^{2\theta} e^{\alpha_0 s}|d_{ij}(s)||w_{x_j}(s-\theta) - w_{x_j}(\gamma_h(s-\theta))|ds \\ & + L M_2 e^{-\alpha_0 t} \sum_{j=1}^{m} \int_{2\theta}^{t} |d_{ij}(s)|ds \\ & + L e^{-\alpha_0 t} \sum_{j=1}^{m} \int_0^{t} |d_{ij}(s)| e^{\alpha_0(s-\gamma_h(s-\theta))} e^{\alpha_0 \gamma_h(s-\theta)} \left|\mathcal{E}_{x_j,h}(\gamma_h(s-\theta))\right| ds. \end{aligned} \tag{3.22}$$

Note that the bounds (3.21)-(3.22) are possible by the fact that and M_1 and M_2 defined by (3.8) and (3.9) are independents of s. Now, considering the notation for

$\|\mathbb{E}\|_\infty$ (see (3.2) and (3.5)), multiplying (3.21) and (3.22) by $e^{\alpha_0 t}$, summing on i, and adding the resulting inequalities, we get

$$
\begin{aligned}
e^{\alpha_0 t}\sum_{i=1}^{m}\Big(|\mathcal{E}_{w_{x_i},h}(t)|+|\mathcal{E}_{y_i,h}(t)|\Big) &\le \sum_{i=1}^{m}\Big(|\mathcal{E}_{w_{x_i},h}(0)|+|\mathcal{E}_{y_i,h}(0)|\Big)\\
&\quad + L\,\Omega(\mathbf{w},h,\theta)\|\mathbb{E}\|_\infty\alpha_0^{-1}(\exp(2\alpha_0\theta)-1)+L(M_1+M_2)\|\mathbb{E}\|_\infty\, t\\
&\quad + L\,e^{\alpha_0(\theta+h)}\|\mathbb{E}\|_\infty\int_0^t e^{\alpha_0\gamma_h(s-\theta)}\sum_{i=1}^{m}\Big(|\mathcal{E}_{w_{x_i},h}(\gamma_h(s-\theta))|+|\mathcal{E}_{y_i,h}(\gamma_h(s-\theta))|\Big)ds,
\end{aligned} \tag{3.23}
$$

where Ω is given by (3.16) and $s-\gamma_h(s-\theta)\le\theta+h$. Further, the inequality (3.23) with

$$
V(t)=\sup_{s\in[-\theta,t]}\exp(\alpha_0 s)\sum_{i=1}^{m}\Big(|\mathcal{E}_{w_{x_i},h}(s)|+|\mathcal{E}_{y_i,h}(s)|\Big)
$$

implies

$$
V(t)\le\Gamma(t)+\int_0^t Le^{\alpha_0(\theta+h)}\|\mathbb{E}\|_\infty V(s)ds, \tag{3.24}
$$

where

$$
\Gamma(t) \;=\; V(0)+\alpha_0^{-1}L\Omega(\mathbf{w},h,\theta)\|\mathbb{E}\|_\infty(e^{2\alpha_0\theta}-1)+L(M_1+M_2)\|\mathbb{E}\|_\infty t.
$$

Now, using the Gronwall inequality, in (3.24) we deduce that

$$
V(t)\le\Gamma(t)\exp\Big(\int_0^t Le^{\alpha_0(\theta+h)}\|\mathbb{E}\|_\infty ds\Big)=\Gamma(t)\exp\Big(Le^{\alpha_0(\theta+h)}\|\mathbb{E}\|_\infty t\Big), \tag{3.25}
$$

which implies the inequality (3.17). □

Since it is assumed that the zero solution of (3.4) is asymptotically stable, it implies that $x_i(\phi_i)(t)\to 0, y_i(\psi_i)(t)\to 0$ α_0-exponentially as $t\to\infty$.

Corollary 3.4. Under the hypotheses of Theorem 3.3 we have that $x_i^h(\phi_i)(t)\to 0, y_i^h(\psi_i)(t)\to 0$ exponentially as $t\to\infty$.

Theorem 3.5. *Let $\boldsymbol{\varphi}$ a particular solution $\alpha-$asymptotically stable of CNN* (1.1) *and* $\mathbf{x}$ *any other solution of CNN* (1.1)*. Assume the conditions of Theorem 3.3. Then* $\mathbf{z}^h\to\mathbf{x}-\boldsymbol{\varphi}$ *uniformly on* $\mathbb{R}^+$ *and exponentially as* $t\to\infty$ *or* $\mathbf{z}^h+\boldsymbol{\varphi}\to\mathbf{x}$ *and the error can be estimated by* (3.17) *with* $\mathbf{y}=\mathbf{x}-\boldsymbol{\varphi}$ *as* $h\to 0$.

In the constant coefficients situation, we have the following result.

Corollary 3.6. For autonomous system corresponding to system (1.1), consider the functions $v_{x_i}^h$ and $v_{y_i}^h$ are solution of following system;

$$\frac{dv_{x_i}^h(t)}{dt} = -a_i v_{x_i}^h(t) + \sum_{j=1}^{m} b_{ij}\Big[f_j\Big((v_{y_j}^h \circ \gamma_h + \varphi_{y_j})(t-\theta)\Big) - f_j(\varphi_{y_j}(t-\theta))\Big], \tag{3.26a}$$

$$\frac{dv_{y_i}^h(t)}{dt} = -c_i v_{y_i}^h(t) + \sum_{j=1}^{m} d_{ij}\Big[g_j\Big((v_{x_j}^h \circ \gamma_h + \varphi_{x_j})(t-\theta)\Big) - g_j(\varphi_{x_j}(t-\theta))\Big], \tag{3.26b}$$

$$v_{x_i}^h(\ell h) = \Phi_{x_i}(\ell h),\ \ \ell = -k, \cdots, 0, \tag{3.26c}$$

$$v_{y_i}^h(\ell h) = \Phi_{y_i}(\ell h),\ \ \ell = -l, \cdots, 0. \tag{3.26d}$$

Similar results of Theorem 3.3 hold for the system (3.26) with M_1 and M_2 given as

$$M_1 \quad = \quad \|\mathbf{\Phi}\|_0 \left[\sum_{i=1}^{m} |a_i| + \frac{e^{\alpha_0\theta}\sum_{i,j=1}^{m} L_{f_i}|b_{ij}|}{\alpha_0\theta}\right] \tag{3.27}$$

and

$$M_2 \quad = \quad \|\mathbf{\Phi}\|_0 \left[\sum_{i=1}^{m} |c_i| + \frac{e^{\alpha_0\theta}\sum_{i,j=1}^{m} L_{g_i}|d_{ij}|}{\alpha_0\theta}\right]. \tag{3.28}$$

4. NUMERICAL EXAMPLES

In this section we present two numerical examples. However, before to present the details we note that the sequences v_i^h and w_i^h in Theorem 2.2 can be rewritten. Indeed, we have that if $v_i^h(n) = x_i^h(nh)$ and $w_i^h(n) = y_i^h(nh)$, then the sequences satisfy the following difference equations

$$\begin{aligned} v_i^h(n+1) &= e^{-\int_{nh}^{(n+1)h} a_i(u)du} v_i^h(n) \\ &+ \int_{nh}^{(n+1)h} e^{-\int_{s}^{(n+1)h} a_i(u)du} \Big\{\sum_{j=1}^{m} b_{ij}(s) f_j(w_j^h(r-k_{ij}))) + I_i(s)\Big\} ds, \end{aligned} \tag{4.1a}$$

$$\begin{aligned} w_i^h(n+1) &= e^{-\int_{nh}^{(n+1)h} c_i(u)du} w_i^h(n) \\ &+ \int_{nh}^{(n+1)h} e^{-\int_{s}^{(n+1)h} c_i(u)du} \Big\{\sum_{j=1}^{m} d_{ij}(s) g_j(v_j^h(r-l_{ij})) + J_i(s)\Big\} ds, \end{aligned} \tag{4.1b}$$

with the initial condition $v_i^h(n) = \phi_i(nh), w_i^h(n) = \psi_i(nh)$. The problem (4.1) is a discretization of the original system (1.1).

4.1. CONSTANT COEFFICIENTS

Consider the following bidirectional neural network model of the form

$$\frac{dx(t)}{dt} = -6x(t) + 0.5\tanh(y(t-0.2)) + 20, \tag{4.2a}$$
$$\frac{dy(t)}{dt} = -5y(t) + 0.25\tanh(x(t-0.1)) + 30. \tag{4.2b}$$

The corresponding discrete time analogue is given as

$$x^h(n+1) = e^{-\int_{nh}^{(n+1)h} 6du} x(n) + \int_{nh}^{(n+1)h} e^{-\int_{s}^{(n+1)h} 6du} \Big\{0.5f(y^h(n-k)) + 20\Big\} ds, \tag{4.3a}$$
$$y^h(n+1) = e^{-\int_{nh}^{(n+1)h} 5du} y(n) + \int_{nh}^{(n+1)h} e^{-\int_{s}^{(n+1)h} 5du} \Big\{0.25f(x^h(n-l)) + 30\Big\} ds, \tag{4.3b}$$

for $n \in \mathbb{Z}^+$, $h > 0$, $k = [0.2/h]$, $l = [0.1/h]$. For $h = 1$, the Table 1 displays the comparison between the actual solution and approximated solution of (4.3).

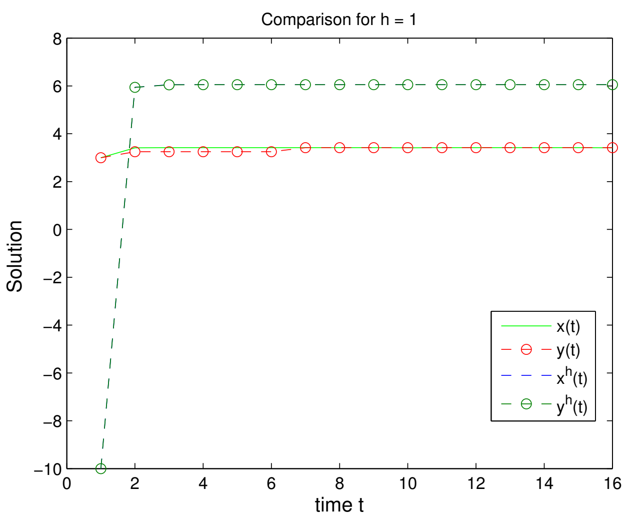

Figure 1: Actual versus approximated solution for $h = 1$ in the case of Example 1 given on Subsection 4.1, see (4.2), (4.3) and also Table 1.

The Table 2 and Figure 2 displays the solutions for $h = 0.20$. Figure 2 has been plotted taking $n = 60$ values into consideration.

Actual versus approximated solution				
t	$x(t)$	$y(t)$	$x^h(t)$	$y^h(t)$
0	3.0000	-10.0000	3.0000	-10.0000
1	3.4117	5.9436	3.249380	5.941610
2	3.4167	6.0494	3.249998	6.049024
3	3.4167	6.0499	3.249999	6.049748
4	3.4167	6.0499	3.250000	6.049849
5	3.4167	6.0499	3.250000	6.049850
6	3.4173	6.0499	3.416252	6.049850
7	3.4202	6.0499	3.416665	6.049850
8	3.4167	6.0499	3.416666	6.049850
9	3.4153	6.0499	3.416666	6.049892
10	3.4167	6.0499	3.416666	6.049892
11	3.4170	6.0499	3.416666	6.049892
12	3.4193	6.0499	3.416666	6.049892
13	3.4167	6.0499	3.416666	6.049892
14	3.4159	6.0499	3.416666	6.049892

Table 1: Actual versus approximated solution for $h = 1$ in the case of Example 1 given on Subsection 4.1, see (4.2), (4.3) and also Figure 1.

Table 3 given below illustrates the difference between the actual solution and approximated solution for different values of h, i.e. $h = 1$ and $h = 0.20$, where absolute error $= |exact - approximate|$ for different values of n and h.

From Table 2, it can be observed that as value of n is increased, the approximated solution is coming closer to the actual solution or we can say that the approximated solution is converging to actual solution for large t. Thus we can make error estimate for such case.

The equilibrium point for the system (4.2) is $(x^*, y^*) = (3.4167, 6.0499)$. Translating the system by substituting $x = 3.4167 + \bar{x}$ and $y = 6.0499 + \bar{x}$, and dropping bars for convenience, we get

$$\begin{aligned} x'(t) &= -6x(t) + 0.5\bar{f}(y(t-0.2)), \\ y'(t) &= -6y(t) + 0.5\bar{f}(x(t-0.2)), \end{aligned}$$

where $\bar{f}(x(t))) = f(\bar{x}(t) + 3.4167) - f(3.4167)$ and $\bar{f}(y(t))) = f(\bar{y}(t) + 6.0499) - f(6.0499)$. Now for translated system for $h = 0.20$, we have $L = 1, \theta = 0.2, \alpha = 5, \|E\|_\infty = 0.75$. Then $\alpha_0 = \alpha - e^{\alpha\theta}\|E\|_\infty = 2.96128863 > 0$ and thus $\rho = \alpha_0 - e^{\alpha_0(\theta+h)}\|E\|_\infty L = 0.5095 > 0$. Hence parameters satisfy the conditions of Theorem 3.2 and 3.3 for the error estimation. Furthermore, we have $\|B\|_\infty = 0.5$, $\|D\|_\infty =$

Actual versus Approximated solution				
t	$x(t)$	$y(t)$	$x^h(t)$	$y^h(t)$
0	3.0000	-10.0000	3.0000	-10.0000
1	3.4116	5.9436	3.1747	0.1454
1.2	3.4152	6.0111	3.2273	3.8777
2.0	3.4167	6.0494	3.2273	3.8777
2.4	3.4167	6.0499	3.3098	5.2507
2.8	3.4167	6.0499	3.3098	5.2507
3.2	3.4173	6.0499	3.3844	5.7559
3.6	3.4202	6.0499	3.3844	5.7559
4.2	3.4167	6.0499	3.4069	5.9417
4.6	3.4162	6.0499	3.4069	5.9417
5.4	3.4175	6.0499	3.4137	6.0101
5.8	3.4167	6.0499	3.4137	6.0101
6.0	3.4173	6.0499	3.4137	6.0101
6.2	3.4203	6.0499	3.4158	6.0352
6.4	3.4166	6.0499	3.4158	6.0352

Table 2: Actual versus approximated solution for $h = 0.20$ in the case of constant coefficients example given on Subsection 4.1, see (4.2), (4.3) and also Figure 2.

$0.25, \|a\|_\infty = 0.6, \|c\|_\infty = 0.5$. Using (3.27) and (3.28) to estimate M_1 and M_2, we have

$$M_1 = 97.8470, \quad M_2 = 74.9235.$$

Now, for some large $K_1, K_2 > 0$, estimating $\|w - w^h\|$ and $\Omega(w, h, \theta)$ by $K_1 h$ and $K_2 h$, we obtain

$$\begin{aligned} \mathcal{C}(t) &\leq K_1 h + 0.75(97.8470 + 74.9235)t + \frac{1}{2.9613}(0.75)K_2 h(e^{2(0.2)(2.96128863.-1)}) \\ &\leq K_1 h + 0.5745 K_2 h + 129.5779t, \end{aligned}$$

which give the uniform smallness, here K_1, K_2 are Lipschitz constants associated to initial conditions for $s \geq 2\theta$. Thus using Theorem 3.3, the estimate $|w_{x_i}(t) - w^h_{x_i}(t)| + |w_{y_i}(t) - w^h_{y_i}(t)| \leq \mathcal{C}e^{-\rho t}$ holds, where $\mathcal{C}$ and ρ are defined above. Since $\rho > 0$, from the estimate, it can be concluded that the approximate solution converges exponentially to the actual solution for large t.

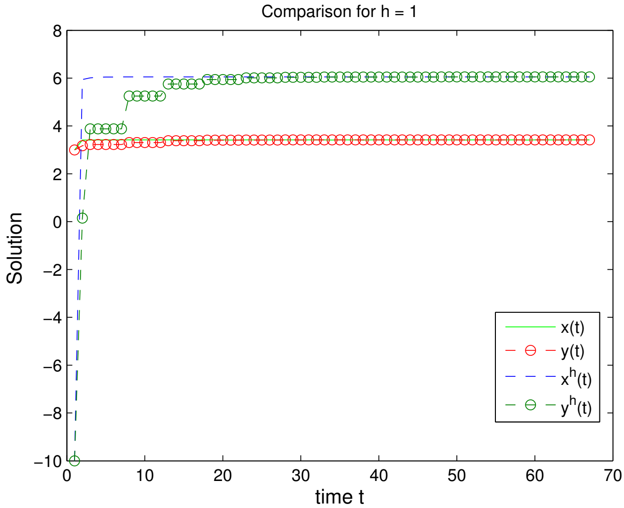

Figure 2: Actual versus approximated solution for $h = 0.20$ in the case of Example 1 given on Subsection 4.1, see (4.2), (4.3) and also Table 2.

4.2. VARIABLE COEFFICIENTS

Consider the following non-autonomous bidirectional neural network model given by

$$\frac{dx(t)}{dt} = -a(t)x(t) + b(t)\tanh(y(t-\sigma)) + I(t), \quad (4.4a)$$

$$\frac{dy(t)}{dt} = -c(t)y(t) + d(t)\tanh(x(t-\tau)) + J(t), \quad (4.4b)$$

where $a(t) = 12 + 2\sin t$, $c(t) = 8 + \cos t$, $b(t) = 6\cos t$, $d(t) = 5\sin t$, $I(t) = 5\sin t$, $J(t) = 10\cos t$, $\tau = 1$, $\sigma = 2$.

The discrete-time analogue corresponding to (4.4) is given as

$$x^h(n+1) = e^{-\int_{nh}^{(n+1)h} a(u)du} x(n) + \int_{nh}^{(n+1)h} e^{-\int_s^{(n+1)h} a(u)du} \Big\{ b(s) f(y^h(n-k)) + I(s) \Big\} ds, \quad (4.5a)$$

$$y^h(n+1) = e^{-\int_{nh}^{(n+1)h} c(u)du} y(n) + \int_{nh}^{(n+1)h} e^{-\int_s^{(n+1)h} c(u)du} \Big\{ d(s) f(x^h(n-l)) + J(s) \Big\} ds, \quad (4.5b)$$

for $n \in \mathbb{Z}^+$, $h > 0$, $k = [\sigma/h]$, $l = [\tau/h]$. The discrete-time analogue is studied for $h = 1$ with the initial conditions given by $(1, 1.5)$, $s \in [-4, 0]$. The Table 4 and Figure 3 compares the values and convergence behaviour of the actual solution and approximated solution for (4.4) and (4.5).

t	$\lvert\mathcal{E}_{w_x,h}(t)\rvert$	$\lvert\mathcal{E}_{w_y,h}(t)\rvert$
0	0	0
1	0.1623	0.0020
2	0.1667	0.0004
3	0.1667	0.0002
4	0.1667	0.0001
5	0.1667	0.0001
6	0.0010	0.0001
7	0.0035	0.0001
8	0.0000	0.0001
9	0.0014	0.0000
10	0.0000	0.0000
10	0.0000	0.0000
11	0.0003	0.0000
12	0.0026	0.0000
13	0.0000	0.0000

For $h = 1$

t	$\lvert\mathcal{E}_{w_x,h}(t)\rvert$	$\lvert\mathcal{E}_{w_y,h}(t)\rvert$
0	0	0
1.0	0.2369	5.7982
1.2	0.1879	2.1334
2.0	0.1894	2.1717
2.4	0.1069	0.7992
2.8	0.1069	0.7992
3.2	0.0323	0.2940
3.6	0.0323	0.2940
4.2	0.0098	0.1082
4.6	0.0093	0.1082
5.4	0.0038	0.0398
5.8	0.003	0.0398
6.0	0.0036	0.0398
6.2	0.0045	0.0147
6.4	0.0008	0.0147

For $h = 0.20$

Table 3: Error between actual and approximated solution for $h = 1$ and $h = 0.20$ for systems (4.2) and (4.3), see Figures 1,2 and 3 and Tables 1-2.

Table 5 indicates the difference between the actual solution and approximated solution for $h = 1$, where absolute error $= |exact - approximate|$ for different values of n.

Now consider the case for $b(t) = 0.6\cos t, d(t) = 0.5\sin t, \tau = 0.1, \sigma = 0.2$ with similar initial conditions. The Table 6 and Figure 4 compares the values and convergence behaviour of the actual solution and approximated solution for $h = 0.10$ for the systems (4.4) and (4.5).

Table 7 indicates the difference between the actual solution and approximated solution for $h = 0.10$, where absolute error $= |exact - approximate|$ for different values of n.

Furthermore, error estimation for this case is as follows. Using (3.3) and (3.4), let $w = (w_1, w_2)^T$ be the solution of the translated system given by

$$w'(t) = -\mathbb{A}(t)w(t) + \mathbb{E}(t)[\mathbf{f}((w+\phi)(t-\theta)) - \mathbf{f}(\phi(t-\theta))],$$

where $\mathbb{A}(t) = \mathrm{diag}(12 + 2\sin(t), 8 + \cos(t))$, $\mathbb{E}(t) = \mathrm{diag}(6\cos(t), 5\sin(t))$.

We have $\alpha = 7, \|E\|_\infty = 1.1, \|a\|_\infty = 14, \|B\|_\infty = 0.6, \|c\|_\infty = 9, \|D\|_\infty = 0.5, \|\phi\|_0 = 2.5$ and $L = 1$. Thus $\alpha_0 = 2.5393$ and $\rho = 0.1830 > 0$. Since $\rho > 0$,

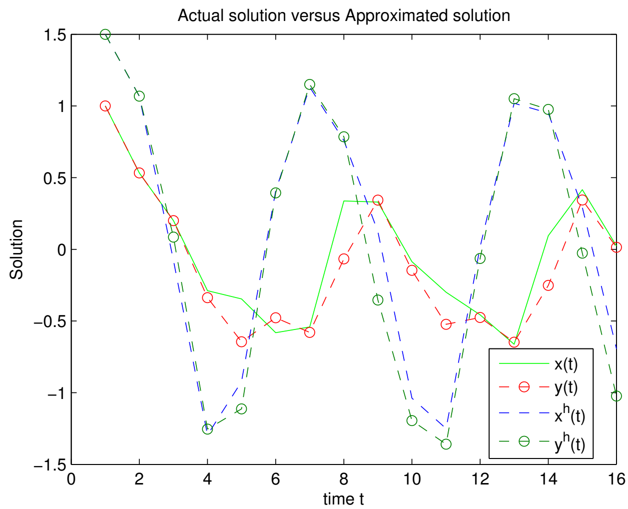

Figure 3: Actual versus approximated solution for $h = 1$ in the case of Example 2 given on Subsection 4.2, see (4.4), (4.5) and also Table 4.

Actual versus Approximated solution				
t	$x(t)$	$y(t)$	$x^h(t)$	$y^h(t)$
0	1.0000	1.5000	1.0000	1.5000
1	0.5328	1.0685	0.5324	1.0682
2	0.2003	-0.0777	0.2002	0.0851
3	-0.2883	-1.2955	-0.3369	-1.2539
4	-0.3458	-0.9236	-0.6448	-1.1131
5	-0.5815	0.4070	-0.4771	0.3938
6	-0.5415	1.1314	-0.5801	1.1503
7	0.3372	0.7585	-0.0670	0.7851
8	0.3299	0.1228	0.3436	-0.3538
9	-0.0860	-1.0380	-0.1464	-1.1948
10	-0.2993	-1.2505	-0.5233	-1.3597
11	-0.4572	0.0174	-0.4753	-0.0635
12	-0.6621	1.0183	-0.6473	1.0512
13	0.0945	0.9506	-0.2512	0.9761
14	0.4159	0.2928	0.3440	-0.0267

Table 4: Actual versus approximated solution for $h = 1$ in the case of Example 2 given on Subsection 4.2, see (4.4), (4.5) and also Figure 3.

t	$\lvert\mathcal{E}_{w_x,h}(t)\rvert$	$\lvert\mathcal{E}_{w_y,h}(t)\rvert$
0	0	0
1	0.0004	0.0003
2	0.0001	0.1628
3	0.0486	0.0416
4	0.2990	0.1895
5	0.1044	0.0132
6	0.4042	0.0266
7	0.0137	0.4766
8	0.0604	0.1568
9	0.2240	0.1092
10	0.0181	0.0809

Table 5: Error betewen actual and approximated solution for $h = 1$ for the systems (4.4) and (4.5), see Figure 3 and Table 4.

Actual versus Approximated solution				
t	$x(t)$	$y(t)$	$x^h(t)$	$y^h(t)$
0	1.0000	1.5000	1.0000	1.5000
1	0.3128	0.7410	0.3469	1.2695
1.1	0.3274	0.6569	0.1487	1.1750
1.2	0.3390	0.5653	0.1520	1.1750
1.3	0.3476	0.4665	0.1553	1.1749
1.4	0.3535	0.3612	0.1586	1.1747
1.5	0.3566	0.2500	0.1618	1.1745
1.6	0.3572	0.1337	0.1637	1.1740
1.7	0.3555	0.0133	0.1656	1.1734
1.8	0.3516	-0.1103	0.1675	1.7276
1.9	0.3456	-0.2590	0.1694	1.1720
2	0.3376	-0.3623	0.1779	1.1720

Table 6: Actual versus approximated solution for $h = 1$ in the case of Example 2 given on Subsection 4.2, see (4.4), (4.5) and also Figure 4.

we can approximate solution of our system through estimation. Approximating M_1 and M_2 (for $h = 0.1$) as

$$\begin{aligned} M_1 &= \|\phi\|_0(\|a\|_\infty + Le^{\alpha_0\theta}\|B\|_\infty)h = 3.7493, \\ M_2 &= \|\phi\|_0(\|c\|_\infty + Le^{\alpha_0\theta}\|D\|_\infty)h = 2.4577. \end{aligned}$$

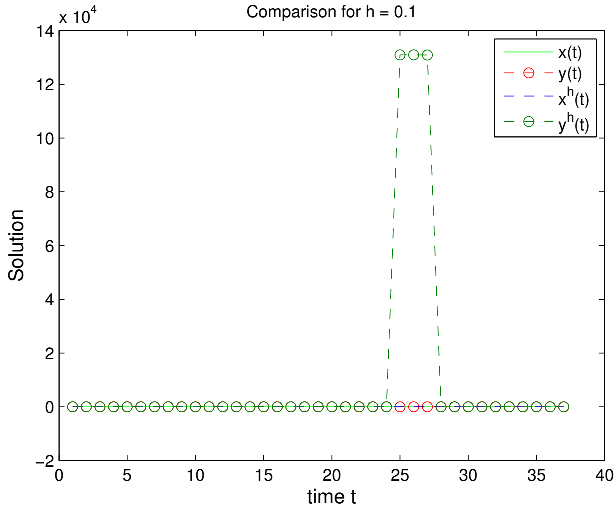

Figure 4: Actual versus approximated solution for $h = 0.1$ in the case of Example 2 given on Subsection 4.2, see (4.4), (4.5) and also Table 6.

t	$\lvert\mathcal{E}_{w_x,h}(t)\rvert$	$\lvert\mathcal{E}_{w_y,h}(t)\rvert$
0	0	0
1	0.0341	0.5285
1.1	0.1787	0.5181
1.2	0.1870	0.6097
1.3	0.1923	0.7084
1.4	0.1949	0.8135
1.5	0.1948	0.9245
1.6	0.1935	1.0403
1.7	0.1899	1.1601
1.8	0.1841	1.6173
1.9	0.1762	0.913
2	0.1597	0.8097

Table 7: Error between actual and approximated solution for $h = 0.1$ for the systems (4.4) and (4.5), see Figure 4 and Table 6.

In the similar manner as done in Example 4.1, estimating $\|w - w^h\|$ and $\Omega(w, h, \theta)$ by $K_1 h$ and $K_2 h$ $(K_1, K_2 > 0)$, we obtain

$$\begin{aligned}\mathcal{C}(t) &\leq K_1 h + 3.4139t + 0.7630K_2 h \\ &\leq 0.1K_1 + 0.0763K_2 + 3.4139t \quad \text{for} \quad h = 0.1.\end{aligned}$$

Thus from Corollary 3.6, we have $|w_{x_i}(t) - w^h_{x_i}(t)| + |w_{y_i}(t) - w^h_{y_i}(t)| \leq \mathcal{C}e^{-\rho t}$, or we can say that approximated solution converges exponentially to actual solution for large t.

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