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WAVELET TRANSFORMS AND QUASI-INTERPOLATION OPERATOR

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ABSTRACT: In this paper, we survey the literature on the Fourier and wavelet transforms in both the continuous and discrete cases. A few new results have been obtained but the tone is intended to be expository. Finally, we have discussed the Feichtinger space S_0. It is dense in $L^2(R)$, much larger than the Schwartz space, and it is a Banach space. Moreover, we have proved some results using the Schoenberg's quasi-interpolation techniques in S_0.

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1. INTRODUCTION

The time-evolution of the frequencies is not reflected in the Fourier transform, at least not directly. In theory, a signal can be reconstructed from its Fourier transform, but the transform contains information about the frequencies of the signal over all times instead of showing how the frequencies very with time. The wavelet transform acts as a time and frequency localization operator. Several authors such as Daubechies et al [1], Duffin and Schaeffer [3], R. Coifman, A.J.E.M. Janssen, S. Mallat, J. Morlet, P. Tehmitchian and others have extensively developed the theory of continuous and discrete wavelet transforms.

Frazier and Jawerth [11] developed a discrete wavelet transform which allowed functions in a large class of spaces besides just $L^2(R)$ to be analyzed. Later, H. Feichtinger realized that the same could be done for the Gabor case Feichtinger [6], and then, together with K. Grochenig, unified the Gabor and wavelet transforms into a single theory, showing that a large class of transforms give rise to discrete representations of functions Feichtinger and Grochening [7], Feichtinger and Grochening [8], Feichtinger and Grochening [9].

The present paper contains three sections. Section 1, i.e., Introduction gives some definitions and auxiliary results on Fourier transform. Section 2 deals with some new

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results on continuous wavelet transform. An inversion formula has been obtained in $L^p(R), p \geq 1$ using the approximate identity. Finally, in Section 3 we have discussed the Feichtinger space as a Banach algebra under convolution and obtained some new results using the technique of quasi-interpolation. An inversion formula for short time Fourier transform also have been obtained.

1.1. **FOURIER TRANSFORMS.** The Fourier transform of a function $f \in L^1(R)$ is

$$\widehat{f}(\xi) = \int_R e^{-2\pi i \xi x} dx, \quad \xi \in R. \tag{1.1.1}$$

If $f \in L^1(R)$ and $\widehat{f}(\xi) \in L^1(R)$, then the inverse Fourier transform of $\widehat{f}(\xi)$ is defined by

$$f(x) = \int_R \widehat{f}(\xi) e^{2\pi i \xi x} d\xi\,, \tag{1.1.2}$$

for almost every $x \in R$. If f is continuous, then (1.1.1) holds for every x.

To define the Fourier transform of functions $f \in L^2(R)$ we can find functions $f_n \in L^1(R) \bigcap L^2(R)$ such that $f_n \to f \in L^2(R)$ by real analysis techniques. One way of choosing the f_n as

$$f_n(x) = \begin{cases} f\,, & -n \leq x \leq n\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

then

$$\widehat{f}(\xi) \lim_{n\to\infty} \int_{-n}^{n} f(x) e^{-2\pi i \xi x} dx\,,$$

this limit is in the Hilbert space $L^2(R)$, not a usual pointwise limit.

1.2. **DEFINITIONS AND AUXILIARY RESULTS.**

1.2.1. *Translation, modulation, and dilation.* Given a function f we define the following operators.

Translation: $T_a f(x) = f(x-a)$ for $a \in R$.

Modulation: $E_a f(x) = e^{2\pi i a x} f(x)$ for $a \in R$.

Dilation: $D_a f(x) = |a|^{-1/2} f(x/a)$ for $a \in R \setminus \{0\}$.

Each of these is a unitary operator from $L^2(R)$ onto itself. R represents the set of real numbers. A property is said to hold almost everywhere, denoted a.e. if the set of points where it fails has Lebesgue measure zero. All functions f are defined on the real line and are complex valued, unless otherwise indicated.

Definition 1.2.2. Given $1 \leq p < \infty$, we define the Lebesgue space $L^p(R) = \{f : ||f||_p = \left(\int_R |f(x)|^p\right)^{1/p} < \infty\}$. For $p = \infty$, we take $L^\infty(R) = \{f : ||f||_\infty =$

ess $\sup_{x\in R}|f(x)| < \infty\}$. The essential supremum of a real-valued function f is ess $\sup_{x\in R} f(x) = \inf\{x \in R : f(x) \leq \lambda$ a.e$\}$. It is known that for $1 \leq p \leq \infty, L^p(R)$ is a Banach space with norm $||.||_p$ and that $L^2(R)$ is a Hilbert space with inner product $< f, g >= \int_R f(x)\bar{g(x)}dx$. In view of Cauchy-Schwarz inequality it gives $| < f, g > | \leq ||f||_2||g||_2$.

Definition 1.2.3. Let $\varphi \in L^1(R)$ such that $\int_R \varphi(x)dx = 1$. Then $\varphi_\varepsilon(x) = \varepsilon^{-1}\varphi(x/\varepsilon)$ is called an approximate identity if:

(i) $\int_R \varphi_\varepsilon(x)dx = 1$,

(ii) $\sup_{\varepsilon>0} \int_R |\varphi_\varepsilon(x)|dx < \infty$,

(iii) $\lim_{\varepsilon>0} \int_{|x|>\delta} |\varphi_\varepsilon(x)|dx = 0$, $\forall\delta > 0$.

Properties (i) and (ii) are obvious due to

$$\int_R \varphi_\varepsilon(x)dx = \int_R \varepsilon^{-1}\varphi(x/\varepsilon)dx = \int_R \varphi(x/\varepsilon)d(x/\varepsilon) = 1.$$

For (iii), we have

$$\int_{|x|>\delta} \varphi_\varepsilon(x)du = \int_{|x|>\delta} (1/\varepsilon)\varphi(x/\varepsilon)du = \int_\delta^\infty (1/\varepsilon)\varphi(x/\varepsilon)du + \int_{-\infty}^{-\delta} (1/\varepsilon)\varphi(x/\varepsilon)du.$$

Putting $y = x/\varepsilon$, we get

$$\lim_{\varepsilon\to 0} \int_{\delta/\varepsilon}^\infty \varphi(y)dy = \int_{-\infty}^{-\delta/\varepsilon} \varphi(y)dy = 0.$$

Definition 1.2.4. If $\varphi \in L^1(R)$ with $\widehat{\varphi}(0) = 1$ and we define $\varphi_n(x) = n\varphi(nx)$, where $n = 1/\varepsilon$ as $n \to \infty, \varepsilon \to 0$. Then the sequences of functions $\{\varphi_n\}_{n=1}^\infty$ is an approximate identity if:

(1) $\int_R \varphi_n(x)du = 1$ for all n,

(2) $\sup_n \int_R |\varphi_n(x)|du < \infty$,

(3) $\lim_{n\to\infty} \int_{|x|>\delta} |\varphi_n(x)|du = 0$ for every $\delta > 0$.

Let us consider the class $S(R)$ of rapidly decreasing C^∞-functions on R i.e., Schwartz class such that

$$S(R) = \left\{ f : R \to R, \sup_{x\in R} \left(x^n \frac{d^m}{dx^m} f \right)(x) < \infty \right\};\ n, m \in N \bigcup \{0\}.$$

We know that if $f \in S(R)$ then $\hat{f} \in S(R)$ and $S(R) \subset L^p(R)$. If $\rho \in S(R) \implies |\rho(x)| \leq \frac{C}{1+|x|^n}$. For $1 \leq p \leq \infty$,

$$\int_R |\rho(x)|^p dx \leq \int_R \frac{C^p dx}{(1 + |x|^n)^p} < \infty$$

$$\Rightarrow \rho \in L^p(R).$$

Define a sequence $\{\rho_N\}$ such that

$$\rho_N(x) = \begin{cases} f(x)\,, & -N \le x \le N\,, \\ 0\,, & \text{otherwise,} \end{cases}$$

$\Rightarrow \exists \rho_N \in S(R), f \in L^p(R)$ such that $\int_R |\rho_N - f|^p dx \to 0$ as $N \to \infty$, it proves that $S(R)$ is dense in $L^p(R)$.

Remark 1.2.1. If $0 \le \varphi(x) \in S(R)$. Then $\varphi_n(x) = n\varphi(nx)$ is an approximate identity.

Remark 1.2.2. If $\{\varphi_n\}_{n=1}^{\infty}$ is an approximate identity and $1 \le p \le \infty$, then $\lim_{n\to\infty} ||f * \varphi_n - f||_p = 0$ for every $f \in L^p(R)$.

To prove the Remark 1.2.2, first we shall prove the following lemma.

Lemma 1.2.1. *If $f \in L^1(R)$ and $\varphi \in S(R)$ then $\varphi * f \in S(R)$.*

Proof. We have

$$\varphi * f = \int_R \varphi(y) f(x-y)dy\,,$$

or

$$\frac{d^n}{dx^n}(\varphi * f) = \int_R \varphi(y) \frac{d^n}{dx^n} f(x-y)dy\,,$$

or

$$|x|^n \frac{d^n}{dx^n}(\varphi * f) = |x|^n \int_R f(x-y) \frac{d^n}{dx^n}\varphi(y)dy\,.$$

Putting $x - y = z$, we get

$$= \int_R f(y)|x|^n \frac{d^n}{dx^n}\varphi(x-y)dy.$$

Since $|x-y| \le |x| + |y| \le \frac{3|x|}{2}$, so we get

$$= \int_{|y|>|x|/2} f(y)|x|^n \frac{d^n}{dx^n}\varphi(x-y)dy + \int_{|y|\le|x|/2} f(y)|x|^n \frac{d^n}{dx^n}\varphi(x-y)dy \to 0. \qquad \square$$

Proof of Remark 1.2.2. Consider

$$\left[\int_R |(\varphi_n * f)(x) - f(x)|^p dx\right]^{1/p}$$

$$= \left[\int_R dx \left|\int_R \varphi_n(x-y) f(y)dy - f(x)\right|^p\right]^{1/p}$$

$$= \left[\int_R dx \left|\int_R \varphi_n(y) f(x-y)dy - f(x)\right|^p\right]^{1/p}.$$

Using $f(x) = \int_R f(x)\varphi_n(y)dy$ in above we obtain

$$\left[\int_R dx \left|\int_R \varphi_n(y)(f(x-y) - f(x))dy\right|^p\right]^{1/p}$$

$$\leq \left[\int_R dx \int_{|y|>\delta} |\varphi_n(y)|^p |f(x-y)-f(x)|^p dy\right]^{1/p}$$

$$+ \left[\int_R dx \int_{|y|\leq\delta} |\varphi_n(y)|^p |f(x-y)-f(x)|^p dy\right]^{1/p}$$

$$\leq \int_{|y|>\delta} dy|\varphi_n(y)| \left[\int_R dx|f(x-y)-f(x)|^p\right]^{1/p}$$

$$+ \int_{|y|\leq\delta} dy|\varphi_n(y)| \left[\int_R dx|f(x-y)-f(x)|^p\right]^{1/p}$$

$$\leq \int_{|y|>\delta} dy|\varphi_n(y)| \,(2||f||_p)$$

$$+ \int_{|y|\leq\delta} dy|\varphi_n(y)| \sup_{|y|<\delta} \left[\int_R |f(x-y)-f(x)|^p dx\right]^{1/p}.$$

Using limit $n \to \infty$, the right hand side tends to zero since

$$\sup_{|y|<\delta} \left[\int_R |f(x-y)-f(x)|^p dx\right]^{1/p} \to 0.$$

Hence the proof is completed. □

2. MAIN RESULTS

2.1. **WAVELETS. Definition 2.1.1.** (Wavelet) Wavelets constitutes a family of functions derived from one single function $g \in L^2(R)$ and indexed by two labels, one for position and one for frequency. We define

$$g_{a,b}(x) = |a|^{1/2} g\left(\frac{x-b}{a}\right), \quad a \neq 0, b \in R.$$

g is admissible if $C_g = \int_R \frac{|\widehat{g}(\xi)|^2}{|\xi|} < \infty$.

Definition 2.1.2. (Continuous Wavelet Transform) If $g \in L^2(R)$, then the integral transformation W_g defined on $L^2(R)$ by

$$W_g[f](a,b) = < f, g_{a,b} > = \int_R f(x)\overline{g_{a,b}(x)}dx = < f, D_a T_b g >$$

for all $f \in L^2(R)$ is called a continuous wavelet transform of $f(x)$.

Now we shall prove the following result.

Theorem 2.1.1. *Let $f \in L^2(R)$. Then:*
(i) The mapping $(a,b) \to g_{a,b} : R_+ \times R \to L^2(R)$ is continuous.
(ii)The operator W_g is continuous on $R \times R$.
(iii) W_g is bounded.

Proof. (i) We have

$$g_{a,b} = [D_a(T_b)](g) = D_a g(x-b) = |a|^{-1/2} g\left(\frac{x-b}{a}\right),$$

i.e., $g_{a,b}(x)$ is the composition of two functions transformation T_b and dilation D_a. To prove the continuity of mapping $(a,b) \to g_{a,b}$, it is sufficient to prove that both functions T_b and D_a are continuous.

Let $T_b(g) = g(x-b)$. We have to show that $||T_b(g) - T_{b'}(g)|| \to 0$ if $|g_b - g_{b'}| \to 0$. We have

$$||T_b(g) - T_{b'}(g)|| = \int_R |g(x-b) - g(x-b')|^2 dx$$
$$= \int_R |g(X) - g(X+b-b')|^2 dX < \varepsilon/2.$$

Since for $n > n_0, |g_n(x) - g_{n_0}(x)| < \varepsilon/2$. It follows that T_b is continuous.

Now for dialation, we have for $a > a'$

$$||D_a(g) - D_{a'}(g)|| = \int_R ||a|^{-1/2} g(x/a) - |a'|^{-1/2} g(x/a')|^2 dx$$
$$= \int_R ||a|^{-1/2} g(X) - |a'|^{-1/2} g(aX/a')|^2 a dX$$
$$= \int_R |g(X) - g(aX/a')|^2 dX < \varepsilon/2$$

since for $n > n_0, |g(x/n) - g(x/n_0)| < \varepsilon/2$.

Hence the proof of (i) part is completed.

For (ii) and (iii) we have

$$|W_g[f](a,b)| = |< f, D_a T_b g >| \leq ||f||_2 D_a T_b g||_2 = ||f||_2 ||g||_2,$$

$W_g f$ is a bounded function of a, b. Since f and $g_{a,b}$ are continuous it follows that $W_g[f](a,b)$ is also continuous. □

The wavelet transform W_g is a generalization of the ordinary L^2-Fourier transform. W_g would therefore like to have an inversion formula for the W_g-transform analogous to that for the ordinary Fourier transform. If $f \in L^1(R)$ then it is not necessary that f also belongs to $L^1(R)$. Therefore in view point of this problem we shall use the approximate identity to define the inversion formula for $W_g(f)$.

Theorem 2.1.2. *Suppose $g \in L^p(R)$ is admissible with $C_g = 1$. Let $\{\varphi_n\}_{n=1}^{\infty}$ be an approximate identity such that $\varphi_n \in S(R)$ and $\varphi_n(x) = \varphi_n(-x)$ for all x. Then* $\lim_{n\to\infty} ||f - f_n||_p = 0$ *for all $f \in L^p(R)$, where $1 \leq p \leq \infty$,*

$$f_n(x) = \int_{-\infty}^{\infty} \int_0^{\infty} W_g f(a,b) (\varphi_n * D_a T_b(g))(x) \frac{dadb}{a^2}$$

and

$$f(x)=\int_{-\infty}^{\infty}\int_{0}^{\infty}W_gf(a,b)D_aT_b(g)(x)\frac{dadb}{a^2}.$$

Proof. We have

$$\begin{aligned}(f*\varphi_n)(x)&=\int_R f(t)\varphi_n(x-t)dt=<f,\overline{T_x\varphi_n}>=<W_gf,W_g(\overline{T_x\varphi_n})>\\&=\int_{-\infty}^{\infty}\int_{0}^{\infty}W_xf(a,b)<D_aT_b(g),\overline{T_x\varphi_n}>\frac{dadb}{a^2}\\&=\int_{-\infty}^{\infty}\int_{0}^{\infty}W_xf(a,b)(\varphi_n*D_aT_bg)(x)\frac{dadb}{a^2}.\end{aligned}$$

Since φ_n is an approximate identity so for $f\in L^p(R),\varphi_n*f\rightarrow f\in L^p(R)$. If $g_{a,b}(x)$ is the wavelet in $L^p(R)$ then $\varphi_n*g_{a,b}$ also a wavelet in $L^p(R)$. So, in view of Remark 1.2.2., we get

$$\lim_{n\rightarrow\infty}||f*\varphi_n-f||_p=0$$

for $f\in L^p(R)$. □

2.2. **FEICHTINGER SPACE S_n.** In this section we shall discuss the Feichtinger space S_n which is dense in the space L^2 of square integrable functions on R, but first we introduce the following definition.

Definition 2.2.1. Functions defined on R that vanish outside a closed (finite) interval are said to have compact support. If t stands for time and $f(t)$ has compact support, then we say f is time limited. For $f\in L^1(R)$, if f has compact support, we say that f is band limited. Clearly, any function with compact support vanishes at infinity.

2.2.1. *The Feichtinger Space.* The Feichtinger space S_0 shares several properties with the Schwartz space $S(R)$, yet S_0 is much larger, it does not depend on differentiability and it is a Banach space. Time frequency shifts and the Fourier transform are isometrics on S_0. It was introduced in Feichtinger [4], Reiter and Stegeman [14]. All its members are continuous and integrable functions. S_0 is the space of all functions on R which are represented in the time-frequency domain by integrable functions Feichtinger and Grochenig [5]. We say that the norm of a function f in S_0 is the L^1-norm of its short time Fourier transform V_gf with respect to the Gaussian window g, i.e.,

$$||f||_{S_0}=\int\int_{R^2}|V_gf(x,\omega)|dxd\omega,$$

where $g(t)=e^{-\pi t^2}$ and the short time Fourier transform is defined by

$$V_gf(x,\omega)=\int_R f(t)\overline{g(t-x)}e^{-2\pi i\omega t}dt=<f,E_\omega T_x(g)>,$$

where $x, \omega \in R$. Here the Gaussian function g is replaced by an arbitrary non-zero function from S_0. For example the triangle function, the trapezoidal function, or any Schwartz function. The sufficient condition for f to be in $S_0 = S_0(R)$ is that f, f' and f'' are in L^1, see Okoudjou [13]. Yet a function in S_0 need not be differentiable. For example, compactly supported function is in S_0 if and only if its Fourier transform is integrable. Because of the Fourier invariance of S_0, any integrable band limited function is in S_0. We have proved that if $\varphi \in L^1(R)$ then φ_ε is an approximate identity with the property that $\widehat{\varphi_\varepsilon} \in L^1(R)$ which implies that $\varphi_\varepsilon \in S_0$. We have also proved that if $\varphi \in S(R)$ then φ_ε is an approximate identity. So if $\varphi_\varepsilon \in S(R)$ then $\widehat{\varphi_\varepsilon} \in S(R)$ and $S(R)$ is dense in $L^p(R), p \geq 1$ it gives that $\varphi_\varepsilon \in L^1(R)$ and hence $\varphi_\varepsilon \in S_0$ For more on S_0, see Feichtinger and Grochenig [5], Feichtinger and Grochening [10], Frazier and Jawerth [12].

Now we shall prove the following result.

Theorem 2.2.1. *Let $g, f \in S_0$. Then:*

(i) The mapping $(x, \omega) \to E_\omega T_x(g) : R \times R \to L^1(R)$ is continuous.

(ii)The operator $V_g f$ is continuous.

Proof. We have

$$E_\omega T_x(g) = g(t-x)e^{2\pi i \omega t} .$$

$E_\omega T_x$ is the composition of two functions transformation T_x and modulation E_ω. To prove the continuity of the mapping $(x, \omega) \to E_\omega T_x$ it is sufficient to prove that both functions E_ω and T_x are continuous. We have already proved that T_x is continuous so it remains to prove the continuity of E_ω. For this we have to prove that

$$|E_\omega(g) - E_{\omega'}(g)| \to 0, \omega > \omega', \quad \text{if } |\omega - \omega'| \to 0.$$

$$= \int_R |(e^{2\pi i \omega t} - e^{2\pi i \omega' t})g(t)|^2 dt$$

$$\leq \int |(1 - e^{2\pi i (\omega - \omega')t} g(t)| dt < \epsilon/2$$

as $|\omega - \omega'| < \varepsilon/2$.

(ii) Since f and $E_\omega T_x(g)$ are continuous it follows that $V_g f =< f, E_\omega T_x g >$ is also continuous. □

Now we shall prove the inversion formula for $V_g f$.

Theorem 2.2.2. *Suppose $g \in S_0$ and φ_ε be an approximate identity such that $\varphi_\varepsilon \in S_0$ and $\varphi_\varepsilon(x) = \varphi_\varepsilon(-x)$ for all x. Then*

$$||f - f_\varepsilon||_2 \to 0 \quad as\, \varepsilon \to 0 \, ,$$

where

$$f_\varepsilon(t) = \int_R \int_R V_g f(x,\omega) < \varphi_\varepsilon * E_\omega T_x(g) > (t) dx d\omega$$

and

$$f(t) = \int_R \int_R V_g f(x,\omega) E_\omega T_x(g)(t) dx d\omega.$$

Proof. We have

$$f * \varphi_\varepsilon(x) = \int_R f(t)\varphi_\varepsilon(x-t)dt$$

$$=< f, \overline{T_x\varphi_\varepsilon} >=< V_g f, V_g(\overline{T_x\varphi_\varepsilon}) >$$

$$\int\int_{R^2} V_g f(x,\omega) < E_\omega T_x(g), \overline{T_x\varphi_\varepsilon} > dx d\omega$$

$$\int\int_{R^2} V_g f(x,\omega)(\varphi_\varepsilon * E_\omega T_x(g))(t) dx d\omega.$$

Since $f * \varphi_\varepsilon \to f$ in S_0 and $\varphi_\varepsilon * E_\omega T_x(g) \to E_\omega T_x(g) \in S_0$ and S_0 is sense in $L^2(R)$. It gives that $||f * \varphi_\varepsilon - f||_2 \to 0$ as $\varepsilon \to 0$.

The space S_0 is a Banach algebra under convolution contains approximate units obtained by dilation. □

Theorem 2.2.3. *Suppose $v \in S_0$ with $v(0) = 1$. Given $r > 0$, let $v_r(t) = v(t/r)$ for $t \in R$. Then*

$$||v_r * f - f||_{S_0} \to 0 \quad as \;\; r \to \infty,$$

for all $f \in S_0$.

Proof. Let $v_r = \widehat{\varphi_n}, r = 1/h, v(0) = 1 \Rightarrow \widehat{\varphi}(0) = 1$. We have for $g = \varphi_\varepsilon \in S(R)$

$$||v_r * f - f||_{S_0} = \int\int_{R^2} |V_{\varphi_\varepsilon}(v_r * f - f)| dx d\omega$$

$$|V_{\varphi_\varepsilon}(v_r * f - f)| = |\int_R (v_r f - f)(t)\overline{\varphi_\varepsilon(t-x)}e^{-2\pi i\omega t}dt|$$

$$= \int_R |v_r * f(t) - f(t)\overline{\varphi_\varepsilon(t-x)}|dt$$

$$= \int_R dt|\overline{\varphi_\varepsilon(t-x)} \int_R v_r(t-z)f(z)dz - f(t)|$$

$$= \int_R dt|\overline{\varphi_\varepsilon(t-x)} \int_R v_r(z)f(t-z)dz - f(t)|$$

$$= \int_R dt| \int_R v_r(z)f(t-z)dz\overline{\varphi_\varepsilon(t-z)} - \int_R \overline{\varphi_\varepsilon(t-x)}f(t)dt|$$

$$= |\int_R v_r(z)dz \int_R \overline{\varphi_\varepsilon(t-x)}f(t-z)dt - f(x)|$$

$$= |\int_R v_r(z)dz \int_R \overline{\varphi_\varepsilon(-y)}f(x-y-z)dy - f(x)|$$

$$= |\int_R v_r(z)dz f(x-z)dz - f(x)|$$

$$= |\int_R \widehat{v}_\varphi(z)dzf(x-z)dz - f(x)|$$
$$= |(\widehat{\varphi}_h * f)(x) - f(x)|.$$

Since $\varphi_h \in S(R) \Rightarrow \widehat{\varphi}_h \in S(R)$ and $\widehat{\varphi}_h * f \to f$ as $h \to 0$ or $r \to \infty$. Hence the above expression tends to zero. Hence the proof is completed. □

Schoenberg's quasi-interpolation is a general scheme in approximation theory. Now we define the quasi-interpolation operator and its Fourier transform for $f \in S_0$.

Definition 2.2.3. Given $\varphi \in S_0$ and $h > 0$ let Q_h^φ denote the quasi-interpolation operator, defined by discrete convolution for $f \in S_0$ by

$$Q_h^\varphi f(t) = f * \varphi_h(t) = \sum_{k \in Z} f(hk)\varphi(t/h - k), \quad t \in R\,.$$

Given $v \in S_0$ and $r > 0$ the operator $\widehat{Q}_r^v$ for $f \in S_0$ is defined as

$$\widehat{Q}_r^v \widehat{f}(\omega) = f * \varphi_h(t) = v(\omega/r) \sum_{k \in Z} \widehat{f}(\omega - rk).$$

Theorem 2.2.4. *Let $v \in \widehat{\varphi}$ and $r = 1/h$. Then*

$$Q_h^{\widehat{\varphi}} f = \widehat{Q}_r^v \widehat{f}, f \in S_0.$$

Proof. We have

$$\widehat{Q}_h^\varphi f(t) = \int_R \sum_{k \in Z} f(hk)\varphi(t/h - k)e^{-2\pi i \xi t}dt.$$

Let $t/h - k = x$, we get

$$= \int_R \sum_{k \in Z} f(hk)\varphi(x)e^{-2\pi i \xi(hx+hk)}hdx$$
$$= \int_R \sum_{k \in Z} f(hk)h\varphi(x)e^{-2\pi i \xi hx}e^{-2\pi i \xi hk}dx$$
$$= \sum_{k \in Z} f(hk)h\widehat{\varphi}(h\xi)e^{-2\pi i \xi hk}$$
$$= h\widehat{\varphi}(h\xi) \sum_{k \in Z} f(hk)e^{-2\pi i \xi hk}$$
$$\widehat{\varphi}(\xi/r) \sum_{k \in Z} f(k/r)e^{-2\pi i \xi k/r}\frac{1}{r}.$$

In view of the Poisson summation formula and its Fourier transform for $a > 0$ and $f \in S_0$

$$\sum_{k \in Z} \widehat{f}(\omega - ka) = \frac{1}{a}\sum_{k \in Z} f(k/a)e^{-2\pi i \omega k/a}, \omega \in R,$$

with absolute convergence of both series. We obtain

$$= v(\xi/r)\sum_{k\in Z}\widehat{f}(\xi - rk) = \widehat{Q}_r^v\widehat{f}.$$

Remark 2.2.1. The quasi-interpolation converges for functions from S_0 indeed in the norm of S_0.

Theorem 2.2.5. *(i) Suppose that $\varphi \in S_0$ satisfies $\widehat{\varphi}(k) = \delta_{k,0}$, for $k \in Z$. Then for all $f \in S_0$ we have*

$$||Q_h^{\varphi}(f) - f||_{S_0} \to 0, h \to 0.$$

(ii) Suppose that $v \in S_0$ satisfies $v(k) = \delta_{k,0}$ for $k \in Z$. Then for all $f \in S_0$, we have

$$||\widehat{Q}_r^v(f) - f||_{S_0} \to 0, r \to \infty.$$

Remark 2.2.2. In the consequence of Theorem 2.2.4 the statement (i) and (ii) in Theorem 2.2.5 are equivalent, since the Fourier transform is an isometry on S_0.

Proof of Theorem 2.2.5. To prove (i) we have

$$||Q_h^{\varphi}(f) - f||_{S_0} = \int\int_{R^2} |V_{\varphi_\epsilon}(f * \varphi_h)(t) - f(t)| dx d\omega$$

$$|V_{\varphi_\epsilon}(f * \varphi_h - f)(x,\omega)| = |\int_R (f * \varphi_h - f)(t)\overline{\varphi_\varepsilon(t-x)}e^{-2\pi i\omega t}dt|$$

$$= \int_R |\left(\sum_{k\in Z} f(hk)\varphi(t/h - k) - f\right)\overline{\varphi_\varepsilon(t-x)}dt|$$

$$= |\sum_{k\in Z}\int_R f(hk)\varphi(t/h-k)\overline{\varphi(t-x)}dt - \int_R f(t)\overline{\varphi_\varepsilon(t-x)}dt|$$

$$= |\int_R \overline{\varphi_\varepsilon(y)}\sum_{k\in Z} f(hk)\varphi\left(\frac{z-y-k}{h}\right)dy - \int_R f(z-y)\overline{\varphi_\varepsilon(y)}dy|$$

$$= |f * \varphi_h(z-y) - f(z-y)| \to 0, h \to 0.$$

Hence the proof is completed.

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