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SOURCE TERMS AND MULTIPLICITY OF SOLUTIONS IN A NONLINEAR ELLIPTIC EQUATION

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ABSTRACT: We are concerned with the multiplicity of solutions of a nonlinear elliptic equation. We investigate relations between the multiplicity of solutions and source terms in the Dirichlet problem.

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1. INTRODUCTION

Let Ω be a bounded set in $\mathbf{R^n}(n \geq 1)$ with smooth boundary $\partial\Omega$ and let A denote the elliptic operator

$$A = \sum_{1 \leq i,j \leq n} a_{i,j}(x) D_i D_j, \tag{1.1}$$

where $a_{ij} = a_{ji} \in C^\infty(\bar{\Omega})$.

We consider a semilinear elliptic boundary value problem under the Dirichlet boundary condition

$$Au + bu^+ - au^- = h(x) \quad \text{in} \quad \Omega, \tag{1.2}$$
$$u = 0 \quad \text{on} \quad \partial\Omega.$$

Here A is a second order elliptic differential operator and a mapping from $L^2(\Omega)$ into itself with compact inverse, with eigenvalues $-\lambda_i$, each repeated as often as multiplicity. We denote ϕ_n to be the eigenfuction corresponding to $\lambda_n (n = 1, 2, \cdots)$, and ϕ_1 is the eigenfuction such that $\phi_1 > 0$ in Ω and the set $\{\phi_n | \ n = 1, 2, 3 \cdots\}$ is an orthonormal set in H, where H is a Hilbert space with inner product

$$(u, v) = \int_\Omega uv, \quad u, v \in L^2(\Omega).$$

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We suppose that $\lambda_1 < a < \lambda_2 < b < \lambda_3$. Under these assumptions, we have a concern with the multiplicity of solutions of (1.2) when h is generated by two eigenfunctions ϕ_1 and ϕ_2. Then equation (1.2) is equivalent to

$$Au + bu^+ - au^- = h \quad \text{in} \quad H, \tag{1.3}$$

where $h = t_1\phi_1 + t_2\phi_2 (t_1, t_2 \in \mathbf{R})$. Hence we will study the equation (1.3). To study equation (1.3), We use the contraction mapping principle to reduce the problem from an infinite dimensional space in H to a finite dimensional one.

Let V be the two dimensional subspace of H spanned by $\{\phi_1, \phi_2\}$ and W be the orthogonal complement of V in H. Let P be an orthogonal projection H onto V. Then every element $u \in H$ is expressed as

$$u = v + w,$$

where $v = Pu, w = (I - P)u$. Hence equation (1.3) is equavelent to a system

$$Aw + (I - P)(b(v + w)^+ - a(v + w)^-) = 0 \tag{1.4}$$

$$Av + P(b(v + w)^+ - a(v + w)^-) = t_1\phi_1 + t_2\phi_2. \tag{1.5}$$

Here we look on (1.4) and (1.5) as a system of two equation in the two unknows v and w.

For fixed $v \in V$, (1.4) has a unique solution $w = \theta(v)$. Furthermore, $\theta(v)$ is Lipschitz continuous (with respect to the L^2-norm) in terms of v.

The study of the multipicity of solution of (1.3) is reduced to the study of the multipicity of solutions of an equivalent problem

$$Av + P(b(v + \theta(v))^+ - a(v + \theta(v))^-) = t_1\phi_1 + t_2\phi_2 \tag{1.6}$$

defined on the two dimensional subspace V spanned by $\{\phi_1, \phi_2\}$.

While one feels intuively that (1.6) ought to be easier to solve than (1.3), there is the disadvantage of an implicitly defined term $\theta(v)$ in the equation. However, in our case, it turns out that we know $\theta(v)$ for some special $v's$.

If $v \geq 0$ or $v \leq 0$, then $\theta(v) \equiv 0$. For example, let us take $v \geq 0$ and $\theta(v) = 0$. Then equation (1.4) reduces to

$$A0 + (I - P)(bv^+ - av^-) = 0,$$

which is satisfied because $v^+ = v, v^- = 0$ and $(I - P)v = 0$, since $v \in V$. Since the subspace V is spanned by $\{\phi_1, \phi_2\}$ and ϕ_1 is a positive eigenfuction, there exists a cone C_1 defined by

$$C_1 = \{v = c_1\phi_1 + c_2\phi_2 \;\; | \;\; c_1 \geq 0, |c_2| \leq qc_1\}$$

for some $q > 0$ so that $v \geq 0$ for all $v \in C_1$ and a cone C_3 defined by

$$C_3 = \{v = c_1\phi_1 + c_2\phi_2 \;\; | \;\; c_1 \leq 0, |c_2| \leq q|c_1|\},$$

so that $v \leq 0$ for all $v \in C_3$.

Thus, even if we do not know $\theta(v)$ for all $v \in V$, we know $\theta(v) \equiv 0$ for $v \in C_1 \cup C_3$. Now we define a map $\Pi : V \to V$ given by

$$\Pi(v) = Av + P(b(v+\theta(v))^+ - a(v+\theta(v))^-), \quad v \in V. \tag{1.7}$$

2. THE NONLINEARITY CROSSES ONE EIGENVALUE

Theorem 2.1. $\Pi(cv) = c\Pi(v)$ *for* $c \geq 0$.

Proof. Let $c \geq 0$. If v satisfies

$$A(\theta(v)) + (I-P)(b(v+\theta(v))^+ - a(v+\theta(v))^-) = 0,$$

then

$$A(c\theta(v)) + (I-P)(b(cv+c\theta(v))^+ - a(cv+c\theta(v))^-) = 0$$

and hence $\theta(cv) = c\theta(v)$. Therefore we have

$$\begin{aligned}\Pi(cv) &= A(cv) + P(b(cv+\theta(cv))^+ - a(cv+\theta(cv))^-)\\ &= cAv + P(b(cv+c\theta(v))^+ - a(cv+c\theta(v))^-)\\ &= c\Pi(v).\end{aligned}$$

We investigate the image of the cones C_1, C_3 under Π. First, we consider the image of cone C_1. If $v = c_1\phi_1 + c_2\phi_2 \geq 0$, we have

$$\begin{aligned}\Pi(v) &= Av + P(b(v+\theta(v))^+ - a(v+\theta(v))^-)\\ &= -c_1\lambda_1\phi_1 - c_2\lambda_2\phi_2 + b(c_1\phi_1 + c_2\phi_2)\\ &= c_1(b-\lambda_1)\phi_1 + c_2(b-\lambda_2)\phi_2.\end{aligned}$$

Thus the image of the rays $c_1\phi_1 \pm qc_1\phi_2 (c_1 \geq 0)$ can explicitly caculated and they are

$$c_1(b-\lambda_1)\phi_1 \pm qc_1(b-\lambda_2)\phi_2 \quad (c_1 \geq 0). \tag{2.1}$$

Therefore If $\lambda_1 < a < \lambda_2 < b < \lambda_3$, then Π maps C_1 onto the cone

$$R_1 = \left\{ d_1\phi_1 + d_2\phi_2 \;\middle|\; d_1 \geq 0, |d_2| \leq q\left(\frac{b-\lambda_2}{b-\lambda_1}\right) d_1 \right\}.$$

Second, we consider the image of the cone C_3. If

$$v = -c_1\phi_1 + c_2\phi_2 \leq 0 \quad (c_1 \geq 0, |c_2| \leq qc_1),$$

the image of the rays $-c_1\phi_1 \pm qc_1\phi_2 (c_1 \geq 0)$ are

$$c_1(\lambda_1 - a)\phi_1 \pm qc_1(\lambda_2 - a)\phi_2 \quad (c_1 \geq 0). \tag{2.2}$$

Therefore, if $\lambda_1 < a < \lambda_2 < b < \lambda_3$, then Π maps the cone C_3 onto the cone

$$R_3 = \left\{ d_1\phi_1 + d_2\phi_2 \;\middle|\; d_1 \leq 0, |d_2| \leq q\left(\frac{\lambda_2 - a}{\lambda_1 - a}\right) d_1 \right\}.$$

Now we set

$$C_2 = \{v = c_1\phi_1 + c_2\phi_2 \ \mid \ c_2 \geq 0, c_2 \geq q|c_1|\},$$
$$C_4 = \{v = c_1\phi_1 + c_2\phi_2 \ \mid \ c_2 \leq 0, |c_2| \geq q|c_1|\},$$

Then the union of C_1, C_2, and C_3, C_4 are the space V.

We remember the map $\Pi : V \to V$ given by

$$\Pi(v) = Av + P(b(v + \theta(v))^+ - a(v + \theta(v))^-), \quad v \in V.$$

Let R_i $(1 \leq i \leq 4)$ be the image of $C_i (1 \leq i \leq 4)$ under Π.

Theorem 2.2. *Let $\lambda_1 < a < \lambda_2 < b < \lambda_3$. If h belongs to R_1, then equation (1.2) has a poitive solution and no negative solution. If h belongs to R_3, then equation (1.2) has a negative solution.*

Proof. From (2.1) and (2.2), if h belongs to R_1, the equation $\Pi(v) = t_1\phi_1 + t_2\phi_2$ has a positive solution in the cone C_1, namely $\frac{t_1}{b-\lambda_1}\phi_1 + \frac{t_2}{b-\lambda_2}\phi_2$, and if h belongs to R_3, the equation $\Pi(v) = t_1\phi_1 + t_2\phi_2$ has a negative solution in C_3, namely $-\frac{t_1}{\lambda_1-a}\phi_1 - \frac{t_2}{\lambda_2-a}\phi_2$. □

Lemma 2.1 means that the images $\Pi(C_2)$ and $\Pi(C_4)$ are the cones in the plane V. Before we investigate the images $\Pi(C_2)$ and $\Pi(C_4)$, we set

$$R_2^* = \left\{ d_1\phi_1 + d_2\phi_2 \ \middle| \ d_2 \geq 0, -q^{-1} \mid \tfrac{\lambda_1-a}{\lambda_2-a} \mid d_2 \leq d_1 \leq q^{-1} \mid \tfrac{b-\lambda_1}{b-\lambda_2} \mid d_2 \right\},$$
$$R_4^* = \left\{ d_1\phi_1 + d_2\phi_2 \ \middle| \ d_2 \leq 0, -q^{-1} \mid \tfrac{\lambda_1-a}{\lambda_2-a} \mid |d_2| \leq d_1 \leq q^{-1} \mid \tfrac{b-\lambda_1}{b-\lambda_2} \mid |d_2| \right\}.$$

Then the union of R_1, R_2^*, R_3, R_4^* is the plane V.

To investigate a relation between the multiplicity of solutions and source terms in a nonlinear elliptic differential equation

$$Au + bu^+ - au^- = h \quad \text{in} \quad H,$$

we consider the restriction $\Pi|_{C_i} (1 \leq i \leq 4)$ of Π to the cone C_i. Let $\Pi_i = \Pi|_{C_i}$, i.e.,

$$\Pi_i : C_i \to V.$$

Theorem 2.3. *For $i = 1, 3$, the image of Π_i is R_i and $\Pi_i : C_i \to R_i$ is bijective.*

Proof. We consider the restriction Π_1. By (2.4), the restriction Π_1 maps C_1 onto R_1. Let l_1 be the segment defined by

$$l_1 = \left\{ \phi_1 + d_2\phi_2 \middle| \ |d_2| \leq q\left(\frac{b-\lambda_2}{b-\lambda_1}\right) \right\}.$$

Then the inverse image $\Pi_1^{-1}(l_1)$ is a segment

$$L_1 = \left\{ \frac{1}{b-\lambda_1}(\phi_1 + c_2\phi_2) \middle| \ |c_2| \leq q \right\}.$$

It follows from Theorem 2.1 that $\Pi_1 : C_1 \to R_1$ is bijective.

Similarly, $\Pi_3 : C_3 \to R_3$ is also a bijection. □

We have investigated next lemma in [5].

Lemma 2.4. *Let Q_2 be one of the sets $R_1 \cup R_4^*$ or $R_2^* \cup R_3$ such that it is contained in $\Pi(C_2)$ and let Q_4 be one of the sets $R_1 \cup R_2^*$ or $R_3 \cup R_4^*$ such that it is contained in $\Pi(C_4)$. Let $\gamma_i (i = 2, 4)$ be any simple path in Q_i with end points on ∂Q_i, where each ray (starting from the origin) in Q_i intersects only one point of γ_i. Then the inverse image $\Pi_i^{-1}(\gamma_i)$ of γ_i is a simple path in C_i with end points on ∂C_i, where any ray (starting from the origin) in C_i intersects only one point of this path.*

By Lemma 2.4, we have the following theorem.

Theorem 2.5. *For $i = 2, 4$, if we let $\Pi_i(C_i) = R_i$, then R_2 is one of the sets $R_1 \cup R_4^*$ or $R_2^* \cup R_3$, and R_4 is one of the sets $R_3 \cup R_4^*$ or $R_1 \cup R_2^*$. Furthermore the restriction Π_i maps C_i onto R_i.*

3. SOLUTIONS AND APPLICATIONS OF CRITICAL POINTS THEORY

We investigate the multipicity of solutions of a nonlinear elliptic differential equation

$$Au + bu^+ - au^- = t\phi_1 \quad \text{in} \quad H, \tag{3.1}$$

where $\lambda_1 < a < \lambda_2 < b < \lambda_3$ and $t > 0$.

Henceforth, let F denote the functional defined by

$$F(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 - G(u) + t\phi_1 u\right] dx, \tag{3.2}$$

where $G(u) = \frac{1}{2}\left(b(u^+)^2 + a(u^-)^2\right)$ and $u \in E$. Then,

$$DF(u)y = F'(u)y = \int_\Omega (\nabla u \cdot \nabla y - g(u)y + t\phi_1 y)\, dx \quad \text{for all} \quad y \in E$$

and solutions of (3.1) coincide with solutions of

$$DF(u) = 0, \tag{3.3}$$

where $g(u) = G'(u) = bu^+ - au^-$.

Therefore, we shall investigate critical points of F.

Theorem 3.1. *Let $\lambda_1 < a < \lambda_2 < b < \lambda_3, h \in V$. Let $v \in V$ be given. Then there exists a unique solution $z \in W$ of the equation*

$$Az + (I - P)(b(v + z)^+ - a(v + z)^- - h) = 0 \quad \textit{in} \quad W. \tag{3.4}$$

If $z = \theta(v)$, then θ is continuous on V and we have $DF(v + \theta(v))(w) = 0$ for all $w \in W$. In particular $\theta(v)$ satisfies a uniform Lipschitz in v with respect to the L^2-norm. If $\tilde{F} : V \to R$ is defined by $\tilde{F}(v) = F(v + \theta(v))$, then $\tilde{F}$ the has continuous Frechét derivative $D\tilde{F}$ with respect to v and

$$D\tilde{F}(v)(r) = DF(v + \theta(v))(r) \quad \textit{for all} \quad r \in V.$$

If v_0 is a critical point of $\tilde{F}$, then $v_0+\theta(v_0)$ is a solution of (3.1) and conversely every solution of (3.1) is $D\tilde{F}(v_0)=0$.

Proof. Let $\lambda_1 < a < \lambda_2 < b < \lambda_3, \alpha = \frac{1}{2}(a+b)$, and $g(u) = bu^+ - au^-$. If $g_1(u) = g(u) - \alpha u$, then equation (3.4) is equavalent to

$$z = (-A-\alpha)^{-1}(I-P)(g_1(v+w)). \tag{3.5}$$

The right hand side of (3.5) defines, for fixed $v \in V$, a Lipschitz mapping of $(I-P)H$ into itself with Lipschitz cosntant $\gamma < 1$. Therefore, by the contraction mapping principle, for given $v \in V$, there exists a unique $z \in (I-P)H$ which satisfies (3.5). If $\theta(v)$ denotes the unique $\in (I-P)H$ which solves (3.5) then θ is continuous (with respect to the L^2-norm) in V. In fact, $z_1 = \theta(v_1)$ and $z_2 = \theta(v_2)$, then we have

$$\begin{aligned} z_1 - z_2 \quad &= (-A-\alpha)^{-1}(I-P)[(g_1(v_1+z_1) - g_2(v_2+z_2)] \\ &= (-A-\alpha)^{-1}(I-P)[(g_1(v_1+z_1) - (g_1(v_1+z_2)] \\ &+(-A-\alpha)^{-1}(I-P)[(g_1(v_1+z_2) - (g_1(v_2+z_2)]\,. \end{aligned}$$

Since $|g_1(u_1) - g_1(u_2)| \le (b-\alpha)|u_1-u_2|$, it follows that if $\beta = \max\{(\lambda_m - \alpha)^{-1} |\, m \ge 3, m \in N\} = (\lambda_3 - \alpha)^{-1} = ||(-A-\delta)^{-1}(I-P)||$, and $\gamma = \beta(b-\alpha) < 1$, then

$$||z_1 - z_2|| \le \gamma\,(||v_1 - v_2|| + ||z_1 - z_2||)\,.$$

Hence

$$||z_1 - z_2|| \le k||v_1 - v_2||, \qquad k = \frac{\gamma}{1-\gamma},$$

which shows that $\theta(v)$ satisfies a uniform Lipschitz condition in v with respect to the L^2 norm. Since θ is continuous on V, $\tilde{F}$ is C^1 with respect to v and

$$D\tilde{F}(v)(r) = DF(v+\theta(v))(r) \quad \text{for all} \quad r \in V. \tag{3.6}$$

Suppose that there exists $v_0 \in V$ such that $D\tilde{F}(v_0) = 0$. From (3.3) and (3.6) it follows that $D\tilde{F}(v_0)(v) = DF(v_0 + \theta(v_0))(v) = 0$ for all $v \in V$. Since

$$\int_\Omega \nabla v \cdot \nabla w = 0 \quad \text{for all} \quad w \in W,$$

we have

$$DF(v+\theta(v))(w) = 0 \quad \text{for all} \quad w \in W.$$

Since H is direct sum of V and W, it follows that $DF(v_0+\theta(v_0)) = 0$ in H. Therefore, $u = v_0 + \theta(v_0)$ is a solution of (3.1).

Conversely our reasoning shows thet if u is a solution of (3.1) and $v = Pu$, then $D\tilde{F}(v) = 0$ in V. □

Let $\lambda_1 < a < \lambda_2 < b < \lambda_3$ and h belongs to the cone R_1. Then equation (3.1) has a positive solution u_p in the cone C_1. By Theorem 3.1, u_p can be written by $u_p = v_p + \theta(v_p)$. Since $v_p \in C_1, \theta(v_p) = 0$. Therefore we have $u_p = v_p$. Similary, if $h \in R_3$, then (3.1) has a negative solution u_n and $u_n = v_n + \theta(v_n)$, where $\theta(v_n) = 0$.

Theorem 3.2. *Let $\lambda_1 < a < \lambda_2 < b < \lambda_3$. Then we have:*

(a) Let $t = b - \lambda_1 (h = (b - \lambda_1)\phi_1)$. Then equation (3.1) has a positive solution v_p and there exists a small open neighborhood B_p of v_p in C_1 such that in B_p, v_p is a strict local point of maximum of $\tilde{F}$.

(b) $t = \lambda_1 - a (h = (\lambda_1 - a)\phi_1)$. Then equation (3.1) has a negative solution v_n and there exists a small open neighborhood B_n of v_n in C_3 such that in B_n, v_n is a saddle point of $\tilde{F}$.

Proof. (a) Let $t = b - \lambda_1 (h = (b - \lambda_1)\phi_1)$. Then equation (3.1) has a $u_p = \phi_1$ which is of the form $u_p = v_p + \theta(v_p)$ (in this case $\theta(v_p) = 0$) and $I + \theta$, where I is an identity map on V, is continuous. Since v_p is in the interior of C_1, there exists a small open neighborhood B_p of v_p in C_1. We note that $\theta(v) = 0$ in B_p. Therefore, if $v = v_p + v^* \in B_p$, then we have

$$\begin{aligned}
\tilde{F}(v) &= \tilde{F}(v_p + v^*) \\
&= \int_\Omega \left[\frac{1}{2}(|\nabla(v_p + v^*)|^2 - b((v_p + v^*)^+)^2 - a((v_p + v^*)^-)^2) + h(v_p + v^*)\right] dx \\
&= \frac{1}{2}\int_\Omega (|\nabla v^*|^2 - bv^{*2})dx + \int_\Omega [\nabla v_p \cdot \nabla v^* - bv_p v^* + hv^*]\, dx \\
&+ \int_\Omega \left[\frac{1}{2}(|\nabla v_p|^2 - bv_p^2) + hv_p\right] dx \\
&= \frac{1}{2}\int_\Omega (|\nabla v^*|^2 - bv^{*2})dx + \int_\Omega [\nabla v_p \cdot \nabla v^* - bv_p v^* + hv^*]\, dx + C,
\end{aligned}$$

where $C = \int_\Omega \left[\frac{1}{2}(|\nabla v_p|^2 - bv_p^2) + hv_p\right] dx = F(u_p) = \tilde{F}(v_p)$.

If $v \in V$ and $v = c_1\phi_1 + c_2\phi_2$, then we have

$$\begin{aligned}
||v||_0^2 = \int_\Omega |\nabla v|^2 dx = \sum_{i=1}^{2} c_i^2 \lambda_i < \lambda_2 \sum_{i=1}^{2} c_i^2 \\
= \lambda_2 \int_\Omega v^2 dx = \lambda_2 ||v||^2. \qquad (3.7)
\end{aligned}$$

Let $v^* = c_1\phi_1 + c_2\phi_2$ and let $v = v_p + v^* \in B_p$. Then

$$\int_\Omega [\nabla v_p \cdot \nabla v^* - bv_p v^* + hv^*]\, dx = 0.$$

By (3.7),

$$\tilde{F}(v) - \tilde{F}(v_p) = \frac{1}{2}\int_\Omega (|\nabla v^*|^2 - bv^{*2})dx < (\lambda_2 - b)\int_\Omega v^2 dx.$$

Since $\lambda_2 < b$, it follows that for $t = b - \lambda_1$, v_p is a strict local point of maximum for $\tilde{F}(v)$.

(b) Let $t = \lambda_1 - a (h = (\lambda_1 - a)\phi_1)$. Then equation (3.1) has a negative solution $u_n = -\phi_1$ which is of the form $u_n = v_n + \theta(v_n)$, where $\theta(v_n)$ and $-I + \theta$ is continuous in V. Since v_n is the interior, IntC_3, of C_3. We note that $\theta(v) = 0$ in B_n. Therefore, if $v = v_n + v_* \in B_n$, then we have

$$\begin{aligned}
\tilde{F}(v) &= \tilde{F}(v_n + v_*) \\
&= \int_\Omega \left[\frac{1}{2}(|\nabla(v_n + v_*)|^2 - a((v_n + v_*)^-)^2) + h(v_n + v_*)\right] dx \\
&= \frac{1}{2}\int_\Omega (|\nabla v_*|^2 - av_*^2)dx + \int_\Omega [\nabla v_n \cdot \nabla v_* - av_n v_* + hv_*]\, dx + \tilde{F}(v_n)\,.
\end{aligned}$$

Let $v_* = c_1\phi_1 + c_2\phi_2$. Then for $v = v_n + v_*$, we have

$$\int_\Omega [\nabla v_n \cdot \nabla v_* - av_n v_* + hv_*]\, dx = 0.$$

Therefore,

$$\begin{aligned}
\tilde{F}(v) - \tilde{F}(v_n) &= \frac{1}{2}\int_\Omega (|\nabla v_*|^2 - av_*^2)dx \\
&= \frac{1}{2}(c_1^2(\lambda_1 - a) + c_2^2(\lambda_2 - a)).
\end{aligned}$$

The above equation implies that v_n is a saddle point of $\tilde{F}$. □

Theorem 3.3. *Let $h \in V$ and let $\lambda_1 < a < \lambda_2 < b < \lambda_3$. For fixed t the functional $\tilde{F}$, defined on V, satisfies the Palais-Smale condition: Any sequence $\{v_n\}_1^\infty \subset V$ for which $\tilde{F}(v_n)$ is bounded and $D\tilde{F}(v_n) \to 0$ possesses a convergent subsequence.*

Proof. It is enought to show that if $\{v_n\}_1^\infty$ is a sequence in V such that $\{D\tilde{F}(v_n)\}_1^\infty$ is bounded, then the sequence of norms $\{||v_n||_0\}_1^\infty$ is bounded. Assuming the contrary, we may suppose that $\{D\tilde{F}(v_n)\}_1^\infty$ is bounded and $||v_n||_0 \to \infty$ as $n \to \infty$. Since all norms on the finite dimensional space V equivalent it follows that $||v_n|| \to \infty$ as $n \to \infty$, where $||\cdot||$ is $L^2(\Omega)$ norm. If for each $n \geq 1$ we set $z_n = \theta(v_n)$ and $u_n = v_n + \theta(v_n)$, then $||u_n|| \to \infty$ as $n \to \infty$. Therefore, since $||v_n||/||u_n||^2 \to 0$ as $n \to \infty$, $\tilde{F}(v_n)(v_n)/||v_n||^2 \to 0$ as $n \to \infty$. Since $\tilde{F}(v_n)(v) = F(u_n)(v)$ for all $v \in V$, so setting $w_n = u_n/||u_n||$. We conclude that

$$\int_\Omega \left[(\nabla w_n \cdot \nabla v_n - bw_n^+ v_n + aw_n^- v_n + t\phi_1(v_n/||u_n||))/||u_n||\right] dx \to 0 \qquad (3.8)$$

$$\text{as} \quad n \to \infty.$$

We see that

$$\int_\Omega (\nabla u_n \cdot \nabla z_n - bu_n^+ z_n + au_n^- z_n + t\phi_1 z_n)dx = 0 \quad \text{for all} \quad n. \qquad (3.9)$$

Dividing the left-hand side (3.9) by $||u_n||^2$, adding to the left-hand side of (3.8) and using $w_n = v_n/||u_n|| + z_n/||u_n||$, we see that (3.8) can be rewritten in the form

$$\int_\Omega \left[|\nabla w_n|^2 - b(w_n^+)^2 - a(w_n^-)^2 + t\phi_1 w_n/||u_n||\right] dx \to 0 \quad \text{as} \quad n \to \infty.$$

Since $||w_n|| = 1$ for all this implies that

$$||w_n||_0^2 = \int_\Omega |\nabla w_n|^2 dx$$

is bounded independently of n. Therefore, we may assume, without loss of generality, that $\{w_n\}_1^\infty$ converges weakly to $w \in W$. Since the injection from H into $L^2(\Omega)$ is compact, it follows that $\{w_n\}_1^\infty$ converges strongly in $L^2(\Omega)$ and $||w|| = 1$. If $z \in W$, then, by the proof of Theorem 3.1,

$$\int_\Omega (\nabla u_n \cdot \nabla z - bu_n^+ z + au_n^- z + t\phi_1 z)dx = 0.$$

Dividing by $||u_n||$ we have

$$\int_\Omega (\nabla w_n \cdot \nabla z - bw_n^+ z + aw_n^- z + t\phi_1 z/||u_n||)dx = 0 \tag{3.10}$$

for all n. Letting $n \to \infty$ in the last equation, we conclude that

$$\int_\Omega (\nabla w \cdot \nabla z - bw^+ z + aw^- z)dx = 0. \tag{3.11}$$

Let $v \in V$. We see that

$$D\tilde{F}(v_n)(v) = \int_\Omega (\nabla u_n \cdot \nabla v - bu_n^+ v + au_n^- v + t\phi_1 v)dx\,.$$

Dividing by $||u_n||$, using the fact $\{D\tilde{I}(v_n)\}_1^\infty$ is bounded, and letting $n \to \infty$, we can obtain

$$\int_\Omega (\nabla w \cdot \nabla v - bw^+ v + aw^- v)dx = 0. \tag{3.12}$$

Since (3.11) holds for arbitray $z \in W$ and (3.12) holds for arbitrary $v \in V$ and H is direct sum of V and W, we conclude that

$$\int_\Omega (\nabla w \cdot \nabla y - bw^+ y + aw^- y)dx = 0 \quad \text{for all} \quad y \in H\,.$$

By (3.3), w is a solution of

$$Aw + bw^+ - aw^- = 0, \quad w|_{\partial\Omega} = 0. \tag{3.13}$$

Since $||w|| = 1$, this contradicts the assumption that (3.13) has only the trivial solution (cf. [9]). Hence the sequence $\{V_n\}_1^\infty$ is bounded and the lemma is proved. □

Let $\hat{V}$ be the vector space spanned by an eigenfunction ϕ_2. Let $\hat{W}$ denote the orthogonal complement of $\hat{V}$ and let $\hat{P} : H \to \hat{V}$ denote the orthogonal projection of H onto $\hat{V}$. By the use of (3.1), (3.2) and Theorem 3.1, we have the following statements.

Given $\hat{v} \in \hat{V}$ and $t \in \mathbf{R}$, there exists a unique solution $\hat{z} = \hat{\theta}(\hat{v})$ of

$$A\hat{z} + (I - \hat{P})g(\hat{v} + \hat{z}) = t\phi_1, \hat{z}|_{\partial\Omega} = 0,$$

where $\hat{z} \in \hat{W}$.

If $\hat{z} = \hat{\theta}(\hat{v})$, then $\hat{\theta}$ is continuous on $\hat{V}$. Let $\hat{F}_0(\hat{v})$ denote the functional defined by $\hat{F}_0(\hat{v}) = F(\hat{v}+\hat{\theta}(\hat{v}))$. Then $\hat{F}_0$ has a continuous Frechét derivative $D\hat{F}_0$ with respect to $\hat{v}$ and u is a solution of equation (3.1) if and only if $u = \hat{v}+\hat{\theta}(\hat{v})$ and $D\hat{F}_0(\hat{v}) = 0$, where $\hat{v} = \hat{P}u$. By Theorem 3.3, for each fixed t the functional $\hat{F}_0$ satisfies the Palais-Smale condition.

By Theorem 3.1, the functional $\hat{F}_0(\hat{v})$ satisfy the following lemma.

Lemma 3.4. *If $t > 0$ there exists $\alpha = \alpha(t) > 0$ such that if $\hat{v} \in \hat{V}$ and $\|\hat{v}\|_0 < \alpha(t)$, then $\hat{\theta}(\hat{v}) = t\phi_1/(b-\lambda_1)$ for $t > 0$ and the point $\hat{v} = 0$ is a stric local point of maximum for $\hat{F}_0$.*

Lemma 3.5. *For $k > 0$ and $t = 0$, $\hat{F}_0(k\hat{v}) = k^2\hat{F}_0(\hat{v})$.*

Proof. Since g is positively homogeneous of degree one, it follows that if $\hat{v} \in \hat{V}, \hat{z} \in \hat{W}$ and $A\hat{z} + (I - \hat{P})g(\hat{v} + \hat{z}) = 0, \hat{z}|_{\partial\Omega} = 0$, then $A(k\hat{z}) + (I - \hat{P})g(k\hat{v} + k\hat{z}) = 0$. Therefore, $\hat{\theta}(k\hat{v}) = k\hat{\theta}(\hat{v})$. We see that $F_0(ku) = k^2F(u)$ for $u \in H$ and $k > 0$. Hence, $\hat{F}_0(k\hat{v}) = F(k\hat{v} + \hat{\theta}(k\hat{v})) = k^2F(\hat{v} + \hat{\theta}(\hat{v})) = k^2\hat{F}_0(\hat{v})$. □

Lemma 3.6. *Let $\lambda_1 < a < \lambda_2 < b < \lambda_3$. Then we have:*

(a) For $t = 0$, $\hat{F}_0(\hat{v}) > 0$ for all $\hat{v} \in \hat{V}$ with $\hat{v} \neq 0$.

(b) For $t > 0$, $\hat{F}_0(\hat{v}) \to \infty$ as $\|\hat{v}\|_0 \to \infty$.

(c) For fixed $t > 0$, $\tilde{F}(v) \to \infty$ along a ϕ_2-axis.

Proof. With Lemma 3.5 and [7], we have (a) and (b).

(c) For fixed t we see that $F(\hat{v} + \hat{\theta}(\hat{v})) = F(v + \theta(v))$. Let $\tilde{F}|_{\hat{V}}$ be the restriction of $\tilde{F}$ to the $\hat{V}$. Then $\tilde{F}|_{\hat{V}} = \hat{F}_0$. By (b), if $t > 0$, then $\tilde{F}(v) \to \infty$ as along a ϕ_2-axis. □

Lemma 3.7. *Let $\lambda_1 < a < \lambda_2 < b < \lambda_3$ and $t = b - \lambda_1$ and $q^2 \mid \lambda_2 - a \mid > \mid \lambda_1 - a \mid$. Then we have $\tilde{F}(v) \to +\infty$ as $\|v\|_0 \to \infty$ along a boundary ray of C_3.*

Proof. Let $v = v_p + v_* \in C_3$ and $v_* = c_1\phi_1 + c_2\phi_2$. Then we have

$$\tilde{F}(v) = \int_\Omega \left[\frac{1}{2}(|\nabla(v_p + v_*)|^2 - a((v_p + v^*)^-)^2) + (b - \lambda_1)\phi_1(v_p + v_*)\right] dx.$$

We note that $v_p + v_* \in \partial C_3$ if and only if $c_2 = q(c_1 + 1), c_1 \leq -1$. It can be shown easily the following holds

$$\begin{aligned}\tilde{F}(v) &= \frac{1}{2}((\lambda_1 - a)c_1^2 + q^2(\lambda_2 - a)c_1^2) \\ &+ (q^2(\lambda_2 - a) + (b - a))c_1 + \frac{1}{2}((\lambda_2 - a)q^2 + (b - a)) + C,\end{aligned}$$

where $C = \int_\Omega \left[\frac{1}{2}(|\nabla v_p|^2 - bv_p^2) + (b - \lambda_1)\phi_1 v_p\right] dx$. Hence if $v \in \partial C_3$, then we have $\tilde{F}(v) \to +\infty$ as $c_1 \to -\infty$. □

Theorem 3.8. *Let $\lambda_1 < a < \lambda_2 < b < \lambda_3$ and $t = b - \lambda_1$. Then $\tilde{F}(v)$ has a critical point in IntC_1, and at least one critical point in IntC_2, and at least one critical point in IntC_4.*

Proof. We denote that $-\tilde{F}(v) = \tilde{F}_*(v)$. By Theorem 3.2 (a), if $t = b - \lambda_1$, then there exists a small open neighborhood B_p of v_p in C_1 such that in $B_p, v_p = \phi_1$ is a strict local point of maximum for $\tilde{F}(v)$. Hence v_p is a stric local point of minimum for $\tilde{F}_*(v)$ in C_1. By Lemma 3.6 (c), $\tilde{F}_*(v) \to -\infty$ as $\|v\|_0 \to \infty$ along a ϕ_2-axis. and $\tilde{F}_* \in C^1(V, \mathbf{R})$ satisfies the Palais-Smale condition.

Since $\tilde{F}_*(v) \to -\infty$ as $\|v\|_0 \to \infty$ along a ϕ_2-axis, we can choose v_0 on ϕ_2-axis such that $\tilde{F}_*(v_0) < \tilde{F}_*(v_p)$. Let Γ be the set of all paths in V joining v_p and v_0. We write

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma} \tilde{F}_*(v).$$

The fact that in B_p, v_p is a strict local point of minimum of $\tilde{F}_*$, the fact that $\tilde{F}_*(v) \to -\infty$ as $\|v\|_0 \to \infty$ along a ϕ_2-axis, the fact $\tilde{F}_*$ satisfies the Palais-Smale condition, and the Mountain Pass Theorem imply that

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma} \tilde{F}_*(v)$$

is a critical value of $\tilde{F}_*$ (see Mountain Pass Theorem and [3, 9]). When $\lambda_1 < a < \lambda_2 < b < \lambda_3$ and $t = b - \lambda_1$, equation (3.1) has a unique positive solution v_p and no negsative solution. Hence there exists a criticl point v_3, in $\mathrm{Int}(C_2 \cup C_4)$, of $\tilde{F}_*$ such that

$$\tilde{F}_*(v_3) = c.$$

We prove that if $v_3 \in \mathrm{Int}C_4$ such that $\tilde{F}_*(v_3) = c$, then there exists another critical point $v \in \mathrm{Int}C_2$ of $\tilde{F}_*$. Suppose $v_3 \in \mathrm{Int}C_4$. Since $\tilde{F}_*(v) \to -\infty$ as $\|v\|_0 \to \infty$ along a ϕ_2-axis, we can choose v_1 on this ϕ_2-axis such that $\tilde{F}_*(v_1) < \tilde{F}_*(v_p)$. Let Γ_1 be the set of all paths in $C_1 \cup C_2 \cup C_3$ joining v_p and v_1. We write

$$c' = \inf_{\gamma \in \Gamma_1} \sup_{\gamma} \tilde{F}_*(v).$$

We note that $\tilde{F}_*(v) \to \infty$ as $\|v\|_0 \to \infty$ along a negative ϕ_1-axis or along a boundary ray, $c_2 = q(c_1 + 1)(c_1 \geq -1)$, of C_1, where $v = v_p + c_1\phi_1 + c_2\phi_2 \in \partial C_1$.

Let us fix ε, η as in Deformation Lemma with $E = V, F = \tilde{F}_*, c = c', K_{c'} = \phi$ and taking $\varepsilon < \frac{1}{2}(c' - \tilde{F}_*(v_p))$. Taking $\gamma \in \Gamma_1$ such that $\sup_\gamma \tilde{F}_* \leq c'$. From Deformation Lemma (see [3]), $\eta(1, \cdot) \circ \gamma \in \Gamma_1$ and

$$\sup \tilde{F}_*(\eta(1, \cdot) \circ \gamma) \leq c' - \varepsilon < c',$$

which is a contradiction. Therefore there exists a critical point v_4 of $\tilde{F}_*$ at leval c' such that $v_4 \in C_1 \cup C_2 \cup C_3$ and $\tilde{F}_*(v_4) = c'$. Since equation (3.1) has a unique positive solution v_p and no negative solution when $\lambda_1 < a < \lambda_2 < b < \lambda_3$ and $t = b - \lambda_1 (> 0)$, the critical point v_4 belongs to $\mathrm{Int}C_2$.

Similarly, we have that if $v_3 \in \mathrm{Int}C_2$ with $\tilde{F}_*(v_3) = c$, then $\tilde{F}_*(v)$ has another critical point in $\mathrm{Int}C_4$. The crical point of $\tilde{F}_*$ if and only if the critcal point of $\tilde{F}$. Hence this completes the theorem. □

Theorem 3.9. *Let* $\lambda_1 < a < \lambda_2 < b < \lambda_3$. *For* $1 \leq i \leq 4$, *let* $\Pi(C_i) = R_i$. *Then* $R_2 = R_1 \cup R_4^*$ *and* $R_4 = R_1 \cup R_2^*$.

Proof. Let $h \in V$. We note that v is a solution of the equation

$$\Pi(v) = Av + P(b(v+\theta(v))^+ - a(v+\theta(v))^-) = h \text{ in } V$$

if and only if v is a critical point of $\tilde{F}$. Hence it follows from Theorem 3.8 that $R_2 \cap R_1 \neq \emptyset$. Since R_2 is one of sets $R_1 \cup R_4^*$ or $R_3 \cup R_2^*$, R_2 must be $R_1 \cup R_4^*$.

On the other hand, it follows from Theorem 3.8 that $R_4 \cap R_1 \neq \emptyset$. Since R_4 is one of sets $R_1 \cup R_2^*$ or $R_3 \cup R_4^*$, R_4 must be $R_1 \cup R_2^*$. □

By Theorem 2.2, Theorem 2.3, Theorem 2.4 and Theorem 3.9, we obtain the main theorem of the equation (1.2).

Theorem 3.9. *Let $\lambda_1 < a < \lambda_2 < b < \lambda_3$. Then we have the following:*

(a) If $h \in IntR_1$, then equation (1.2) has a positive solution and at least two change sign solutions.

(b) If $h \in \partial R_1$, then equation (1.2) has a positive solution and at least one change sign solution.

(c) If $h \in IntR_i^*(i = 2, 4)$, then equation (1.2) has at least one change sign solution.

(d) If $h \in IntR_3^*$, then equation (1.2) has only the negative solution.

(e) If $h \in \partial R_3$, then equation (1.2) has a negative solution.

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