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INVESTIGATIONS ON A LOGISTIC MODEL WITH WEIGHTED EXPONENTIAL GOMPERTZ TYPE CORRECTION. SOME APPLICATIONS

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ABSTRACT: In this paper we will consider the possibility of approximating the input function $s(t)$ in the differential model $y'(t) = ky(t)s(t)$; $y(t_0) = y_0$ with the Gupta–Kundu [7] type correction. We prove upper and lower estimates for the one–sided Hausdorff approximation of the Heaviside step–function $h_{t^*}(t)$ by means of a new logistic family. Numerical examples, illustrating our results are given.

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Key Words: nutrient supply $s(t)$, Weighted exponential Gompertz type correction, Heaviside step function, Hausdorff distance, Upper and lower bounds

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1. INTRODUCTION

Dynamical models consisting of a systems of "reaction" differential equations are commonly used in chemistry, there the differential equations are called *reaction equations*. In chemistry reaction differential equations are induced by chemical reactions networks via reaction kinetic principles, such as *mass action kinetics* [1]–[4].

In the present work we propose a new sigmoidal class of growth functions. We prove upper and lower estimates for the one–sided Hausdorff approximation of the Heaviside step–function $h_{t^*}(t)$ by means of a new logistic family.

The proposed model can be successfully used to approximating data from tumor growth, epidemics, population dynamics, debugging theory and computer viruses propagation theory.

2. PRELIMINARIES

Definition 1. The shifted Heaviside step function is defined by

$$h_{t^*}(t) = \begin{cases} 0, & \text{if} \quad t < t^*, \\ [0,1], & \text{if} \quad t = t^*, \\ 1, & \text{if} \quad t > t^* \end{cases} \tag{1}$$

Definition 2. [5], [6] The Hausdorff distance (the H–distance) $\rho(f, g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,

$$\rho(f, g) = \max\{ \sup_{A \in F(f)} \inf_{B \in F(g)} ||A - B||, \sup_{B \in F(g)} \inf_{A \in F(f)} ||A - B||\}, \tag{2}$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t, x)|| = \max\{|t|, |x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $\mathbb{R}^2$ is $||A-B|| = \max(|t_A - t_B|, |x_A - x_B|)$.

3. MAIN RESULTS.

3.1. A NEW LOGISTIC DIFFERENTIAL MODEL

In the case of continuous bioreactor, the nutrient supply is considered as an input function $s(t)$ as follows:

$$\frac{dy(t)}{dt} = ky(t)s(t)$$

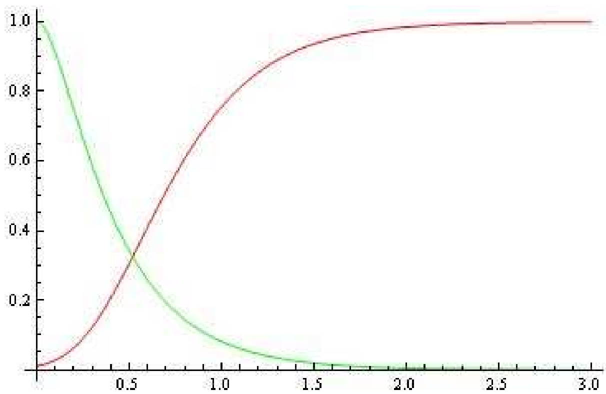

Figure 1: The functions $M(t)$–(red) and $s(t)$–(green) for $k = 10$; $\alpha = 2.1$; $\beta = 2.9$.

where s is additional specified.

In this Section we will consider the possibility of approximating the input function $s(t)$ with the Gupta–Kundu [7] type correction and study the following model:

$$\begin{cases} \frac{dy(t)}{dt} = ky(t)\dfrac{1}{\alpha}e^{-\beta t}\left(\alpha + 1 - e^{-\alpha\beta t}\right) \\ \\ y(t_0) = y_0 \end{cases} \tag{3}$$

where $\alpha,\ \beta > 0$.

The general solution of the differential equation (3) is of the following form:

$$y(t) = y_0 e^{\frac{e^{-(1+\alpha)\beta t} k\left(\frac{(-1-\alpha)e^{\alpha\beta t}}{\beta} + \frac{1}{\beta(1+\alpha)}\right)}{\alpha} - \frac{e^{-(1+\alpha)\beta t_0} k\left(\frac{(-1-\alpha)e^{\alpha\beta t_0}}{\beta} + \frac{1}{\beta(1+\alpha)}\right)}{\alpha}}. \tag{4}$$

Without loss of generality, we consider the following class of this family for:

$$t_0 = 0;\ y_0 = e^{\frac{k\left(\frac{-1-\alpha}{\beta} + \frac{1}{\beta(1+\alpha)}\right)}{\alpha}}$$

$$M(t) = e^{\frac{e^{-(1+\alpha)\beta t} k\left(\frac{(-1-\alpha)e^{\alpha\beta t}}{\beta} + \frac{1}{\beta(1+\alpha)}\right)}{\alpha}}. \tag{5}$$

The function $M(t)$ and the "input function" $s(t)$ are visualized on Fig. 1.

It is important to study the characteristic - "super saturation" of the model to the horizontal asymptote.

Denoting by t^* the unique positive solution of the nonlinear equation $M(t^*) = \frac{1}{2}$.

The one–sided Hausdorff distance d between the function $h_{t^*}(t)$ and the sigmoid - (5) satisfies the relation

$$M(t^* + d) = 1 - d. \tag{6}$$

The following theorem gives upper and lower bounds for d

Theorem. Let $y_0 < 1$ and

$$\begin{aligned} &p = -\tfrac{1}{2}, \\ &q = 1 + \tfrac{k}{2}e^{-\beta t^*}\left(1 + \tfrac{1}{\alpha} - ke^{-\beta t^*(1+\alpha)}\left(1 + \tfrac{1}{1+\alpha} + \tfrac{1}{\alpha(1+\alpha)}\right)\right) \\ &\gamma = 2.1q. \end{aligned} \tag{7}$$

For the one–sided Hausdorff distance d between $h_{t^*}(t)$ and the sigmoid (5) the following inequalities hold for the condition - $\gamma > e^{1.05}$:

$$d_l = \frac{1}{\gamma} < d < \frac{\ln \gamma}{\gamma} = d_r. \tag{8}$$

Proof. Let us examine the function:

$$F(d) = M(t^* + d) - 1 + d. \tag{9}$$

From $F'(d) > 0$ we conclude that function F is increasing.
Consider the function

$$G(d) = p + qd. \tag{10}$$

From Taylor expansion we obtain $G(d) - F(d) = O(d^2)$.
Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see Fig. 2).
In addition $G'(d) > 0$.
Further, for $\gamma > e^{1.05}$ we have

$$G(d_l) < 0; \quad G(d_r) > 0.$$

This completes the proof of the theorem.

Approximations of the $h_{t^*}(t)$ by model (5) for various k, α and β are visualized on Fig. 3–Fig. 4.

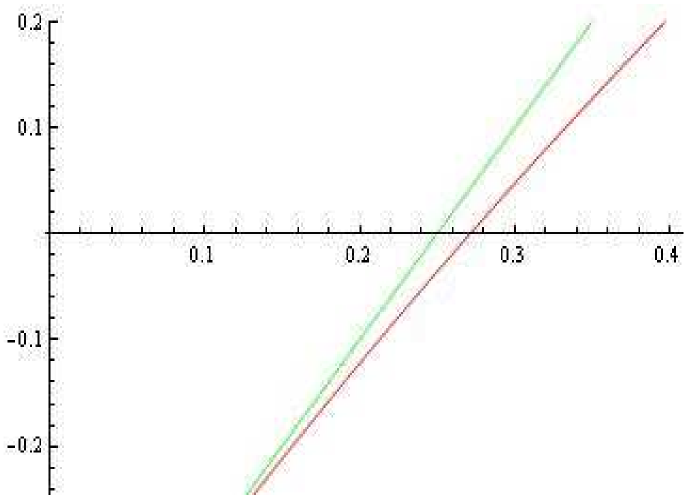

Figure 2: The functions $F(d)$ and $G(d)$ for $k = 10$; $\alpha = 2.1$; $\beta = 2.9$.

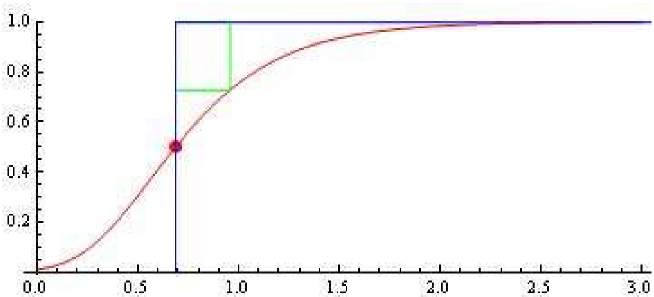

Figure 3: The model (5) for $k = 10$; $\alpha = 2.1$; $\beta = 2.9$; $t^* = 0.686988$; Hausdorff distance $d = 0.271086$; $d_l = 0.237892$; $d_r = 0.341598$.

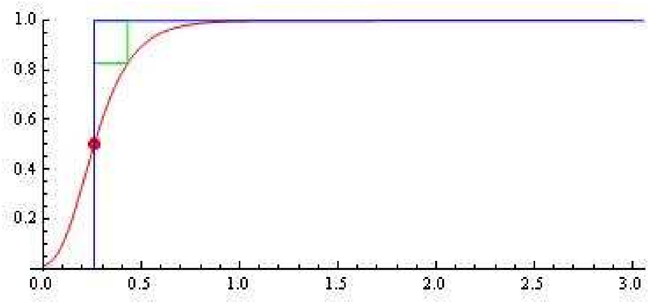

Figure 4: The model (5) for $k = 28$; $\alpha = 2.9$; $\beta = 7.6$; $t^* = 0.258765$; Hausdorff distance $d = 0.171563$; $d_l = 0.131099$; $d_r = 0.266368$.

From the graphic it can be seen that the "saturation" is faster.

3.2. APPROXIMATING THE "GROWTH DATA (MEAN HEIGHT) OF SUNFLOWER PLANTS"

We analyze experimental growth data (mean height) of sunflower plants (DSP) (see, for example [10]):

$$data\_DSP$$

$$:= \{\{14, 36.4\}, \{28, 98.1\}, \{49, 205.5\}, \{56, 228.3\}, \{70, 250.5\},$$

$$\{84, 254.5\}\}.$$

For $\alpha = 0.12$, $\beta = 0.066$, $k = 0.122257$ and $\omega = 273.67$ we obtain the fitted model $M^*(t) = \omega M(t)$ (see, Fig. 5).

4. CONCLUSION.

We will explicitly note that similar approximation and modeling results associated with the use of "input function" $S(t)$ with weighted exponential Gompertz type correction [8]–[9]:

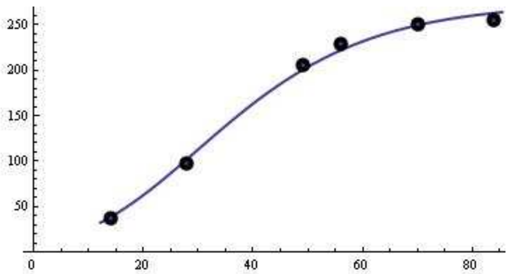

Figure 5: The fitted model $M^*(t)$.

$$S(t) = 1 - \frac{\alpha+1}{\alpha}\left(1 - e^{-\sigma(e^{\lambda t}-1)}\right)^{\beta} + \frac{1}{\alpha}\left(1 - e^{-\sigma(e^{\lambda t}-1)}\right)^{\beta(\alpha+1)}$$

can be obtained with the mathematical apparatus outlined in this chapter and here we will miss them.

We propose a software module within the programming environment *CAS Mathematica* for the analysis of the considered family of functions.

For some recent results see [11]–[40].

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