Communications in Applied Analysis **11** (2007), no. 2, 169–188

GLOBAL EXISTENCE AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE FOURTH ORDER NONLINEAR SCHRÖDINGER TYPE EQUATION

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Communicated by S.I. Nenov

ABSTRACT: We study the global existence and asymptotic behavior in time of solutions to the fourth order nonlinear Schrödinger type equation in one space dimension. The nonlinear interaction is the power type interaction with degree three, and it is a summation of a gauge invariant term and non-gauge-invariant terms. We prove the existence of modified wave operators for this equation with small final states. Here the modification of wave operator is only derived from the gauge invariant nonlinearity.

Dedicated to Professor Mitsuhiro Nakao on his sixtieth birthday.

AMS (MOS) Subject Classification: 35B40, 35Q55

1. INTRODUCTION

This paper is concerned with the global existence and asymptotic behavior of solution to the fourth order nonlinear Schrödinger type equation:

$$i\partial_t u - \frac{1}{4}\partial_x^4 u = \mathcal{N}(u,\overline{u}), \qquad t,x \in \mathbb{R}, \tag{1.1}$$

where the nonlinear term is a summation of the power type nonlinearities:

$$\mathcal{N}(u,\overline{u}) = \lambda_0|u|^2u + \lambda_1 u^3 + \lambda_2|u|^2\overline{u} + \lambda_3\overline{u}^3,$$

u is a complex valued unknown function and $\lambda_0 \in \mathbb{R}$ and $\lambda_j \in \mathbb{C}$ for $j = 1,2,3$. We shall show the existence and uniqueness of global solutions for (1.1) which approach a given modified free profile.

Received December 12, 2006

We briefly explain on the notion of the short and long range scattering for nonlinear equation (1.1). Let $v(t,x) = W(t)\phi(x)$ be the solution to linearized equation of (1.1):

$$i\partial_t v - \frac{1}{4}\partial_x^4 v = 0, \qquad t, x \in \mathbb{R}, \tag{1.2}$$

and $v(0,x) = \phi(x)$. When the solution u to the nonlinear equation (1.1) behaves as $t \to \infty$ like the solution to the linear equation (1.2), we call the nonlinear term $\mathcal{N}$ is the short range interaction, otherwise we call it the long range one. Concerning the short and long range scattering for the (second order) nonlinear Schrödinger equation, see e.g., Barab [1], Ginibre and Ozawa [5], Ginibre and Velo [6], Hayashi and Naumkin [7], Hayashi et al [11], Moriyama et al [13], Ozawa [14], Shimomura [17], Shimomura and Tonegawa [18], Shimomura and Tsutsumi [19], Strauss [20], Tsutsumi [21] and Tsutsumi and Yajima [23]. According to the above literatures, the borderline of the power between the short and long range scattering depends on the decay properties of the fundamental solution. More precisely, for the n-dimensional nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u = \mu|u|^{p-1}u, \qquad t \in \mathbb{R}, x \in \mathbb{R}^n,$$

the pointwise decay of the fundamental solution is $t^{-n/2}$ and the power of the borderline between short and long range scattering is $p = 1 + 2/n$ (see e.g., Cazenave [3], Chapter 7 and Ginibre [4])

Concerning the fourth order equation (1.1), by the method of stationary phase the solution $W(t)\phi$ to the equation (1.2) with initial data ϕ behaves as

$$W(t)\phi \sim \frac{1}{\sqrt{3it}}\frac{1}{|\chi(t)|}\exp\left(\frac{3}{4}it\chi(t)^4\right)\hat{\phi}\left(\chi(t)\right),$$

as $t \to \pm\infty$, where $\chi(t) = |\frac{x}{t}|^{-\frac{2}{3}}\frac{x}{t}$ and $\hat{\phi}(\xi)$ is the Fourier transform of ϕ with respect to space variable. Therefore if ϕ satisfies $|\hat{\phi}(\xi)| = o(|\xi|^{\alpha})$ as $|\xi| \to 0$ with suitable $\alpha > 0$, then $W(t)\phi$ decays like $t^{-\frac{1}{2}}$ in L_x^∞ as the one dimensional Schrödinger equation (see Segata [15]). Therefore the power three of the nonlinear terms in (1.1) is critical between the short and long range scattering with mean zero final data (in general, the order of pointwise decay for the function $W(t)\phi$ is $t^{-1/4}$ (see Ben Artzi-Koch-Saut Ben-Artzi et al [2])). In Segata [15], the first author proved the existence of modified wave operators of (1.1) with small final states for the case where $\lambda_0 \neq 0$ and $\lambda_1 = \lambda_2 = \lambda_3 = 0$. (Furthermore in Segata and Shimomura [16], the authors showed the existence of a unique solution to the equation (1.1) which approaches a suitable modified free profile without any size restrictions on given final data, when $\lambda_0 \in \mathbb{C}$, $\operatorname{Im}\lambda_0 < 0$ and $\lambda_1 = \lambda_2 = \lambda_3 = 0$). More precisely, it was shown in Segata [15] that for given ϕ_+, there exists a unique solution $u \in C(\mathbb{R}, L_x^2)$ to (1.1) such that for $3/8 < \alpha < 1$,

$$\left\| u(t) - \sqrt{2\pi} V(t,x) \exp\left(-\frac{i\lambda}{3} \left| \hat{\phi}_+ (\chi(t)) \right|^2 |\chi(t)|^{-2} \log |t| \right) \hat{\phi}_+ (\chi(t)) \right\|_{L^2_x} = \mathcal{O}(t^{-\alpha}),$$

as $t \to \infty$, where $V(t,x)$ is the fundamental solution to (1.2). From this result, the nonlinear interaction $\lambda_0 |u|^2 u$ is the long range interaction. Our interest in the present paper is whether the other nonlinear terms in (1.1) are the long range or not.

To see this we note that the first term in the nonlinear terms satisfies gauge invariant property, namely, $F(e^{i\theta}u, \overline{e^{i\theta}u}) = F(u, \overline{u})$ for any $\theta \in \mathbb{R}$ and the other nonlinear terms do not satisfy this condition. We decompose the nonlinear terms due to this property:

$$\mathcal{N}(u, \overline{u}) = \mathcal{N}_g(u, \overline{u}) + \mathcal{N}_{ng}(u, \overline{u}),$$

$$\mathcal{N}_g(u, \overline{u}) = \lambda_0 |u|^2 u, \qquad \mathcal{N}_{ng}(u, \overline{u}) = \lambda_1 u^3 + \lambda_2 |u|^2 \overline{u} + \lambda_3 \overline{u}^3.$$

As we shall see Theorem 1.1 below, $\mathcal{N}_g$ is the long range interaction and the other nonlinear terms $\mathcal{N}_{ng}$ are short range interactions. Therefore the gauge invariant property of nonlinear terms plays an important role in the asymptotic behavior of solutions to (1.1).

To state our main theorem, we introduce several notations. We denote $\{W(t)\}_{t \in \mathbb{R}}$ the free evolution group generated by the linear oprator $-i\partial_x^4/4$:

$$W(t)u_0 = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi - \frac{i}{4}t\xi^4} \hat{u}_0(\xi) d\xi = \int_{\mathbb{R}} V(t, x-y) u_0(y) dy,$$

where the function $V(t,x)$ is the fundamental solution to linearized equation of (1.1):

$$V(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi - \frac{i}{4}t\xi^4} d\xi.$$

Let $H^{s,\alpha}$ be the weighted Sobolev space defined by

$$H^{s,\alpha} = \{\phi \in \mathcal{S}'; \|\phi\|_{H^{s,\alpha}} = \|(1+|x|^2)^{\alpha/2}(1-\partial_x^2)^{s/2}\phi\|_{L^2} < \infty\},$$

$s, \alpha \in \mathbb{R}$ and $\dot{H}^s$ be the homogeneous Sobolev space

$$\dot{H}^s = \{\phi \in \mathcal{S}'; \|\phi\|_{\dot{H}^s} = \|(-\partial_x^2)^{s/2}\phi\|_{L^2} < \infty\}, \quad s \in \mathbb{R}.$$

For $1 \le p, q \le \infty$ and an interval $I \subset \mathbb{R}$, we define

$$L_t^p(I, L_x^q) = \left\{ f;\ \|f\|_{L_t^p(I, L_x^q)} = \left(\int_I \|f(t)\|_{L_x^q}^p dt \right)^{\frac{1}{p}} < \infty \right\}.$$

Let us define the class of final data $\phi_\pm$.

$$\begin{aligned} \mathcal{D} &\equiv \{\phi \in \mathcal{S}'; \phi \in H^{0,4} \text{ and } x^k\phi \in \dot{H}^{k-12},\ k = 0,1,2,3,4\}, \\ \|\phi\|_{\mathcal{D}} &\equiv \|\phi\|_{H^{0,4}} + \sum_{k=0}^{4} \|x^k \phi\|_{\dot{H}^{k-12}}. \end{aligned} \tag{1.3}$$

For a given final data $\phi_\pm$, we introduce the following asymptotic profile

$$v_\pm(t,x) = \frac{1}{\sqrt{3it}} \frac{1}{|\chi(t)|} \exp\left(\frac{3}{4} it\chi(t)^4 + iS^\pm(t,\chi(t))\right) \hat{\phi}_\pm(\chi(t)), \tag{1.4}$$

$t, x \neq 0$, where $\hat{\phi}_\pm$ are the Fourier transforms of $\phi_\pm$ with respect to space variable and

$$S^\pm(t,\chi(t)) = \mp\frac{\lambda_0}{3}|\hat{\phi}_\pm(\chi(t))|^2|\chi(t)|^{-2}\log|t|, \qquad |t| \geq e.$$

The main result is as follows.

Theorem 1.1. (i) *Let $\phi_+ \in \mathcal{D}$ and $\|\phi_+\|_{\mathcal{D}}$ be sufficiently small, where $\mathcal{D}$ is the set defined by* (1.3). *Then the equation* (1.1) *has a unique solution $u \in C([0,\infty); L^2(\mathbb{R})) \cap L^8_{loc}((0,\infty); L^\infty(\mathbb{R}))$ satisfying*

$$\sup_{t\geq e}(t^\alpha \|u(t) - v_+(t)\|_{L^2_x}) \;<\; \infty, \tag{1.5}$$

$$\sup_{t\geq e}\left\{t^\alpha\left(\int_t^\infty \|u(\tau) - v_+(\tau)\|^8_{L^\infty_x}\, d\tau\right)^{\frac{1}{8}}\right\} \;<\; \infty, \tag{1.6}$$

where $3/8 < \alpha < 1$ and $v_+(t,x)$ is the modified free dynamics given by (1.4).
(ii) *Let $\phi_+ \in \mathcal{D}$ and $\|\phi_+\|_{\mathcal{D}}$ be sufficiently small. Then the equation* (1.1) *has a unique solution $u \in C((-\infty,0]; L^2(\mathbb{R})) \cap L^8_{loc}((-\infty,0); L^\infty(\mathbb{R}))$ satisfying*

$$\sup_{t\leq -e}(|t|^\alpha \|u(t) - v_-(t)\|_{L^2_x}) \;<\; \infty,$$

$$\sup_{t\leq -e}\left\{|t|^\alpha\left(\int_{-\infty}^t \|u(\tau) - v_-(\tau)\|^8_{L^\infty_x}\, d\tau\right)^{\frac{1}{8}}\right\} \;<\; \infty,$$

where $3/8 < \alpha < 1$ and $v_-(t,x)$ is the modified free dynamics given by (1.4).

Remark 1.1. The modified wave operator $\Omega_+ : \phi_+ \mapsto u(0)$ for the positive time to the equation (1.1) is well-defined on a suitable small ball of $\mathcal{D}$, where u is the solution obtained in the first half of Theorem 1.1. Similarly, the existence of a modified wave operator for the negative time to the equation (1.1) follows from the second half of Theorem 1.1.

Remark 1.2. If $\phi \in \mathcal{D}$, then $\hat{\phi}(\xi)$ is almost flat near $\xi = 0$, more precisely, $\hat{\psi}(\xi)$ behaves like $|\xi|^\alpha$, $\alpha > \frac{23}{2}$ near $\xi = 0$.

Remark 1.3. The function $(3it)^{-\frac{1}{2}}|\chi(t)|^{-1}\exp(\frac{3}{4}it\chi(t)^4)\hat{\phi}_\pm(\chi(t))$ is the leading term of the function $W(t)\phi_\pm$. Therefore roughly speaking, the first half of Theorem 1.1 says that

$$u(t) \sim \exp\left(-\frac{\lambda_0}{3}\left|\hat{\phi}_+(\chi(t))\right|^2 \chi(t)^{-2}\log|t|\right) W(t)\phi_+, \qquad \text{as } t \to \infty.$$

From this, we see that if $\lambda_0 = 0$, then the solutions obtained in Theorem 1.1 are asymptotically free.

Here we briefly outline the proof of Theorem 1.1. We consider the case of the positive time. For the negative time, we can treat analogously. Let $\mathcal{L} = i\partial_t - \frac{1}{4}\partial_x^4$. If we assume that the solution to (1.1) satisfies $u(t) \sim v_+(t)$ as $t \to \infty$, then (1.1) is equivalent to the integral equation of Yang-Ferdman type:

$$u(t) - \tilde{v}_+(t) = i \int_t^\infty W(t-\tau)\{\mathcal{N}(u,\overline{u})(\tau) - \mathcal{L}\tilde{v}_+(\tau)\}\, d\tau, \tag{1.7}$$

where $\tilde{v}_+(t,x) = \psi(t)v_+(t,x)$ and $\psi \in C^\infty([0,\infty))$ with $\psi(t) = 1$ if $t \geq 2e$ and $\psi(t) = 0$ if $0 \leq t \leq e$ (in order to avoid a singularity of the function v_+, we multiply v_+ by a cut-off function $\psi(t)$). To show the existence of a solution for (1.1) which satisfies (1.5) and (1.6) in Theorem 1.1, we apply the Banach Fixed Point Theorem to this integral equation (1.7). More precisely, we mainly show that the map

$$\Phi u(t) \equiv \tilde{v}_+(t) + i \int_t^\infty W(t-\tau)\{\mathcal{N}(u,\overline{u})(\tau) - \mathcal{L}\tilde{v}_+(\tau)\}\, d\tau \tag{1.8}$$

is a contraction on the complete metric space

$$X^\rho = \Bigg\{ u \in C([0,\infty), L_x^2(\mathbb{R}));$$
$$\sup_{t\geq 0}(t+1)^\alpha \left\{ \|u(t) - \tilde{v}_+(t)\|_{L_x^2} + \left(\int_t^\infty \|u(\tau) - \tilde{v}_+(\tau)\|_{L_x^\infty}^8 \, d\tau \right)^{1/8} \right\} \leq \rho \Bigg\}$$

with the metric

$$\begin{aligned} &\|u_1 - u_2\|_X \\ &= \sup_{t\geq 0}(t+1)^\alpha \left\{ \|u_1(t) - u_2(t)\|_{L_x^2} + \left(\int_t^\infty \|u_1(\tau) - u_2(\tau)\|_{L_x^\infty}^8 \, d\tau \right)^{1/8} \right\}, \end{aligned}$$

if $\|\phi_+\|_{\mathcal{D}}$ and ρ are sufficiently small. To guarantee that Φ is a contraction on X^ρ, we split the right hand side of (1.8) into three parts:

$$\begin{aligned} \Phi u(t) - \tilde{v}_+(t) &= i \int_t^\infty W(t-\tau)\{\mathcal{N}(u,\overline{u})(\tau) - \mathcal{N}(\tilde{v}_+,\overline{\tilde{v}}_+)(\tau)\}\, d\tau \\ &\quad - i \int_t^\infty W(t-\tau)\{\mathcal{L}\tilde{v}_+(\tau) - \mathcal{N}_g(\tilde{v}_+,\overline{\tilde{v}}_+)(\tau)\}\, d\tau \\ &\quad + i \int_t^\infty W(t-\tau)\mathcal{N}_{ng}(\tilde{v}_+,\overline{\tilde{v}}_+)(\tau)\, d\tau. \end{aligned} \tag{1.9}$$

The estimate for the first term in the right hand side of (1.9) easily follows from the Strichartz estimates for the free evolution group $\{W(t)\}_{t\in\mathbb{R}}$ (see Lemma 2.1 and Proposition 2.1 below). The estimate for the second term is essentially obtained by Segata [15], Proposition 2.2. We note that the modification of the free dynamics comes from the estimate for the second term. The main task in this paper is the estimation on X^ρ for the third term. Roughly speaking, the non-gauge-invariant terms have the oscillation factors which induce the additional time decay thanks to the integration by parts. By making use of those properties we are able to estimate the third term without the modification of the free dynamics. This is carried out in Lemmas 3.1, 3.2 and 3.3 (see Section 3 below).

As we mentioned above, hereafter we only consider the case $t > 0$, because the case $t < 0$ is treated analogously.

The outline of this paper is as follows. In Section 2, we solve the Cauchy problem at infinite initial time for the equation (1.7). We give the estimate for the third term in the right hand side of (1.9) in Subsection 3.1. In Subsection 3.2, we guarantee Theorem 1.1.

2. THE CAUCHY PROBLEM AT INFINITE INITIAL TIME

In this section, we solve the Cauchy problem at infinite initial time for (1.7) (see Proposition 2.1 below). To prove Proposition 2.1 we use the following Strichartz estimates on the free evolution group $\{W(t)\}_{t\in\mathbb{R}}$.

Lemma 2.1. (see Kenig et al [12]) *Let I be an interval of $\mathbb{R}$ (bounded or not) and $t_0 \in \overline{I}$. If (q_i, r_i) satisfy $8 \le q_i \le \infty$, $2 \le r_i \le \infty$ and $\frac{4}{q_i} + \frac{1}{r_i} = \frac{1}{2}$, $(i = 1, 2)$. Then*

$$\left\| \int_{t_0}^{t} W(t-t')f(t')dt' \right\|_{L_t^{q_1}(I;L_x^{r_1})} \le C\|f\|_{L_t^{q_2'}(I;L_x^{r_2'})} \tag{2.1}$$

where p' is the Hölder conjugate exponent of p and C depends on q and not on I.

Let

$$\mathcal{R}(u, \overline{u}) = \mathcal{L}u - \mathcal{N}(u, \overline{u}), \tag{2.2}$$

where $\mathcal{L} = i\partial_t - \frac{1}{4}\partial_x^4$ and $\mathcal{N}(u, \overline{u}) = \lambda_0|u|^2u + \lambda_1u^3 + \lambda_2|u|^2\overline{u} + \lambda_3\overline{u}^3$. $\mathcal{R}(u, \overline{u})$ is a difference between the left hand side and right hand side in (1.1).

Proposition 2.1. *Assume that there exists $\delta > 0$ such that $F(t, x)$ satisfies*

$$\|F(t)\|_{L_x^\infty} \le \delta(t+1)^{-\frac{1}{2}}, \tag{2.3}$$

and

$$\sup_{t\ge 0}(t+1)^{\alpha}\left\{\left\|\int_t^\infty W(t-\tau')\mathcal{R}(F,\overline{F})(\tau')d\tau'\right\|_{L_x^2} + \left\|\int_\tau^\infty W(\tau-\tau')\mathcal{R}(F,\overline{F})(\tau')d\tau'\right\|_{L_\tau^8(t,\infty,L_x^\infty)}\right\} \le \delta, \tag{2.4}$$

where $3/8 < \alpha < 1$ and that δ is sufficiently small. Then (1.1) has a unique solution $u \in C([0,\infty), L_x^2(\mathbb{R}))$ such that

$$\sup_{t\ge 0}(t+1)^{\alpha}(\|u(t) - F(t)\|_{L_x^2} + \|u(\tau) - F(\tau)\|_{L_\tau^8(t,\infty,L_x^\infty)}) < \infty.$$

Proof of Proposition 2.1. By (1.1), u satisfies

$$\begin{aligned} u-F &= i\int_t^\infty W(t-\tau)\{\mathcal{N}(u,\overline{u})(\tau)-\mathcal{L}F(\tau)\}d\tau \\ &= i\int_t^\infty W(t-\tau)\{\mathcal{N}(u,\overline{u})(\tau)-\mathcal{N}(F,\overline{F})(\tau)\}d\tau \\ &\quad -i\int_t^\infty W(t-\tau)\mathcal{R}(F,\overline{F})(\tau)d\tau. \end{aligned} \tag{2.5}$$

For $T\ge 0$, we define the following complete metric space:

$$\begin{aligned} X_T &\equiv \{u\in C([T,\infty),L_x^2(\mathbb{R})); \\ &\qquad \sup_{t\ge T}(t+1)^\alpha(\|u(t)-F(t)\|_{L_x^2}+\|u(\tau)-F(\tau)\|_{L_\tau^8(t,\infty,L_x^\infty)})<\infty\} \end{aligned}$$

with the metric

$$\begin{aligned} &\|u_1-u_2\|_{X_T} \\ &\equiv \sup_{t\ge T}\{(t+1)^\alpha(\|u_1(t)-u_2(t)\|_{L_x^2}+\|u_1(\tau)-u_2(\tau)\|_{L_\tau^8(t,\infty,L_x^\infty)})\}, \end{aligned}$$

and for $\rho>0$ and $T\ge 0$, we define the following closed subset of X_T:

$$\begin{aligned} \widetilde{X}_T^\rho &\equiv \{u\in C([T,\infty),L_x^2(\mathbb{R})); \\ &\qquad \sup_{t\ge T}(t+1)^\alpha(\|u(t)-F(t)\|_{L_x^2}+\|u(\tau)-F(\tau)\|_{L_\tau^8(t,\infty,L_x^\infty)})\le\rho\}. \end{aligned}$$

We show the existence of a unique solution to the equation (1.1) in X_0. We define the map

$$\begin{aligned} \Phi u(t) &\equiv F(t)+i\int_t^\infty W(t-\tau)\{\mathcal{N}(u,\overline{u})(\tau)-\mathcal{N}(F,\overline{F})(\tau)\}d\tau \\ &\quad -i\int_t^\infty W(t-\tau)\mathcal{R}(F,\overline{F})(\tau)d\tau. \end{aligned}$$

We prove the existence of solution u to (1.1) by showing that Φ is a contraction map on $\widetilde{X}_0^\rho$ if δ and ρ are sufficiently small. Firstly, we prove that $u\in\widetilde{X}_0^\rho$ then $\Phi(u)\in\widetilde{X}_0^\rho$. Let $u\in\widetilde{X}_0^\rho$. Since

$$|\mathcal{N}(u,\overline{u})-\mathcal{N}(F,\overline{F})|\le C(|u-F|^3+|F||u-F|^2+|F|^2|u-F|).$$

Combining this inequality and the Strichartz estimate (Lemma 2.1) we have

$$\begin{aligned} &\|(\Phi u)(t)-F(t)\|_{L_x^2}+\left(\int_t^\infty\|(\Phi u)(\tau)-F(\tau)\|_{L_x^\infty}^8 d\tau\right)^{\frac{1}{8}} \\ \le\ & C\left(\int_t^\infty\||u-F|^3(\tau)\|_{L_x^1}^{\frac{8}{7}}d\tau\right)^{\frac{7}{8}}+C\int_t^\infty\||F||u-F|^2(\tau)\|_{L_x^2}d\tau \\ &+C\int_t^\infty\||F|^2|u-F|(\tau)\|_{L_x^2}d\tau \\ &+\left\|\int_t^\infty W(t-\tau)\mathcal{R}(F,\overline{F})(\tau)d\tau\right\|_{L_x^2} \end{aligned}$$

$$
\begin{aligned}
&+\left\|\int_\tau^\infty W(\tau-\tau')\mathcal{R}(F,\overline{F})(\tau')d\tau'\right\|_{L^8_\tau(t,\infty,L^\infty_x)}\\
\equiv\ & I_1(t)+I_2(t)+I_3(t)+I_4(t)+I_5(t). \qquad (2.6)
\end{aligned}
$$

By Hölder's inequality, we have

$$
\begin{aligned}
I_1(t) &\le C\left(\int_t^\infty \|u(\tau)-F(\tau)\|^{\frac{8}{7}}_{L^\infty_x}\||u-F|^2(\tau)\|^{\frac{8}{7}}_{L^1_x}d\tau\right)^{\frac{7}{8}}\\
&\le C\left(\int_t^\infty \|u(\tau)-F(\tau)\|^8_{L^\infty_x}d\tau\right)^{\frac{1}{8}}\left(\int_t^\infty \||u-F|^2(\tau)\|^{\frac{4}{3}}_{L^1_x}d\tau\right)^{\frac{3}{4}}\\
&= C\left(\int_t^\infty \|u(\tau)-F(\tau)\|^8_{L^\infty_x}d\tau\right)^{\frac{1}{8}}\left(\int_t^\infty \|u(\tau)-F(\tau)\|^{\frac{8}{3}}_{L^2_x}d\tau\right)^{\frac{3}{4}}\\
&\le C\|u-F\|^3_{X_0}(t+1)^{-3\alpha+\frac{3}{4}}. \qquad (2.7)
\end{aligned}
$$

Similarly, by Hölder's inequality and the assumption (2.3), we obtain

$$
\begin{aligned}
I_2(t) &\le C\int_t^\infty \||F||u-F|(\tau)\|_{L^2_x}\|u(\tau)-F(\tau)\|_{L^\infty_x}d\tau\\
&\le C\left(\int_t^\infty \||F||u-F|(\tau)\|^{\frac{8}{7}}_{L^2_x}d\tau\right)^{\frac{7}{8}}\left(\int_t^\infty \|u(\tau)-F(\tau)\|^8_{L^\infty_x}d\tau\right)^{\frac{1}{8}}\\
&\le C\left(\int_t^\infty \|F(\tau)\|^{\frac{8}{7}}_{L^\infty_x}\|u(\tau)-F(\tau)\|^{\frac{8}{7}}_{L^2_x}d\tau\right)^{\frac{7}{8}}\\
&\qquad\qquad\times\left(\int_t^\infty \|u(\tau)-F(\tau)\|^8_{L^\infty_x}d\tau\right)^{\frac{1}{8}}\\
&\le C\delta\|u-F\|^2_{X_0}(t+1)^{-2\alpha+\frac{3}{8}}. \qquad (2.8)
\end{aligned}
$$

and

$$
\begin{aligned}
I_3(t) &\le C\int_t^\infty \||F|^2(\tau)\|_{L^\infty_x}\|u(\tau)-F(\tau)\|_{L^2_x}d\tau\\
&= C\int_t^\infty \|F(\tau)\|^2_{L^\infty_x}\|u(\tau)-F(\tau)\|_{L^2_x}d\tau\\
&\le C\delta^2\|u-F\|_{X_0}(t+1)^{-\alpha}. \qquad (2.9)
\end{aligned}
$$

By the assumption (2.4), we have

$$
I_4(t)+I_5(t)\le\delta(t+1)^{-\alpha}. \qquad (2.10)
$$

Collecting (2.6)–(2.10), we obtain

$$
\begin{aligned}
&\|\Phi u(t)-F(t)\|_{L^2_x}+\|\Phi u(\tau)-F(\tau)\|_{L^8_\tau(t,\infty,L^\infty_x)}\\
\le\ & C(1+t)^{-\alpha}(\|u-F\|^3_{X_0}(1+t)^{-2\alpha+3/4}+\delta^2\|u-F\|_{X_0}+\delta)\\
\le\ & C(1+t)^{-\alpha}(\rho^3(1+t)^{-2\alpha+3/4}+\delta^2\rho+\delta) \qquad (2.11)
\end{aligned}
$$

for $t \geq 0$, and hence

$$\sup_{t\geq 0}(t+1)^{\alpha}(\|\Phi u(t)-F(t)\|_{L_x^2}+\|\Phi u(\tau)-F(\tau)\|_{L_\tau^8(t,\infty,L_x^\infty)}) \\ \leq C(\rho^3+\delta^2\rho+\delta).$$

Therefore by choosing ρ and δ sufficiently small, we have $\Phi u \in \widetilde{X}_0^\rho$. By similar way, we have

$$\|\Phi u_1 - \Phi u_2\|_{X_0} \leq \frac{1}{2}\|u_1-u_2\|_{X_0}$$

for $u_1, u_2 \in \widetilde{X}_0^\rho$ with sufficiently small numbers ρ and δ. Hence Banach's Fixed Point Theorem yields that (1.1) has a unique solution $u \in \widetilde{X}_0^\rho$ for sufficiently small ρ and δ. It remains to prove the uniqueness in X_0. Let $T \geq 0$ and $u_1 \in X_0$ and $u_1 \in X_0$ be solutions to the equation (1.1) (then $u_1, u_2 \in X_T$ for any $T \geq 0$). As in the derivation of the first inequality of (2.11), we have

$$\begin{aligned}&\|u_1-u_2\|_{X_T}\\ &\quad\leq\ C((\|u_1-F\|_{X_T}^2+\|u_2-F\|_{X_T}^2)(1+T)^{-2\alpha+3/4}+\delta^2)\|u_1-u_2\|_{X_T}.\end{aligned}$$

Since $3/8 < \alpha < 1$, we see that if $\delta > 0$ is sufficiently small and $T \geq 0$ is sufficiently large, then

$$\|u_1-u_2\|_{X_T} \leq 0.$$

Therefore for sufficiently small $\delta > 0$ and sufficiently large $T \geq 0$, $u_1(t) = u_2(t)$ when $t \geq T$. The local well-posedness in L^2 of the equation (1.1) implies $u_1(t) = u_2(t)$ when $0 \leq t \leq T$, where T is determined above. These facts yield the uniqueness in X_0, if $\delta > 0$ is sufficiently small. Therefore this proposition is proved. □

3. REMAINDER ESTIMATES AND PROOF OF THEOREM 1.1

In this section, we give the estimate for the third term in the right hand side of (1.9), and we prove Theorem 1.1.

3.1. ESTIMATES FOR ASYMPTOTICS AND REMAINDER TERMS.

To prove Theorem 1.1, it suffices to show that $\tilde{v}_+(t,x) = \psi(t)v_+(t,x)$ defined by (1.4) satisfies two inequalities (2.3) and (2.4). To see this, we give several propositions and lemmas. Concerning the term:

$$-i\int_t^\infty W(t-\tau)\{\mathcal{L}\tilde{v}_+(\tau)-\mathcal{N}_g(\tilde{v}_+,\overline{\tilde{v}}_+)(\tau)\}d\tau\,,$$

we have the following estimate.

Proposition 3.1. *If $\phi_+ \in \mathcal{D}$, then*

$$\|\tilde{v}_+(t)\|_{L^\infty_x} \leq C(t+1)^{-\frac{1}{2}}\|\phi_+\|_{\mathcal{D}}, \tag{3.1}$$

$$\begin{aligned}&\|\mathcal{L}\tilde{v}_+(t) - \mathcal{N}_g(\tilde{v}_+, \overline{\tilde{v}}_+)(t)\|_{L^2_x}\\ &\qquad \leq C(t+1)^{-2}(\log(t+1))^4\|\phi_+\|_{\mathcal{D}}(1+\|\phi_+\|^8_{\mathcal{D}}),\end{aligned} \tag{3.2}$$

for any $t \geq 0$.

This proposition is obtained by same argument as Proposition 2.2 in Segata [15]. Therefore we omit the proof.

Next we estimate the following term:

$$w(t,x) \equiv i\int_t^\infty W(t-\tau)\mathcal{N}_{ng}(\tilde{v}_+, \overline{\tilde{v}_+})(\tau)d\tau.$$

Since

$$\begin{aligned}
&\mathcal{N}_{ng}(\tilde{v}_+, \overline{\tilde{v}_+})(t,x)\\
&= \frac{\lambda_1}{3\sqrt{3}}\psi(t)^3(it)^{-\frac{3}{2}}\frac{1}{|\chi(t)|^3}\exp\left(\frac{9}{4}it\chi(t)^4\right)\exp\left(3iS^+\left(t,\chi(t)\right)\right)\hat{\phi}_+\left(\chi(t)\right)^3\\
&\quad -\frac{i\lambda_2}{3\sqrt{3}}\psi(t)^3(it)^{-\frac{3}{2}}\frac{1}{|\chi(t)|^3}\exp\left(-\frac{3}{4}it\chi(t)^4\right)\exp\left(-iS^+\left(t,\chi(t)\right)\right)\\
&\qquad\qquad \times\left|\hat{\phi}_+\left(\chi(t)\right)\right|^2\overline{\hat{\phi}_+\left(\chi(t)\right)}\\
&\quad +\frac{i\lambda_3}{3\sqrt{3}}\psi(t)^3(it)^{-\frac{3}{2}}\frac{1}{|\chi(t)|^3}\exp\left(-\frac{9}{4}it\chi(t)^4\right)\exp\left(-3iS^+\left(t,\chi(t)\right)\right)\\
&\qquad\qquad \times\overline{\hat{\phi}_+\left(\chi(t)\right)}^3,
\end{aligned}$$

we estimate $w(t,x)$ by considering the general form of above three terms:

$$h(it)\exp\left(\frac{3}{4}i\omega t\chi(t)^4\right)\exp\left(if\quad\left(\chi(t)\right)\log t\right)g\left(\chi(t)\right).$$

Lemma 3.1. *Let $\omega \neq 1$, $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and let $h \in C^1([0,\infty))$ satisfy* $\operatorname{supp} h \subset [e,\infty)$ *and $|h(it)| + t|h'(it)| \leq C(1+t)^{-3/2}$ for $t \geq 0$. Then we have for $t \geq 0$,*

$$\begin{aligned}
&\int_t^\infty W(t-\tau)h(i\tau)\exp\left(\frac{3}{4}i\omega\tau\chi(\tau)^4\right)\exp\left(if\left(\chi(\tau)\right)\log\tau\right)g\left(\chi(\tau)\right)d\tau\\
=\ &\frac{1}{1-\omega^3}\int_t^\infty h(i\tau)\exp\left(\frac{3}{4}i\omega\tau\chi(\tau)^4\right)\exp\left(if\left(\chi(\tau)\right)\log\tau\right)g\left(\chi(\tau)\right)d\tau\\
&-\frac{i}{1-\omega^3}\int_t^\infty W(t-\tau)\left\{\int_\tau^\infty h(i\tau')\left(i\omega^3\partial_{\tau'} - \frac{1}{4}\partial_x^4\right)\exp\left(\frac{3}{4}i\omega\tau'\chi(\tau')\right)\right.\\
&\qquad\qquad\left.\times\exp\left(if\left(\chi(\tau')\right)\log\tau'\right)g\left(\chi(\tau')\right)\,d\tau'\right\}d\tau\\
&+\frac{i\omega^3}{1-\omega^3}\int_t^\infty W(t-\tau)\left\{\int_\tau^\infty h'(i\tau')\exp\left(\frac{3}{4}i\omega\tau'\chi(\tau')^4\right)\right.
\end{aligned}$$

$$\times \exp\left(if\left(\chi(\tau')\right)\log\tau'\right) g\left(\chi(\tau')\right) d\tau' \Bigg\} \, d\tau. \tag{3.3}$$

Proof of Lemma 3.1. Let $t \geq 0$. Firstly, we note the following equality:

$$\begin{aligned} & W(-\tau)F(\tau) \\ = \; & -\partial_\tau \left\{ W(-\tau) \int_\tau^T F(\tau')\, d\tau' \right\} + \frac{i}{4} W(-\tau) \int_\tau^T \partial_x^4 F(\tau')\, d\tau', \end{aligned} \tag{3.4}$$

where $t \leq \tau \leq T$. Integrating the equality (3.4) with respect to τ over the interval (t, T), applying $W(t)$ to the resulting equality and letting $T \to \infty$, we have

$$\begin{aligned} & \int_t^\infty W(t-\tau)F(\tau) d\tau \\ = \; & \int_t^\infty F(\tau')\, d\tau' + \frac{i}{4} \int_t^\infty W(t-\tau) \left(\int_\tau^\infty \partial_x^4 F(\tau')\, d\tau' \right) d\tau. \end{aligned} \tag{3.5}$$

By substituting

$$F(\tau) = h(i\tau) \exp\left(\frac{3}{4} i\omega\tau\chi(\tau)^4\right) \exp\left(if\left(\chi(\tau)\right)\log\tau\right) g\left(\chi(\tau)\right),$$

into (3.5) and integration by parts, we have

$$\begin{aligned} & \int_t^\infty W(t-\tau) h(i\tau) \exp\left(\frac{3}{4} i\omega\tau\chi(\tau)^4\right) \exp\left(if\left(\chi(\tau)\right)\log\tau\right) g\left(\chi(\tau)\right)\, d\tau \\ = \; & \int_t^\infty h(i\tau') \exp\left(\frac{3}{4} i\omega\tau'\chi(\tau')^4\right) \exp\left(if\left(\chi(\tau')\right)\log\tau'\right) g\left(\chi(\tau')\right)\, d\tau' \\ & + \frac{i}{4} \int_t^\infty W(t-\tau) \Bigg[\int_\tau^\infty h(i\tau') \partial_x^4 \Bigg\{ \exp\left(\frac{3}{4} i\omega\tau'\chi(\tau')^4\right) \\ & \qquad \times \exp\left(if\left(\chi(\tau')\right)\log\tau'\right) g\left(\chi(\tau')\right) \Bigg\} d\tau' \Bigg] d\tau \\ = \; & \int_t^\infty h(i\tau') \exp\left(\frac{3}{4} i\omega\tau'\chi(\tau')\right) \exp\left(if\left(\chi(\tau')\right)\log\tau'\right) g\left(\chi(\tau')\right)\, d\tau' \\ & - i \int_t^\infty W(t-\tau) \Bigg[\int_\tau^\infty h(i\tau') \left(i\omega^3 \partial_{\tau'} - \frac{1}{4}\partial_x^4 \right) \Bigg\{ \exp\left(\frac{3}{4} i\omega\tau'\chi(\tau')\right) \\ & \qquad \times \exp\left(if\left(\chi(\tau')\right)\log\tau'\right) g\left(\chi(\tau')\right) \Bigg\} d\tau' \Bigg] d\tau \\ & - \omega^3 \int_t^\infty W(t-\tau) \Bigg[\int_\tau^\infty h(i\tau') \partial_{\tau'} \Bigg\{ \exp\left(\frac{3}{4} i\omega\tau'\chi(\tau')^4\right) \\ & \qquad \times \exp\left(if\left(\chi(\tau')\right)\log\tau'\right) g\left(\chi(\tau')\right) \Bigg\} d\tau' \Bigg] d\tau \\ = \; & \int_t^\infty h(i\tau') \exp\left(\frac{3}{4} i\omega\tau'\chi(\tau')^4\right) \exp\left(if\left(\chi(\tau')\right)\log\tau'\right) g\left(\chi(\tau')\right)\, d\tau' \\ & - i \int_t^\infty W(t-\tau) \Bigg[\int_\tau^\infty h(i\tau') \left(i\omega^3 \partial_{\tau'} - \frac{1}{4}\partial_x^4 \right) \Bigg\{ \exp\left(\frac{3}{4} i\omega\tau'\chi(\tau')^4\right) \end{aligned}$$

$$\times \exp\left(if\left(\chi(\tau')\right)\log\tau'\right) g\left(\chi(\tau')\right)\Big\} d\tau'\Big] d\tau$$

$$+\omega^3\int_t^\infty W(t-\tau)h(i\tau)\exp\left(\frac{3}{4}i\omega\tau\chi(\tau)^4\right)\exp\left(if\left(\chi(\tau)\right)\log\tau\right) g\left(\chi(\tau)\right)\, d\tau$$

$$+i\omega^3\int_t^\infty W(t-\tau)\Big\{\int_\tau^\infty h'(i\tau')\exp\left(\frac{3}{4}i\omega\tau'\chi(\tau')^4\right)$$

$$\times \exp\left(if\left(\chi(\tau')\right)\log\tau'\right) g\left(\chi(\tau')\right) d\tau'\Big\} d\tau. \tag{3.6}$$

Since the third term in the most right hand side of (3.6) is equal to ω^3 times the most left hand side of (3.6) and $\omega \neq 1$, we have this lemma. $\square$

Concerning the second term in the right hand side of (3.3), we note that the leading term of the function

$$\left(i\omega^3\partial_t - \frac{1}{4}\partial_x^4\right)\exp\left(\frac{3}{4}i\omega t\chi(t)^4\right)\exp\left(if\left(\chi(t)\right)\log t\right) g\left(\chi(t)\right)$$

is

$$-\omega^3 t^{-1}\left(f\left(\chi(t)\right) - \frac{i}{2}\right) g\left(\chi(t)\right)\exp\left(\frac{3}{4}i\omega t\chi(t)^4\right)\exp\left(if\left(\chi(t)\right)\log t\right).$$

More precisely, by setting

$$\begin{aligned}
&\mathcal{R}_g(t,x)\\
&= \left(i\omega^3\partial_t - \frac{1}{4}\partial_x^4\right)\exp\left(\frac{3}{4}i\omega t\chi(t)^4\right)\exp\left(if\left(\chi(t)\right)\log t\right) g\left(\chi(t)\right)\\
&\quad +\omega^3 t^{-1}\left(f\left(\chi(t)\right) - \frac{i}{2}\right) g\left(\chi(t)\right)\exp\left(\frac{3}{4}i\omega t\chi(t)^4\right)\exp\left(if\left(\chi(t)\right)\log t\right),
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
&\mathcal{M}(f,g)\\
&= \sum_{j=2}^{4}\sum_{0\le r\le j}\||x|^{-4j+r+5}\partial_x^r g\|_{L_x^2} + \sum_{j=2}^{4}\sum_{\substack{0\le q+r\le j\\ q\neq 0}}\||x|^{-4j+q+r+5}\partial_x^q f\partial_x^r g\|_{L_x^2}\\
&\quad +\sum_{j=3}^{4}\sum_{\substack{0\le p+q+r\le j\\ p,q\neq 0}}\||x|^{-4j+p+q+r+5}\partial_x^p f\partial_x^q f\partial_x^r g\|_{L_x^2}\\
&\quad +\sum_{j=3}^{4}\sum_{\substack{0\le k+p+q+r\le j\\ k,p,q\neq 0}}\||x|^{-4j+k+p+q+r+5}\partial_x^k f\partial_x^p f\partial_x^q f\partial_x^r g\|_{L_x^2}\\
&\quad +\||x|^{-7}(\partial_x f)^4 g\|_{L_x^2},
\end{aligned} \tag{3.8}$$

we have the following lemma.

Lemma 3.2. *Let $\mathcal{R}_g(t,x)$ be defined by* (3.7)*. For $t \geq 0$, we have*

$$\|\mathcal{R}_g(t)\|_{L^2_x} \leq (t+1)^{-\frac{3}{2}}(\log(t+1))^4\mathcal{M}(f,g), \tag{3.9}$$

where $\mathcal{M}(f,g)$ is given by (3.8)*.*

Remark 3.1. If $f(x) = |x|^{-2}|\hat{\phi}_+(x)|$ and $g(x) = |x|^{-3}\hat{\phi}_+(x)^3$, $|x|^{-3}|\hat{\phi}_+(x)|^2\,\overline{\hat{\phi}_+(x)}$ or $|x|^{-3}\overline{\hat{\phi}_+(x)}^3$, then by Hölder's inequality and Sobolev's embedding

$$\mathcal{M}(f,g) \leq C\|\phi_+\|^3_{\mathcal{D}}(1+\|\phi_+\|^8_{\mathcal{D}}).$$

Proof of Lemma 3.2. By the Leibniz rule, we have

$$\begin{aligned}
&\left(i\omega^3\partial_t - \frac{1}{4}\partial_x^4\right)\exp\left(\frac{3}{4}i\omega t\chi(t)^4\right)\exp\left(if\left(\chi(t)\right)\log t\right)g\left(\chi(t)\right)\\
=\;&\left\{\left(i\omega^3\partial_t - \frac{1}{4}\partial_x^4\right)\exp\left(\frac{3}{4}i\omega t\chi(t)^4\right)\right\}\exp\left(if\left(\chi(t)\right)\log t\right)g\left(\chi(t)\right)\\
&-\sum_{m+n=1}\partial_x^3\exp\left(\frac{3}{4}i\omega t\chi(t)^4\right)\partial_x^m\exp\left(if\left(\chi(t)\right)\log t\right)\partial_x^n g\left(\chi(t)\right)\\
&+i\omega^3\sum_{\substack{\ell+m+n=1\\ \ell\neq 1}}\partial_t^\ell\exp\left(\frac{3}{4}i\omega t\chi(t)^4\right)\partial_t^m\exp\left(if\left(\chi(t)\right)\log t\right)\partial_t^n g\left(\chi(t)\right)\\
&-\frac{1}{4}\sum_{\substack{\ell+m+n=4\\ \ell\neq 3,4}}\frac{4!}{\ell!m!n!}\partial_x^\ell\exp\left(\frac{3}{4}i\omega t\chi(t)^4\right)\\
&\qquad\qquad\times\partial_x^m\exp\left(if\left(\chi(t)\right)\log t\right)\partial_x^n g\left(\chi(t)\right).
\end{aligned}\tag{3.10}$$

Simple calculation yields

$$\begin{aligned}
&\left(i\omega^3\partial_t - \frac{1}{4}\partial_x^4\right)\exp\left(\frac{3}{4}i\omega t\chi(t)^4\right)\\
=\;&\left\{\frac{1}{2}i\omega^3t^{-1} - \frac{5}{36}\omega^2t^{-2}\chi(t)^{-4} - \frac{5}{54}i\omega t^{-3}\chi(t)^{-8}\right\}\exp\left(\frac{3}{4}i\omega t\chi(t)^4\right),
\end{aligned}\tag{3.11}$$

and

$$\begin{aligned}
&\partial_x^3\exp\left(\frac{3}{4}i\omega t\chi(t)^4\right)\\
=\;&\left\{-i\omega^3\chi(t)^3 - \omega^2t^{-1}\chi(t)^{-1} - \frac{2}{9}i\omega t^{-2}\chi(t)^{-5}\right\}\exp\left(\frac{3}{4}i\omega t\chi(t)^4\right).
\end{aligned}\tag{3.12}$$

By substituting the equalities (3.11) and (3.12) into the first and second term into the right hand side of (3.10), we obtain

$$\begin{aligned}
&\left(i\omega^3\partial_t - \frac{1}{4}\partial_x^4\right)\exp\left(\frac{3}{4}i\omega t\chi(t)^4\right)\exp\left(if\left(\chi(t)\right)\log t\right)g\left(\chi(t)\right)\\
= &-\omega^3 t^{-1}\left(f\left(\chi(t)\right) - \frac{i}{2}\right)g\left(\chi(t)\right)\exp\left(\frac{3}{4}i\omega t\chi(t)^4\right)\exp\left(if\left(\chi(t)\right)\log t\right)\\
&+\mathcal{R}_g(t,x), \qquad (3.13)
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{R}_g(t,x)\\
= &-\omega^2 t^{-2}\Bigg\{-\frac{1}{3}\chi(t)^{-3}\partial g\left(\chi(t)\right) - \frac{i}{3}\log t\chi(t)^{-3}\partial f\left(\chi(t)\right)g\left(\chi(t)\right)\\
&\qquad +\frac{5}{36}\chi(t)^{-4}g\left(\chi(t)\right)\Bigg\}\exp\left(\frac{3}{4}i\omega t\chi(t)^4\right)\exp\left(if\left(\chi(t)\right)\log t\right)\\
&-i\omega t^{-3}\Bigg\{-\frac{2}{27}\chi(t)^{-7}\partial g\left(\chi(t)\right) - \frac{2}{27}i\log t\chi(t)^{-7}\partial f\left(\chi(t)\right)g\left(\chi(t)\right)\\
&\qquad +\frac{5}{54}\chi(t)^{-8}g\left(\chi(t)\right)\Bigg\}\exp\left(\frac{3}{4}i\omega t\chi(t)^4\right)\exp\left(if\left(\chi(t)\right)\log t\right)\\
&-\frac{1}{4}\sum_{\substack{\ell+m+n=4\\ \ell\neq 3,4}}\frac{4!}{\ell!m!n!}\partial_x^\ell\exp\left(\frac{3}{4}i\omega t\chi(t)^4\right)\\
&\qquad \times\partial_x^m\exp\left(if\left(\chi(t)\right)\log t\right)\partial_x^n g\left(\chi(t)\right). \qquad (3.14)
\end{aligned}$$

By taking the L^2 norm of (3.14) and changing the variables we have (3.9). $\square$

To estimate the right hand side of (3.3), we prepare the following lemma.

Lemma 3.3. *We assume that* $B \in C^1([0,\infty))$ *satisfies* $\operatorname{supp} B(it) \subset [e,\infty)$ *and* $|B(it)| + |t||B'(it)| \leq C(t+1)^{-m-\frac{1}{2}}$, *where* m *is a positive integer. Then for* $t \geq 0$, *we have*

$$\begin{aligned}
&\left\|\int_t^\infty B(i\tau)\exp\left(\frac{3}{4}i\omega\tau\chi(\tau)^4\right)\exp\left(if\left(\chi(\tau)\right)\log\tau\right)A\left(\chi(\tau)\right)d\tau\right\|_{L_x^2}\\
\leq\ & C(t+1)^{-m}\log(t+1)(\||x|^{-3}A\|_{L_x^2} + \||x|^{-2}\partial_x A\|_{L_x^2}\\
&\qquad +\||x|^{-2}\partial_x fA\|_{L_x^2} + \||x|^{-3}fA\|_{L_x^2}), \qquad (3.15)
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\int_t^\infty B(i\tau)\exp\left(\frac{3}{4}i\omega\tau\chi(\tau)^4\right)\exp\left(if\left(\chi(\tau)\right)\log\tau\right)A\left(\chi(\tau)\right)d\tau\right\|_{L_x^\infty}\\
\leq\ & C(t+1)^{-m-\frac{1}{2}}\log(t+1)(\||x|^{-4}A\|_{L_x^\infty} + \||x|^{-3}\partial_x A\|_{L_x^\infty}\\
&\qquad +\||x|^{-3}\partial_x fA\|_{L_x^\infty} + \||x|^{-4}fA\|_{L_x^\infty}). \qquad (3.16)
\end{aligned}$$

Proof of Lemma 3.3. By the identity

$$\exp\left(\frac{3}{4}i\omega\tau\chi(\tau)^4\right) = \frac{1}{1-\frac{1}{4}i\omega\tau\chi(\tau)^4}\partial_\tau\left\{\tau\exp\left(\frac{3}{4}i\omega\tau\chi(\tau)^4\right)\right\},$$

and the integration by parts, we have

$$\begin{aligned}
&\int_t^\infty B(i\tau)\exp\left(\frac{3}{4}i\omega\tau\chi(\tau)^4\right)\exp\left(if\left(\chi(\tau)\right)\log\tau\right)A\left(\chi(\tau)\right)d\tau\\
=&\int_t^\infty \frac{1}{1-\frac{1}{4}i\omega\tau\chi(\tau)^4}B(i\tau)\exp\left(if\left(\chi(\tau)\right)\log\tau\right)A\left(\chi(\tau)\right)\\
&\qquad\qquad\times\partial_\tau\left\{\tau\exp\left(\frac{3}{4}i\omega\tau\chi(\tau)^4\right)\right\}d\tau\\
=&-\frac{t}{1-\frac{1}{4}i\omega t\chi(t)^4}B(it)\exp\left(\frac{3}{4}i\omega t\chi(t)^4\right)\exp\left(if\left(\chi(t)\right)\log t\right)A\left(\chi(t)\right)\\
&-\int_t^\infty\partial_\tau\left\{\frac{1}{1-\frac{1}{4}i\omega\tau\chi(\tau^4)}B(i\tau)\exp\left(if\left(\chi(\tau)\right)\log\tau\right)A\left(\chi(\tau)\right)\right\}\\
&\qquad\qquad\times\tau\exp\left(\frac{3}{4}i\omega\tau\chi(\tau)^4\right)d\tau.
\end{aligned}$$

$$\begin{aligned}
=&-\frac{t}{1-\frac{1}{4}i\omega t\chi(t)^4}B(it)\exp\left(\frac{3}{4}i\omega t\chi(t)^4\right)\exp\left(if\left(\chi(t)\right)\log t\right)A\left(\chi(t)\right)\\
&+\int_t^\infty\frac{1}{1-\frac{1}{4}i\omega\tau\chi(\tau)^4}\exp\left(\frac{3}{4}i\omega\tau\chi(\tau)^4\right)\exp\left(if\left(\chi(\tau)\right)\log\tau\right)\\
&\quad\times\left\{\frac{i}{12}\omega B(i\tau)\frac{\tau\chi(\tau)^4}{1-\frac{1}{4}i\omega\tau\chi(\tau)^4}A\left(\chi(\tau)\right)-i\tau B'(i\tau)A\left(\chi(\tau)\right)\right.\\
&\quad+\frac{i}{3}B(i\tau)\log\tau\chi(\tau)\partial f\left(\chi(\tau)\right)A\left(\chi(\tau)\right)-iB(i\tau)f\left(\chi(\tau)\right)A\left(\chi(\tau)\right)\\
&\quad\left.+\frac{1}{3}B(i\tau)\chi(\tau)\partial A\left(\chi(\tau)\right)\right\}d\tau.
\end{aligned}\tag{3.17}$$

By taking the L^2 norm for both hand sides of (3.17) with respect to x variable, we have

$$\begin{aligned}
&\left\|\int_t^\infty B(i\tau)\exp\left(\frac{3}{4}i\omega\tau\chi(\tau)^4\right)\exp\left(if\left(\chi(\tau)\right)\log\tau\right)A\left(\chi(\tau)\right)d\tau\right\|_{L^2_x}\\
\le\ & C(t+1)^{-m-\frac{1}{2}}\left\||\chi(t)|^{-4}A\left(\chi(t)\right)\right\|_{L^2_x}\\
&+C\int_t^\infty(\tau+1)^{-m-\frac{3}{2}}\log\tau\Big(\left\||\chi(\tau)|^{-4}A\left(\chi(\tau)\right)\right\|_{L^2_x}\\
&\qquad+\left\||\chi(\tau)|^{-3}\partial A\left(\chi(\tau)\right)\right\|_{L^2_x}+\left\||\chi(\tau)|^{-3}\partial f\left(\chi(\tau)\right)A\left(\chi(\tau)\right)\right\|_{L^2_x}\\
&\qquad+\left\||\chi(\tau)|^{-4}f\left(\chi(\tau)\right)A\left(\chi(\tau)\right)\right\|_{L^2_x}\Big)d\tau.
\end{aligned}\tag{3.18}$$

Here we have used the following inequality

$$\left|\frac{1}{1-\frac{1}{4}i\omega t\chi(t)^4}\right| \le Ct^{-1}\left|\chi(t)\right|^{-4}.$$

By changing the variables in the right hand side of (3.18), we obtain (3.15). Similarly, by taking the L^∞ norm for both hand sides of (3.17) with respect to x variable, we have the inequality (3.16). □

3.2. PROOF OF THEOREM 1.1.

Proof of Theorem 1.1. We prove Theorem 1.1 for the positive time. For the negative time, we can treat analogously. We assume that $\phi_+ \in \mathcal{D}$ and $\|\phi_+\|_D$ is sufficiently small. Let v_+ be the function defined by (1.4). Due to Proposition 2.1, it is sufficient to show that $\tilde{v}_+(t,x) = \chi(t)v_+(t,x)$ satisfies (2.3) and (2.4) with small constant $\delta > 0$, where $\chi \in C^\infty([0,\infty))$ with $\chi(t) = 1$ if $t \ge 2e$ and $\chi(t) = 0$ if $0 \le t \le e$. By (3.1) in Proposition 3.1, v_+ satisfies (2.3) with $\delta = C\|\phi_+\|_{\mathcal{D}}$. To check the estimate (2.4), we split the function $\mathcal{R}(v_+, \overline{v}_+)$ (see (2.2)) into two parts:

$$\mathcal{R}(v_+,\overline{v}_+) = \{\mathcal{L}v_+ - \mathcal{N}_g(v_+,\overline{v}_+)\} - \mathcal{N}_{ng}(v_+,\overline{v}_+). \tag{3.19}$$

Hereafter we choose

$$\begin{aligned}
&(\omega, f(x), g(x), h(it))\\
&= \left(3, -i\lambda_0|x|^{-2}|\hat{\phi}_+(x)|^2, |x|^{-3}\hat{\phi}_+(x)^3, \psi(t)^3(it)^{-\frac{3}{2}}\right)\\
&\quad \left(-1, \frac{i\lambda_0}{3}|x|^{-2}|\hat{\phi}_+(x)|^2, |x|^{-3}|\hat{\phi}_+(x)|^2\overline{\hat{\phi}_+(x)}, \psi(t)^3(it)^{-\frac{3}{2}}\right)\\
\text{or}&\quad \left(-3, i\lambda_0|x|^{-2}|\hat{\phi}_+(x)|^2, |x|^{-3}\overline{\hat{\phi}_+(x)}^3, \psi(t)^3(it)^{-\frac{3}{2}}\right).
\end{aligned} \tag{3.20}$$

The estimate for the first term in the right hand side of (3.19) is obtained by (3.2) in Proposition 3.1. In the rest of this section, we give the estimate for the second term of (3.19).

By Lemma 3.1, we have

$$\begin{aligned}
&\int_t^\infty W(t-\tau)h(i\tau)\exp\left(\frac{3}{4}i\omega\tau\chi(\tau)^4\right)\exp\left(if\left(\chi(\tau)\right)\log\tau\right)g\left(\chi(\tau)\right)d\tau\\
= &\frac{1}{1-\omega^3}\int_t^\infty h(i\tau)\exp\left(\frac{3}{4}i\omega\tau\chi(\tau)^4\right)\exp\left(if\left(\chi(\tau)\right)\log\tau\right)g\left(\chi(\tau)\right)d\tau\\
&+\frac{i\omega^3}{1-\omega^3}\int_t^\infty W(t-\tau)\int_\tau^\infty h(i\tau')\tau'^{-1}\exp\left(\frac{3}{4}i\omega\tau'\chi(\tau')^4\right)\\
&\qquad\times\exp\left(if\left(\chi(\tau')\right)\log\tau'\right)\left(f\left(\chi(\tau')\right)-\frac{i}{2}\right)g\left(\chi(\tau')\right)d\tau'd\tau
\end{aligned}$$

$$
\begin{aligned}
&+\frac{i\omega^3}{1-\omega^3}\int_t^\infty W(t-\tau)\int_\tau^\infty h'(i\tau') \\
&\qquad\times \exp\left(\frac{3}{4}i\omega\tau'\chi(\tau')^4\right)\exp\left(if\left(\chi(\tau')\right)\log\tau'\right)g\left(\chi(\tau')\right)d\tau' d\tau \\
&-\frac{i}{1-\omega^3}\int_t^\infty W(t-\tau)\int_\tau^\infty h(i\tau')\mathcal{R}_g(\tau',x)d\tau' d\tau \\
\equiv\ & J_1(t,x)+J_2(t,x)+J_3(t,x)+J_4(t,x).
\end{aligned}
\tag{3.21}
$$

By Lemma 3.3 with $(A(x),B(it))=(g(x),\chi(t)^3(it)^{-\frac{3}{2}})$ (hence $m=1$), we have

$$
\begin{aligned}
&\|J_1(t)\|_{L^2_x}+\|J_1(\tau)\|_{L^8_\tau(t,\infty,L^\infty_x)} \\
&\leq\ C(t+1)^{-1}\log(t+1)(\||x|^{-3}g\|_{L^2_x}+\||x|^{-2}\partial_x g\|_{L^2_x} \\
&\qquad +\||x|^{-2}\partial_x fg\|_{L^2_x}+\||x|^{-3}fg\|_{L^2_x}+\||x|^{-4}g\|_{L^\infty_x} \\
&\qquad +\||x|^{-3}\partial_x g\|_{L^\infty_x}+\||x|^{-3}\partial_x fg\|_{L^\infty_x}+\||x|^{-4}fg\|_{L^\infty_x}).
\end{aligned}
\tag{3.22}
$$

By the Strichartz estimate (Lemma 2.1) and (3.15) in Lemma 3.3 with $(A(x),B(it))=((f(x)-\frac{i}{2})g(x),h(it)t^{-1})$ or $(g(x),h'(it))$ (hence $m=2$), we obtain

$$
\begin{aligned}
&\sum_{j=2}^{3}(\|J_j(t)\|_{L^2_x}+\|J_j(\tau)\|_{L^8_\tau(t,\infty,L^\infty_x)})\leq C\int_t^\infty\left\|\int_\tau^\infty h(i\tau')\tau'^{-1}\exp\left(\frac{3}{4}i\omega\tau'\chi(\tau')^4\right)\right. \\
&\quad\left.\times\exp\left(if\left(\chi(\tau')\right)\log\tau'\right)\left(f\left(\chi(\tau')\right)-\frac{i}{2}\right)g\left(\chi(\tau')\right)d\tau'\right\|_{L^2_x}d\tau \\
&+C\int_t^\infty\left\|\int_\tau^\infty h'(i\tau')\exp\left(\frac{3}{4}i\omega\tau'\chi(\tau')^4\right)\right. \\
&\quad\left.\times\exp\left(if\left(\chi(\tau')\right)\log\tau'\right)g\left(\chi(\tau')\right)d\tau'\right\|_{L^2_x}d\tau \\
\leq\ & C(t+1)^{-1}\log(t+1) \\
&\quad\times\sum_{\substack{G(x)=(f-\frac{i}{2})g \\ or\ \ g}}(\||x|^{-3}G\|_{L^2_x}+\||x|^{-2}\partial_x G\|_{L^2_x} \\
&\qquad +\||x|^{-2}\partial_x fG\|_{L^2_x}+\||x|^{-3}fG\|_{L^2_x}).
\end{aligned}
\tag{3.23}
$$

Finally, by the Strichartz estimate (Lemma 2.1) and Lemma 3.2, we have

$$
\begin{aligned}
&\leq\ C\int_t^\infty\left\|\int_\tau^\infty h(i\tau')\mathcal{R}_g(\tau')d\tau'\right\|_{L^2_x}d\tau \\
&\leq\ C\int_t^\infty\int_\tau^\infty|h(i\tau')|\|\mathcal{R}_g(\tau')\|_{L^2_x}d\tau' d\tau \\
&\leq\ C(t+1)^{-1}(\log(t+1))^4\mathcal{M}(f,g).
\end{aligned}
\tag{3.24}
$$

By collecting (3.21)–(3.24), we obtain

$$
\begin{aligned}
& \left\| \int_t^\infty W(t-\tau)h(i\tau)\exp\left(\frac{3}{4}i\omega\tau\chi(\tau)^4\right) \right. \\
& \qquad\qquad\qquad \left. \times \exp\left(if\left(\chi(\tau)\right)\log\tau\right) g\left(\chi(\tau)\right) d\tau \right\|_{L_x^2} \\
+ & \left\| \int_\tau^\infty W(\tau-\tau')h(i\tau')\exp\left(\frac{3}{4}i\omega\tau'\chi(\tau')^4\right) \right. \\
& \qquad\qquad\qquad \left. \times \exp\left(if\left(\chi(\tau')\right)\log\tau'\right) g\left(\chi(\tau')\right) d\tau' \right\|_{L_\tau^8(t,\infty,L_x^\infty)} \\
\leq & \; C(t+1)^{-1}(\log(t+1))^4 \Big\{ \||x|^{-4}g\|_{L_x^\infty} + \||x|^{-3}\partial_x g\|_{L_x^\infty} \\
& \qquad + \||x|^{-3}\partial_x fg\|_{L_x^\infty} + \||x|^{-4}fg\|_{L_x^\infty} \\
& \qquad + \sum_{\substack{G(x)=(f-\frac{i}{2})g \\ or\ g}} (\||x|^{-3}G\|_{L_x^2} + \||x|^{-2}\partial_x G\|_{L_x^2} \\
& \qquad + \||x|^{-2}\partial_x fG\|_{L_x^2} + \||x|^{-3}fG\|_{L_x^2}) + \mathcal{M}(f,g) \Big\},
\end{aligned} \tag{3.25}
$$

for $t \geq 0$, where $\mathcal{M}(f,g)$ is defined by (3.8). When f and g are as in (3.20), by the Hölder inequality and the Sobolev embedding, the most right hand side of (3.25) is bunded by $P(\|\phi_+\|_{\mathcal{D}})$, where $P(x)$ is a polynomial in x. By substituting (3.20) into (3.25) and summing up the results we have

$$
\begin{aligned}
& \left\| \int_t^\infty W(t-\tau)\mathcal{N}_{ng}(\tilde{v}_+, \overline{\tilde{v}}_+)(\tau) d\tau \right\|_{L_x^2} \\
& \qquad + \left\| \int_\tau^\infty W(\tau-\tau')\mathcal{N}_{ng}(\tilde{v}_+, \overline{\tilde{v}}_+)(\tau') d\tau' \right\|_{L_\tau^8(t,\infty,L_x^\infty)} \\
\leq & \; Ct^{-1}(\log t)^4 P(\|\phi\|_{\mathcal{D}}).
\end{aligned} \tag{3.26}
$$

By (3.2) in Proposition 3.1 and (3.26), v_+ satisfy (2.4). Hence we obtain the desired result. □

ACKNOWLEDGMENTS

The second author is partially supported by MEXT, Grant-in-Aid for Young Scientists (B) 18740076.

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