Neural, Parallel, and Scientific Computations, **27**, No. 2 (2019), 93-104 ISSN: 1061-5369

ON A LOGISTIC DIFFERENTIAL MODEL

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ABSTRACT: In this paper we consider the logistic differential model $y'(t) = ky(t)s(t)$ with $y(t_0) = y_0$ where $s(t)$ is of the form $e^{-\theta(e^{\theta t}-1)}(1-m+me^{-\theta(e^{\theta t}-1)})$. We study upper and lower estimates for the one–sided Hausdorff approximation of the shifted Heaviside function $h_{t^*}(t)$ by means of the general solution $y(t)$. We will illustrate the advances of the solution $y(t)$ for approximating of: "data on the development of Saccharomyces culture in nutrient medium" [8]–[9] and "Telecommunication System Data" [10]–[11]. Numerical examples using *CAS Mathematica* are given.

Key Words: "Supersaturation" of the model, Heaviside function, Hausdorff distance, Upper and lower bounds

Received: April 20, 2019 ; **Accepted:** July 11, 2019 ;
Published: August 5, 2019. **doi:** 10.12732/npsc.v27i2.4
Dynamic Publishers, Inc., Acad. Publishers, Ltd. https://acadsol.eu/npsc

1. INTRODUCTION

Following the ideas given in [1]–[5], [36] we consider the following logistic differential model:

$$\begin{cases} \frac{dy(t)}{dt} = ky(t)e^{-\theta(e^{\theta t}-1)}(1-m+me^{-\theta(e^{\theta t}-1)}) \\ \\ y(t_0) = y_0 \end{cases} \tag{1}$$

where $\theta > 0$; $k > 0$ and $0 < m < 1$. We prove upper and lower estimates for the one–sided Hausdorff approximation of the shifted Heaviside function $h_{t^*}(t)$ by means of the general solution of this differential equation. We will illustrate the advances of the solution $y(t)$ for approximating and modelling of: "data on the development of Saccharomyces culture in nutrient medium" [8] (see, also [9]) and "Telecommunication System Data" [10] (see, also [11]).

2. PRELIMINARIES

Definition 1. The shifted Heaviside step function is defined by

$$h_{t^*}(t) = \begin{cases} 0, & \text{if} \quad t < t^*, \\ [0,1], & \text{if} \quad t = t^*, \\ 1, & \text{if} \quad t > t^*. \end{cases} \tag{2}$$

Definition 2. [6], [7] The Hausdorff distance (the H–distance) $\rho(f,g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,

$$\rho(f,g) = \max\{ \sup_{A \in F(f)} \inf_{B \in F(g)} ||A - B||, \sup_{B \in F(g)} \inf_{A \in F(f)} ||A - B||\}, \tag{3}$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t,x)|| = \max\{|t|,|x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $\mathbb{R}^2$ is $||A - B|| = \max(|t_A - t_B|, |x_A - x_B|)$.

3. MAIN RESULTS

The general solution of the differential equation (1) is of the following form:

$$y(t) = y_0 e^{\frac{e^{\theta} k (e^{\theta} m Ei(-2e^{\theta t}\theta) - (m-1) Ei(-e^{\theta t}\theta))}{\theta} - \frac{e^{\theta} k (e^{\theta} m Ei(-2e^{\theta t_0}\theta) - (m-1) Ei(-e^{\theta t_0}\theta))}{\theta}} \tag{4}$$

where $Ei(.)$ is the exponential integral function defined by $Ei(z) = -\int_{-z}^{\infty} \frac{e^{-t}}{t} \, dt$ (for $z > 0$), where the principal value of the integral is taken. It is important to study the characteristic - "supersaturation" of the model to the horizontal asymptote. In this Section we prove upper and lower estimates for the one–sided Hausdorff approximation of the Heaviside step–function $h_{t^*}(t)$ by means of families (4). Without loss of generality, we consider the following class of this family for:

$$t_0 = 0; \; y_0 = e^{\frac{e^{\theta} k (e^{\theta} m Ei(-2\theta) - (m-1) Ei(-\theta))}{\theta}}$$

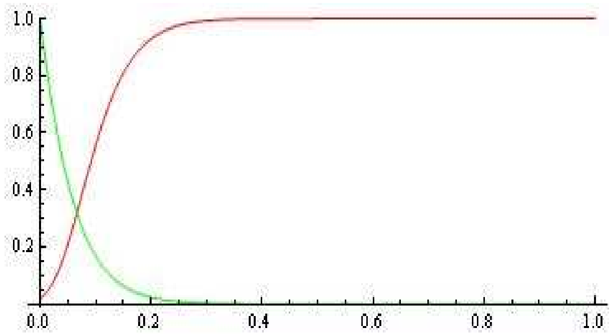

Figure 1: The functions $M(t)$–(red) and $s(t)$–(green) for $k = 70; m = 0.8; \theta = 3$.

$$M(t) = e^{\frac{e^{\theta} k \left(e^{\theta} m Ei(-2e^{\theta t}\theta) - (m-1) Ei(-e^{\theta t}\theta)\right)}{\theta}}. \tag{5}$$

The function $M(t)$ and the "input function" $s(t)$ are visualized on Fig. 1. Denoting by t^* the unique positive solution of the nonlinear equation: $M(t^*) - 0.5 = 0$. The one–sided Hausdorff distance d between the function $h_{t^*}(t)$ and the function(5) satisfies the relation $M(t^* + d) = 1 - d$.

The following theorem gives upper and lower bounds for d

Theorem 1. Let

$$\begin{aligned} &\alpha = -\tfrac{1}{2}, \\ &\beta = 1 + \tfrac{k}{2} e^{\theta(1-e^{\theta t^*})} \left(1 - m + m e^{\theta(1-e^{\theta t^*})}\right) \\ &\gamma = 2.1\beta. \end{aligned} \tag{6}$$

For the one–sided Hausdorff distance d between $h_{t^*}(t)$ and $M(t)$ (5) the following inequalities hold for the condition - $\gamma > e^{1.05}$:

$$d_l = \frac{1}{\gamma} < d < \frac{\ln \gamma}{\gamma} = d_r. \tag{7}$$

Proof. Let us examine the function:

$$F(d) = M(t^* + d) - 1 + d. \tag{8}$$

From $F'(d) > 0$ we conclude that function F is increasing.

Consider the function

$$G(d) = \alpha + \beta d. \tag{9}$$

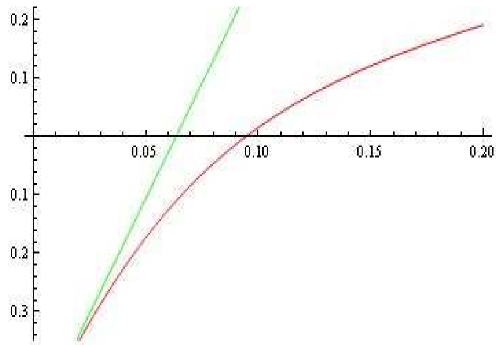

Figure 2: The functions $F(d)$ and $G(d)$ for $k = 70$; $m = 0.8$; $\theta = 3$.

From Taylor expansion we obtain $G(d) - F(d) = O(d^2)$.

Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see Fig. 2).

In addition $G'(d) > 0$.

Further, for $\gamma > e^{1.05}$ we have

$$G(d_l) < 0; \quad G(d_r) > 0.$$

This completes the proof of the theorem.

Approximation of the $h_{t^*}(t)$ by model (5) for $k = 70$; $m = 0.8$; $\theta = 3$ is visualized on Fig. 3.

4. SOME APPLICATIONS

4.1. APPROXIMATING THE "DATA ON THE DEVELOPMENT OF SACCHAROMYCES CULTURE IN NUTRIENT MEDIUM"

We will now analyze a sample of experimental data obtained by the biologist T. Carlson in 1913 about the development of Saccharomyces culture in nutrient medium (see, for example [8], [9]:

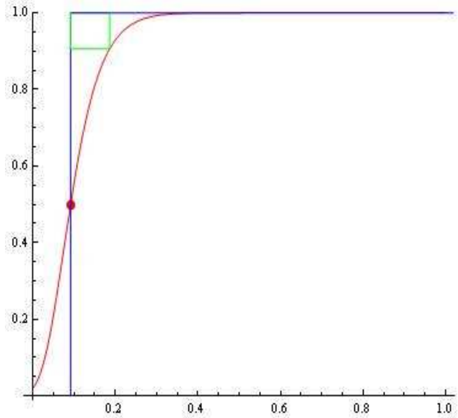

Figure 3: The model (5) for $k = 70$; $m = 0.8$; $\theta = 3$; $t^* = 0.0917241$; Hausdorff distance $d = 0.0944645$; $d_l = 0.0603377$; $d_r = 0.169416$.

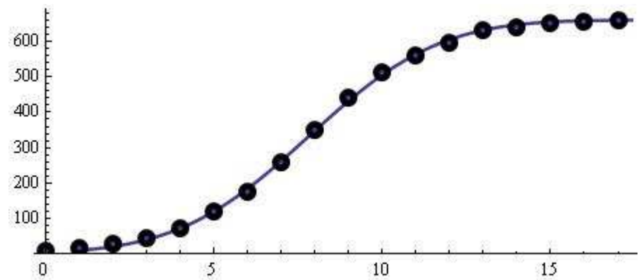

Figure 4: The fitted model $M^*(t)$.

$data\_Carlson$

$:= \{\{0, 9.6\}, \{1, 18.3\}, \{2, 29\}, \{3, 47.2\}, \{4, 71.1\}\{5, 19.1\},$

$\{6, 174.6\}, \{7, 257.3\}, \{8, 350.7\}, \{9, 441\}, \{10, 513.3\}, \{11, 559.7\},$

$\{12, 594.8\}, \{13, 629.4\}, \{14, 640.8\}, \{15, 651.1\}, \{16, 655.9\},$

$\{17, 659.6\}\}.$

After that using the model $M^*(t) = \omega M(t)$ for $\theta = 0.2$, $k = 0.778165$, $m = 0.546432$ and $\omega = 659.6$ we obtain the fitted model (see, Fig. 4).

Week index	Exposure time (cumulative system test hours)	Fault	Cumulative fault
1	356	1	1
2	712	0	1
3	1068	1	2
4	1424	1	3
5	1780	2	5
6	2136	0	5
7	2492	0	5
8	2848	3	8
9	3204	1	9
10	3560	2	11
11	3916	2	13
12	4272	2	15
13	4628	4	19
14	4984	0	19
15	5340	3	22
16	5696	0	22
17	6052	1	23
18	6408	1	24
19	6764	0	24
20	7120	0	24
21	7476	2	26

Figure 5: Table: Phase I system test data [11]

4.2. "TELECOMMUNICATION SYSTEM DATA"

Here we will use the telecommunication system data reported by Zhang in 2002 [10] (see, also [11]) to check the proposed model.

System test data consisting of two releases (Phases 1 and 2) are shown in Tables 1 and 2.

In each phase, the system records the cumulative number of faults by each week.

In both tests, automated test and humaninvolved tests are executed on multiple test beds.

The models $M^*(t)$ for "Phase I system test data" ($\theta = 0.149$, $k = 0.341002$, $m = 0.361086$, $\omega = 26$) and for "Phase II system test data" ($\theta = 0.15$, $k = 0.27003$, $m = 0.0741987$, $\omega = 43$) are visualized on Fig. 6 and Fig. 8, respectively.

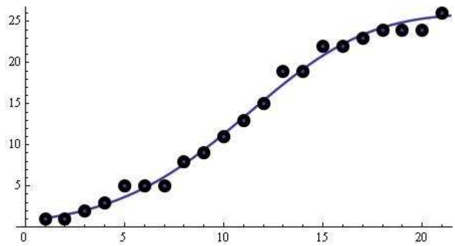

Figure 6: The fitted model $M^*(t)$.

Week index	Exposure time (cumulative system test hours)	Fault	Cumulative fault
1	416	3	3
2	832	1	4
3	1248	0	4
4	1664	3	7
5	2080	2	9
6	2496	0	9
7	2912	1	10
8	3328	3	13
9	3744	4	17
10	4160	2	19
11	4576	4	23
12	4992	2	25
13	5408	5	30
14	5824	2	32
15	6240	4	36
16	6656	1	37
17	7072	2	39
18	7488	0	39
19	7904	0	39
20	8320	3	42
21	8736	1	43

Figure 7: Table: Phase II system test data [11]

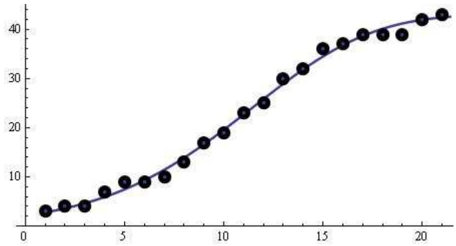

Figure 8: The fitted model $M^*(t)$.

4.3. THE NEW ACTIVATION FUNCTION BASED ON "AMENDMENTS" OF "TRANSMUTED EXPONENTIAL - EXPONENTIAL" TYPE

Definition 3. The sign function of a real number t is defined as follows:

$$sgn(t) = \begin{cases} -1, & \text{if} \quad t < 0, \\ 0, & \text{if} \quad t = 0, \\ 1, & \text{if} \quad t > 0. \end{cases} \tag{10}$$

Let $\lambda = 1$; $\theta = 0.5$.

Definition 4. The new parametric activation function is defined as follows

$$W(t) = \frac{M(t) - M(-t)}{M(t) + M(-t)}. \tag{11}$$

Approximations of the $sgn(t)$ by function $W(t)$ for $\theta = 0.5$, $m = 0.1$ and $k = 10, 20, 30$ are visualized on Fig. 9.

We will note that the study of the Hausdorff's approximation of the sign function by means of this new family can be done in a way given in [15] and we will omit it.

Similarly to the article cited above, recursively generable families of higher order activation functions can also be constructed.

5. CONCLUSION

A special choice of nutrient supply for cell growth in a continuous bioreactor is introduced.

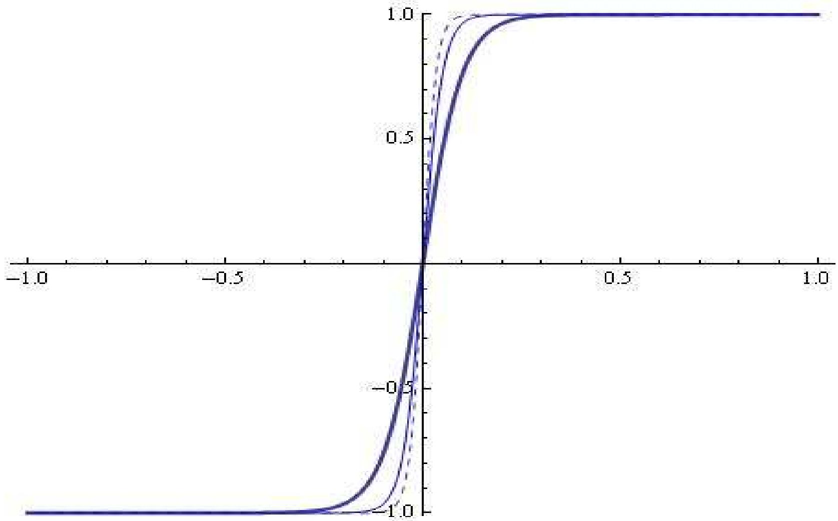

Figure 9: Approximation of the $sgn(t)$ by $W(t)$ for $k = 10$ (thick); $k = 20$ (blue) and $k = 30$ (dashed).

We prove upper and lower estimates for the one–sided Hausdorff approximation of the shifted Heaviside function $h_{t^*}(t)$ by means of the general solution of the differential equation $y'(t) = ky(t)s(t)$ with $y(t_0) = y_0$, where $s(t)$ is the correction of "transmuted exponential - exponential" type.

The general solution $y(t)$ has been applied widely in life testing experiments and debugging theory.

We propose a software module within the programming environment *CAS Mathematica* for the analysis of the considered family of functions.

The module offers the following possibilities:

- calculation of the H-distance between the h_{t^*} and the model $M(t)$;

- generation of the functions under user defined values of the parameters k, λ and θ;

- numerical solution of the differential model (1) and opportunities for comparison with other logistics models;

- software tools for animation and visualization.

For other results, see [12]–[35].

ACKNOWLEDGMENTS

This paper is supported by the National Scientific Program "Information and Communication Technologies for a Single Digital Market in Science, Education and Security (ICTinSES)", financed by the Ministry of Education and Science.

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