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ASYMPTOTIC BEHAVIOR OF OSCILLATORY SOLUTIONS OF n-TH ORDER DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT DEPENDING ON THE UNKNOWN FUNCTION

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ABSTRACT: The paper deals with damped oscillations of the n-th order forced differential equation

$$D_r^{(n)}x(t) + f(t, x(t), x(\Delta(t, x(t)))) = b(t)\,.$$

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1. INTRODUCTION

Consider the n-th order differential equations

$$D_r^{(n)}x(t) + f(t, x(t), x(\Delta(t, x(t)))) = b(t)\,, \qquad t \in J \tag{1}$$

and

$$D_r^{(n)}x(t) + a(t)h(x(\Delta(t, x(t)))) = b(t)\,, \qquad t \in J \tag{2}$$

with deviating argument Δ which depends on the independent variable t as well as on the unknown function x.

Here $n \geq 1$ is an integer, $t \in J = [\alpha, +\infty) \subseteq [0, +\infty) = \mathbb{R}_+$, $a, b : J \to \mathbb{R}$, $f : J \times \mathbb{R}^2 \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$,

$$D_r^{(0)}x(t) = x(t)\,, \qquad D_r^{(i)}x(t) = r(t)(D_r^{(i-1)}x(t))'\,, \qquad i = 1, \ldots, n\,,$$

where $r_i : J \to (0, +\infty)$.

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We notice that some initial oscillation and asymptotic results concerning equation (1) are obtained in the papers of Bainov and Simeonov [1], Bainov et al [2], [3], [4] and Markova and Simeonov [10], [11], [12].

The main results of this paper are stated in eleven theorems. In Theorems 1-7 and Corollaries 1 and 2 sufficient conditions are found under which

$$\lim_{t\to+\infty} D_r^{(k)}x(t) = 0\,, \qquad k = 0, 1, \ldots, n-1 \tag{3}$$

for every oscillatory solution $x(t)$ of equation (1) having a given growth at infinity.

Theorem 8 provides sufficient conditions which garantee that every solution of equation (1) satisfying (3) is nonoscillatory.

In Theorem 9 and Corollaries 3 and 4 the growth of the solutions of equation (2) is estimated in the case when this equation is of retarded type ($\Delta \le t$).

In Theorems 10 and 11 necessary and sufficient conditions are obtained so that all oscillatory solutions of equation (2) satisfy (3).

The main theorems generalize and extend results of Grace and Lalli [6], Greaf et al [7], Kusano and Onose [8], [9] and Singh [13], [14] concerning differential equations with deviating argument Δ which does not depend on x ($\Delta = g(t)$).

2. PRELIMINARY REMARKS

Introduce the following conditions:

H1. $f \in C(J \times \mathbb{R}^2, \mathbb{R})$ and there exists $F \in C(J \times \mathbb{R}_+^2, \mathbb{R}_+)$ such that

$$|f(t, x, y)| \le F(t, |x|, |y|)\,, \qquad t \in J\,, \quad x, y \in \mathbb{R}$$

and

$$F(t, u_1, v_1) \le F(t, u_2, v_2) \quad \text{for} \quad 0 \le u_1 \le u_2 \quad \text{and} \quad 0 \le v_1 \le v_2\,.$$

H2. $r_k \in C(J, (0, +\infty))$, $k = 1, \ldots, n-1$ and $r_n(t) \equiv 1$, $t \in J$.

H3. $b \in C(J, \mathbb{R})$.

H4. $\Delta \in C(J \times \mathbb{R}, \mathbb{R})$.

H5. There exist $\sigma \in C(J, \mathbb{R})$ and $T \in J$ such that

$$\lim_{t\to+\infty} \sigma(t) = +\infty \quad \text{and} \quad \sigma(t) \le \Delta(t, x)\,, \quad t \ge T\,, \quad x \in \mathbb{R}\,.$$

H6. There exist $\tau \in C(J, \mathbb{R})$ and $T \in J$ such that

$$\Delta(t, x) \le \tau(t)\,, \qquad t \ge T\,, \quad x \in \mathbb{R}\,.$$

The domain $\mathcal{D}$ of $D_r^{(n)}$ is defined to be the set of all functions $x : [t_x, +\infty) \to \mathbb{R}$ such that the r-derivatives $D_r^{(k)}x(t)$, $k = 1, \ldots, n$ exist and are continuous on the interval $[t_x, +\infty) \subseteq J$. By a *proper* solution of equation (1) is meant a function $x \in \mathcal{D}$ which satisfies (1) for all sufficiently large t and $\sup\{|x(t)| : t \ge T\} > 0$ for $T \ge t_x$. We assume that equation (1) do possess proper solurions. A proper solution of equation (1) is called *oscillatory* if it has arbitrarily large zeros; otherwise it is called *nonoscillatory.*

3. MAIN RESULTS

Theorem 1. *Assume that:*

1. *Conditions H1-H6 hold and*

$$\int^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |b(s)| ds \dots ds_1 < +\infty \,. \tag{4}$$

2. *There exists a continuous nondecreasing function* $\mu : J \to \mathbb{R}_+$ *such that*

$$\int^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} F(s, c\mu(s), c\mu(\tau(s))) ds \dots ds_1 < +\infty \tag{5}$$

for all $c > 0$.

Then every oscillatory solution $x(t)$ *of equation* (1) *with* $x(t) = O(\mu(t))$ *as* $t \to +\infty$ *satisfies* (3).

Proof. Let $x(t)$ be an oscillatory solution of equation (1) with $x(t) = O(\mu(t))$ as $t \to +\infty$. Then there exist constants $c > 0$ and $T \geq \alpha$ such that $|x(t)| \leq c\mu(t)$ for $t \geq T$. Since $\mu(t)$ is nondecreasing in J it follows from conditions H5 and H6 that

$$|x(\Delta(t, x(t)))| \leq c\mu(\Delta(t, x(t))) \leq c\mu(\tau(t)) \quad \text{for} \quad t \geq T \,. \tag{6}$$

Since $x(t)$ is oscillatory, $D_r^{(k)} x(t)$ is oscillatory for $k = 1, \dots, n-1$. Let $\{t_m\}_{m=1}^{\infty}$ be a sequence of consecutive zeros of $D_r^{(n-1)} x(t)$ and $\beta_m \in (t_m, t_{m+1})$ be such that

$$|D_r^{(n-1)} x(\beta_m)| = \max_{t_m \leq t \leq t_{m+1}} |D_r^{(n-1)} x(t)| \,.$$

Integrating (1) from t_m to β_m we obtain

$$D_r^{(n-1)} x(\beta_m) - D_r^{(n-1)} x(t_m) = -\int_{t_m}^{\beta_m} f(s, x(s), x(\Delta(s, x(s)))) ds + \int_{t_m}^{\beta_m} b(s) ds \,,$$

which, together with (6) and condition H1, gives

$$|D_r^{(n-1)} x(\beta_m)| \leq \int_{t_m}^{\beta_m} F(s, c\mu(s), c\mu(\tau(s))) ds + \int_{t_m}^{\beta_m} |b(s)| ds \,.$$

Summing on m we have

$$\sum_{m=1}^{\infty} |D_r^{(n-1)} x(\beta_m)| \leq \int_{t_1}^{\infty} F(s, \mu(s), \mu(\tau(s))) ds + \int_{t_1}^{\infty} |b(s)| ds < +\infty \,.$$

Consequently $\lim_{m \to +\infty} D_r^{(n-1)} x(\beta_m) = 0$ which implies that $\lim_{t \to +\infty} D_r^{(n-1)} x(t) = 0$. Integrating (1) from t to $+\infty$ we obtain

$$D_r^{(n-1)} x(t) = \int_t^{\infty} f(s, x(s), x(\Delta(s, x(s)))) ds - \int_t^{\infty} b(s) ds \,. \tag{7}$$

We shall prove that $\lim_{t \to +\infty} D_r^{(n-2)} x(t) = 0$.

Let $\{z_m\}_{m-1}^{\infty}$ be a sequence of consecutive zeros of $D_r^{(n-2)}x(t)$ and $\gamma_m \in (z_m, z_{m+1})$ be such that

$$|D_r^{(n-2)}x(\gamma_m)| = \max_{z_m \le t \le z_{m+1}} |D_r^{(n-2)}x(t)| \,.$$

Integrating (7) from z_m to γ_m we obtain

$$D_r^{(n-2)}x(\gamma_m) = \int_{z_m}^{\gamma_m} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} f(s, x(s), x(\Delta(s, x(s))))dsds_{n-1} - \int_{z_m}^{\gamma_m} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} b(s)dsds_{n-1} \,,$$

which implies

$$|D_r^{(n-2)}x(\gamma_m)| \le \int_{z_m}^{\gamma_m} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} F(s, \mu(s), \mu(\tau(s)))dsds_{n-1} + \int_{z_m}^{\gamma_m} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |b(s)|dsds_{n-1} \,.$$

Summing on m we have

$$\sum_{m=1}^{\infty} |D_r^{(n-2)}x(\gamma_m)| \le \int_{z_1}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} F(s, \mu(s), \mu(\tau(s)))dsds_{n-1} + \int_{z_1}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |b(s)|dsds_{n-1} < +\infty \,.$$

Therefore $\lim_{t\to+\infty} D_r^{(n-2)}x(t) = 0$. Integrating (7) from t to $+\infty$ we obtain

$$D_r^{(n-2)}x(t) = -\int_{t}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} f(s, x(s), x(\Delta(s, x(s))))dsds_{n-1} + \int_{t}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} b(s)dsds_{n-1} \,.$$

Continuing the process we deduce that $\lim_{t\to+\infty} D_r^{(k)}(t) = 0$, $k = 0, 1, \ldots, n-1$. $\square$

As a consequence of Theorem 1 we obtain the following two theorems.

Theorem 2. *Assume that:*

1. *Conditions H1-H6 hold and*

$$\int^{\infty} |b(s)|ds < +\infty \,, \qquad \int^{\infty} \frac{1}{r_i(s)}ds < +\infty \,, \qquad i = 1, \ldots, n-1 \,. \tag{8}$$

2. *There exists a continuous and nondecreasing function $\mu : J \to \mathbb{R}_+$ such that*

$$\int^{\infty} F(s, c\mu(s), c\mu(\tau(s)))ds < +\infty \quad \text{for all} \quad c > 0 \,. \tag{9}$$

Then every oscillatory solution $x(t)$ of equation (1) *with $x(t) = O(\mu(t))$ as $t \to +\infty$ satisfies* (3).

Theorem 3. *Assume that:*

1. *Conditions H1-H6 hold and*

$$\int^{\infty} s^{n-1}|b(s)|ds < +\infty\,, \quad r_i(t) \geq r_0 > 0\,,\ i = 1,\dots,n-1\,,\ t \in J\,. \tag{10}$$

2. *There exists a continuous and nondecreasing function $\mu : J \to \mathbb{R}_+$ such that*

$$\int^{\infty} s^{n-1}F(s,\mu(s),\mu(\tau(s)))ds < +\infty \qquad \text{for all} \qquad c > 0\,. \tag{11}$$

Then for every oscillatory solution $x(t)$ of equation (1) with $x(t) = O(\mu(t))$ as $t \to +\infty$ satisfies (3).

Proceeding as in the proof of Theorem 1 one can prove the following theorem.

Theorem 4. *Assume that:*

1. *Conditions H1-H5 and (4) hold.*
2. *There exists a continuous and nonincreasing function $\lambda : J \to \mathbb{R}_+$ such that*

$$\int^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-1}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} F(s,c\lambda(s),c\lambda(\sigma(s)))ds\dots ds_1 < +\infty \tag{12}$$

for all $c > 0$.

Then every oscillatory solution $x(t)$ of equation (1) with $x(t) = O(\lambda(t))$ as $t \to +\infty$ satisfies (3).

Proof. Let $x(t)$ be an oscillatory solution of equation (1) with $x(t) = O(\lambda(t))$ as $t \to +\infty$. Then there exist constants $c > 0$ and $T \geq \alpha$ such that

$$|x(t)| \leq c\lambda(t)\,, \qquad t \geq T\,. \tag{13}$$

Since $\lambda(t)$ is nonincreasing in J it follows from (13) and condition H5 that

$$|x(\Delta(t,x(t)))| \leq c\lambda(\Delta(t,x(t))) \leq c\lambda(\sigma(t))\,, \qquad t \geq T\,.$$

Further the proof is the same as the proof of Theorem 1. □

As a consequence of Theorem 4 we obtain the following two theorems.

Theorem 5. *Assume that:*

1. *Conditions H1-H5 and (8) hold.*
2. *There exists a continuous and nonincreasing function $\lambda : J \to \mathbb{R}_+$ such that*

$$\int^{\infty} F(s,c\lambda(s),c\lambda(\sigma(s)))ds < +\infty \qquad \text{for all} \quad c > 0\,. \tag{14}$$

Then every oscillatory solution $x(t)$ of equation (1) with $x(t) = O(\lambda(t))$ as $t \to +\infty$ satisfies (3).

Theorem 6. *Assume that:*

1. *Conditions H1-H5 and* (10) *hold.*
2. *There exists a continuous and nonincreasing function* $\lambda : J \to \mathbb{R}_+$ *such that*

$$\int^{\infty} s^{n-1} F(s, c\lambda(s), c\lambda(\sigma(s)))ds < +\infty \qquad \text{for all} \qquad c > 0\,. \tag{15}$$

Then every oscillatory solution $x(t)$ *of equation* (1) *with* $x(t) = O(\lambda(t))$ *as* $t \to +\infty$ *satisfies* (3).

From Theorem 4 with $\lambda(t) \equiv 1$ it follows the next theorem.

Theorem 7. *Assume that conditions H1-H5 and* (4) *hold and*

$$\int^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-1}}^{\infty} F(s, c, c)ds \ldots ds_1 < +\infty \qquad \text{for all} \qquad c > 0\,. \tag{16}$$

Then every bounded oscillatory solution of equation (1) *satisfies* (3).

Corollary 1. *Assume that conditions H1-H5 and* (8) *hold and*

$$\int^{\infty} F(s, c, c)ds < +\infty \qquad \text{for all} \qquad c > 0\,. \tag{17}$$

Then every bounded oscillatory solution of equation (1) *satisfies* (3).

Corollary 2. *Assume that conditions H1-H5 and* (10) *hold and*

$$\int^{\infty} s^{n-1} F(s, c, c)ds < +\infty \qquad \text{for all} \qquad c > 0\,. \tag{18}$$

Then every bounded oscillatory solution of equation (1) *satisfies* (3).

Theorem 8. *Assume that:*

1. *Conditions H1-H5 hold.*
2. *There exists a* $c_0 > 0$ *such that either*

$$\lim_{t\to+\infty} \inf \int_T^t [b(s) - F(s, c_0, c_0)]ds > 0\,, \tag{19}$$

or

$$\lim_{t\to+\infty} \sup \int_T^t [b(s) + F(s, c_0, c_0)]ds < 0 \tag{20}$$

for all large T.

Then every solution $x(t)$ *of equation* (1) *satisfying* (3) *is nonoscillatory.*

Proof. Assume the opposite, that there exists an oscillatory solution $x(t)$ of equation (1) such that $\lim_{t\to+\infty} D_r^{(k)} x(t) = 0$, $k = 0, 1, \ldots, n-1$. Then there exists a $T \geq \alpha$ such that

$$D_r^{(n-1)} x(T) = 0\,, \qquad |x(t)| \leq c_0 \quad \text{and} \quad |x(\Delta(t, x(t)))| \leq c_0\,, \quad t \geq T\,.$$

From condition H1 it follows the estimate

$$|f(t,x(t),x(\Delta(t,x(t))))| \le F(t,c_0,c_0)\,, \qquad t \ge T$$

which implies that

$$b(t) - F(t,c_0,c_0) \le b(t) - f(t,x(t),x(\Delta(t,x(t)))) \le b(t) + F(t,c_0,c_0)$$

and

$$b(t) - F(t,c_0,c_0) \le D_r^{(n)}x(t) \le b(t) + F(t,c_0,c_0) \tag{21}$$

for $t \ge T$.

Integrating (21) from T to t we obtain

$$\int_T^t [b(s) - F(s,c_0,c_0)]ds \le D_r^{(n-1)}(t) \le \int_T^t [b(s) + F(s,c_0,c_0)]ds\,, \qquad t \ge T\,.$$

Hence if either (19) or (20) holds, $x(t)$ cannot have arbitrarily large zeros, which is a contradiction. □

Now applying some of the above results to equation (2) we obtain necessary and sufficient conditions so that all oscillatory solutions of equation (2) satisfy (3).

Introduce the functions $R_k(t,T)$, $k = 0,\ldots,n-1$ in the interval $[T,+\infty) \subseteq J$ as follows:

$$R_k(t,T) = \begin{cases} 1\,, & \text{if} \quad k = 0\,, \\ \displaystyle\int_T^t \frac{1}{r_1(s_1)} \int_T^{s_1} \frac{1}{r_2(s_2)} \cdots \int_T^{s_{k-1}} \frac{1}{r_k(s_k)} ds_k \ldots ds_1\,, & \text{if} \quad k > 0\,. \end{cases}$$

Set $R_k(t) = R_k(t,\alpha)$, $k = 0,\ldots,n-1$ and $R(t) = R_{n-1}(t)$.

Introduce the following conditions:

H7. $a \in C(J,\mathbb{R})$.

H8. $h \in C(\mathbb{R},\mathbb{R})$ and there exists a function $H \in C(\mathbb{R}_+,\mathbb{R}_+)$ which is nondecreasing in $\mathbb{R}_+$ and such that

$$|h(x)| \le H(x)\,, \qquad H(xy) \le H(x)H(y) \qquad \text{for} \ \ x > 0\,, \quad y > 0\,, \quad H(0) = 0$$

and

$$\int_{x_0}^x \frac{du}{H(u)} \to +\infty \quad \text{as} \quad x \to +\infty\,, \quad x \ge x_0 > 0\,.$$

H9. There exist $\tau \in C(J,\mathbb{R})$ and $T \in J$ such that

$$\Delta(t,x) \le \tau(t) < t\,, \qquad t \ge T\,, \quad x \in \mathbb{R}\,.$$

H10. $\displaystyle\limsup_{t\to+\infty} \frac{R_k(t)}{R(t)} < +\infty$, $k = 0,1,\ldots,n-2$.

Theorem 9. *Assume that conditions H2-H5 and H7-H10 hold and*

$$\begin{aligned} &\int^{\infty} |b(s)| ds < +\infty \,, \\ &\int^{\infty} |a(s)| H(R(\tau(s))) ds < +\infty \,. \end{aligned} \tag{22}$$

Then every proper solution $x(t)$ of equation (2) *is such that*

$$x(t) = O(R(t)) \quad as \quad t \to +\infty.$$

Proof. Let $x(t)$, $t \geq T_0 \geq \alpha$ be a proper solution of equation (2). From condition H5 it follows that there exists a $T \geq T_0$ such that $\Delta(t, x(t)) \geq \sigma(t) \geq T_0$, $t \geq T$. From (2) we obtain that

$$\begin{aligned} x(t) = &\sum_{k=0}^{n-1} D_r^{(k)} x(T) R_k(t, T) \\ &+ \int_T^t \frac{1}{r_1(s_1)} \int_T^{s_1} \frac{1}{r_2(s_2)} \cdots \int_T^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_T^{s_{n-1}} D_r^{(n)} x(s) ds ds_{n-1} \dots ds_1 \end{aligned}$$

for $t \geq T$, which implies

$$|x(t)| \leq \sum_{k=0}^{n-1} |D_r^{(k)} x(T)| R_k(t) + R(t) \int_T^t |D_r^{(n)} x(s)| ds \,, \qquad t \geq T \,.$$

From condition H10 it follows that there exists a $c > 0$ such that

$$\sum_{k=0}^{n-1} |D_r^{(k)} x(T)| \frac{R_k(t)}{R(t)} \leq c \,, \qquad t \geq T \,.$$

Then

$$\frac{|x(t)|}{R(t)} \leq c + \int_T^t |D_r^{(n)} x(s)| ds \,, \qquad t \geq T \,. \tag{23}$$

Choose $\beta \geq T$ such that $\Delta(t, x(t)) \geq \sigma(t) \geq T$ for $t \geq \beta$. Then from (24) we get

$$\frac{|x(\Delta(t, x(t)))|}{R(\Delta(t, x(t)))} \leq c + \int_T^{\beta} |D_r^{(n)} x(s)| ds + \int_{\beta}^{\Delta(t, x(t))} |D_r^{(n)} x(s)| ds \,, \quad t \geq \beta \,. \tag{24}$$

Set

$$u(t) = \frac{|x(\Delta(t, x(t)))|}{R(\Delta(t, x(t)))} \,, \qquad c_0 = c + \int_T^{\beta} |D_r^{(n)} x(s)| ds + \int_{\beta}^{\infty} |b(s)| ds \,.$$

Then keeping in mind (25) and the inequalities

$$\begin{gathered} \Delta(t, x(t)) \leq t \,, \qquad R(\Delta(s, x(s))) \leq R(\tau(s)) \,, \\ |D_r^{(n)} x(s)| \leq |b(s)| + |a(s)| H(R(\tau(s))) H(u(s)) \,, \end{gathered}$$

we obtain the Bihari-type inequality

$$u(t) \leq c_0 + \int_{\beta}^{t} |a(s)| H(R(\tau(s))) H(u(s)) ds \,, \qquad t \geq \beta \,. \tag{25}$$

Applying to (26) the Bihari's Lemma [5] we conclude that

$$u(t) \leq G^{-1}\left[G(c_0) + \int_\beta^t |a(s)|H(R(\tau(s)))ds\right], \qquad t \geq \beta\,, \tag{26}$$

where $G(u) = \int_{u_0}^u \frac{ds}{H(s)}$, $u \geq u_0 > 0$.

From (27) and condition (23) it follows that

$$\frac{|x(\Delta(t, x(t)))|}{R(\Delta(t, x(t)))} \leq M \qquad \text{for} \quad t \geq \beta \quad \text{and some} \quad M > 0\,.$$

This means that $x(t) = O(R(t))$ as $t \to +\infty$ since $\Delta(t, x(t)) \to +\infty$ as $t \to +\infty$. □

Corollary 3. *Assume that conditions H2-H5, H7-H10 and* (22) *hold, and*

$$\begin{gathered} r_i(t) \geq r_0 > 0\,, \qquad i = 1, \dots, n-1\,, \qquad t \in J\,, \\ \int^\infty |a(s)|H(\tau^{n-1}(s))ds < +\infty\,. \end{gathered} \tag{27}$$

Then every proper solution $x(t)$ of equation (2) *is such that*

$$x(t) = O(t^{n-1}) \quad \textit{as} \quad t \to +\infty.$$

Proof. From (28) we obtain that $R(t) = O(t^{n-1})$ as $t \to +\infty$ and now Corollary 3 follows from Theorem 9. □

Corollary 4. *Assume that conditions H2-H5, H7-H10,* (22) *and* (28) *hold, and*

$$\int^\infty |a(s)|ds < +\infty\,, \tag{28}$$

$$\lim_{t\to+\infty} R(t) < +\infty\,. \tag{29}$$

Then every proper solution of equation (2) *is bounded.*

Proof. Let $x(t)$ be a proper solution of equation (2). It follows from (31) that $R(t) = O(M)$ as $t \to +\infty$ for some $M > 0$. Then by Theorem 9 $x(t) = O(M)$ as $t \to +\infty$, which means that $x(t)$ is bounded. □

Consider the differential equation

$$D_r^{(n)}x(t) + a(t)h(x(\tau(t))) = b(t)\,, \qquad t \in J\,, \tag{30}$$

which is a paricular case of equation (2) with $\Delta = \tau(t)$.

Remark 1. The assertation of Theorem 9 from Grace and Lalli [6] is that all oscillatory solutions of equation (32) are bounded, if conditions H3, H7, H8, (22), (28) and (30) hold,

$$\tau \in C(J, \mathbb{R})\,, \quad \lim_{t\to+\infty} \tau(t) = +\infty\,, \quad 0 < \tau(t) \leq t\,, \quad t \in J \tag{31}$$

and

$$\frac{1}{r_1(t)} = O\left(\frac{1}{t^{n-\gamma}}\right) \quad \text{as} \quad t \to +\infty \quad \text{for some} \quad \gamma \in [0, 1)\,. \tag{32}$$

Obviously, our Corollary 4 (with $\Delta = \tau(t)$) includes Theorem 9 from Grace and Lalli [6]. Moreover, Corollary 4 is applicable in cases, when the same Theorem 9 does not work.

Example 1. Consider the equation

$$(t^{\frac{2}{3}}(t^{\frac{5}{3}}x'(t))')' + a(t)x(\tau(t)) = b(t)\,, \qquad t \geq 1\,, \tag{33}$$

where $a, b, \tau \in C(J, \mathbb{R})$ satisfy conditions (30), (22) and (33, respectively.

Here $n = 3$, $r_1(t) = t^{\frac{5}{3}}$, $r_2(t) = t^{\frac{2}{3}}$ and $\gamma = 3 - \frac{5}{3} = \frac{4}{3} > 1$, that is, condition (34) is violated and Theorem 9 from Grace and Lalli [6] is not applicable to equation (35).

On the other hand, the conditions H2-H5, H7-H9, (22), (28) and (30) of Corollary 4 hold. Moreover,

$$R_1(t) = \int_1^t \frac{1}{r_1(s)} ds = \int_1^\infty \frac{1}{s^{\frac{5}{3}}} ds < +\infty$$

and

$$R(t) = R_2(t) = \int_1^t \frac{1}{s^{\frac{5}{3}}} \int_1^s \frac{1}{u^{\frac{2}{3}}} du ds \leq \int_1^t \frac{1}{s^{\frac{5}{3}}} 3s^{\frac{1}{3}} ds < +\infty\,.$$

Hence the rest conditions H10 and (31) hold and therefore by Corollary 4 every proper solution of equation (35) is bounded.

Theorem 10. *Assume that conditions H2-H5, H7-H10,* (22) *and* (23) *hold,* $a(t) \neq 0$ *for* $t \in J$, *and*

$$\int^\infty R(s)H(R(\tau(s)))|a(s)|ds < +\infty\,. \tag{34}$$

Furthermore, suppose that $b(t)/H(R(\tau(t)))a(t)$ *approaches a finite limit as* $t \to +\infty$.

Then every oscillatory solution $x(t)$ *of equation* (2) *satisfies* (3) *if and only if*

$$\lim_{t \to +\infty} \frac{b(t)}{H(R(\tau(t)))a(t)} = 0\,. \tag{35}$$

Proof. 1. Let (37) hold. Then $R(t)|b(t)| \leq R(t)H(R(\tau(t)))|a(t)|$ for all sufficiently large t. This together with (36) implies

$$\int^\infty R(t)|b(t)| < +\infty$$

and the conclusion follows from Theorem 9 and Theorem 1 (with $F(t, u, v) = a(t)H(v)$ and $\mu(t) = R(t)$).

2. Let $x(t)$ be an oscillatory solution of equation (2) satisfying (3). Assume that (37) is not true, that is,

$$\frac{|b(t)|}{H(R(\tau(t)))|a(t)|} \geq \gamma > 0$$

for all sufficiently large t. Dividing (2) by $a(t)$ and taking the limit as $t \to +\infty$ we conclude that $D_r^{(n)}x(t)$ has one sign for sufficiently large t. Hence $x(t)$ has a constant sign eventually, which is a contradiction. $\square$

Theorem 11. *Assume that conditions H2-H5, H7-H10,* (28) *and* (31) *hold,* $a(t) \neq 0$ *for* $t \in J$*, and*

$$\int^{\infty} s^{n-1}|a(s)|ds < +\infty\,. \tag{36}$$

Furthermore, suppose that $b(t)/a(t)$ *approaches a finite limit as* $t \to +\infty$.

Then every oscillatory solution $x(t)$ *of equation* (2) *satisfies* (3) *if and only if*

$$\lim_{t \to +\infty} \frac{b(t)}{a(t)} = 0\,. \tag{37}$$

Proof. Theorem 11 can be proved proceeding as in the proof of Theorem 10 and by using Corollary 4 and Corollary 2 (with $F(t,u,v) = a(t)H(v)$). We omit the details. □

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