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HIGHER ORDER OF CONVERGENCE VIA GENERALIZED QUASILINEARIZATION METHOD FOR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. The method of quasilinearization has been generalized so that it is applicable to a wide variety of nonlinear problems. This is known as Generalized Quasulinearization Method (GQM method for short). It has all the advantages of the quasilinearization method such as linear iterates and quadratic convergence. Also it has been developed with a weaker condition than asking for the convexity or concavity of the original qasilineaization method. In this work we will focus on the mathematical models which leads to nonlinear parabolic integro-differential equations. These mathematical models are motivated by population models in biology and the Hodgkin-Huxley model in medicine. We consider the situation when the component functions $f(t,x,u)$ and $g(t,x,u)$ of the forcing function satisfy the following conditions: $i)$ $\frac{\partial^{m-1} f(t,x,u)}{\partial u^{m-1}}$ and $\frac{\partial^{m-1} g(t,x,u)}{\partial u^{m-1}}$ exist and they are nondecreasing in u for $m > 2$; $ii)$ $\frac{\partial^{m-1} f(t,x,u)}{\partial u^{m-1}}$ and $\frac{\partial^{m-1} g(t,x,u)}{\partial u^{m-1}}$ are onesided Lipschitzian with respect to u for $m > 2$. We develop two sequences which converge uniformly, monotonically, and rapidly to the unique solution with the rate of convergence m. The earlier known results on cubic and quadrtic convergence can be obtained as special cases of our current result. A numerical example is presented as an application of our theoretical result. This result is a generalization of GQM method to obtain higher order of convergence for nonlinear parabolic integro-differential equations.

Key Words and Phrases: Generalized quasilinearization, higher order of convergence, parabolic integro-differential equation.

AMS subject Classification: 35K57, 35K60

1. INTRODUCTION

The method of quasilinearization [1, 2] combined with the method of upper and lower solutions [6, 8] is an effective and fruitful technique for solving a wide variety of nonlinear differential equations. In [10, 11] we can see the application of the quasilinearization method. It has been extended recently and referred to as a generalized quasilinearization method [9]. In the nuclear reactor model if the effect of the temperature feedback is taken into consideration the neutron flux $u \equiv u(t,x)$ is governed by a Volterra type integro-differential equation [13]. On the other hand, in the study of nerve propagation, a simplified Hodgkin-Huxley model for the propagation of a voltage pulse through a nerve axon is governed by a similar Volterra

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type integro-differential equation [13]. Motivated by the above models we consider nonlinear parabolic integro-differential equations in this paper.
Using generalized quasilinearization method the authors of [3, 5] obtained a quadratic order of convergence for nonlinear integro-differential equations of ordinary and of parabolic type respectively. Assuming convexity assumption of the forcing function they developed linear iterates to obtain the solution of the nonlinear integro-differential problems. However, in [4] the authors have extended monotone method for first order initial value problem to obtain rapid convergence. See also [7] for generalized monotone method. In [12] the authors have obtained cubic convergence for first order ordinary differential equation. In this paper we extend the above results when the $(m-1)$-st derivative of the forcing function is nondecreasing in u and onesided Lipschitzian in u. Using an appropriate iterative scheme and natural lower and upper solutions under suitable conditions, we obtain natural sequences which converge to the unique solution of the nonlinear integro-differential equations of Volterra's type and the rate of convergence is m. Finally, we provide a numerical example to illustrate the application of results obtained.

2. PRELIMINARIES

The following nonlinear second order parabolic integro-differential equation will be considered in this paper.

$$\begin{aligned} \mathcal{L}u &= f(t,x,u(t,x)) + \int_0^t g(t,x,s,u(s,x))ds \quad in \quad Q_T, \\ u(t,x) &= \Phi(t,x), \quad x \in \partial\Omega, \\ u(0,x) &= u_0(x), \ x \in \Omega, \end{aligned} \tag{2.1}$$

where Ω is a bounded domain in R^m with boundary $\partial\Omega \in C^{2+\overline{\alpha}}$ ($\overline{\alpha} \in (0,1)$) and closure $\overline{\Omega}$, $Q_T = (0,T) \times \Omega$, $\overline{Q}_T = [0,T] \times \overline{\Omega}$, $T > 0$. Let $\mathcal{L}$ be a second order differential operator defined by

$$\mathcal{L} = \frac{\partial}{\partial t} - L, \tag{2.2}$$

where

$$L = \sum_{i,j=1}^{m} a_{i,j}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(t,x)\frac{\partial}{\partial x_i}. \tag{2.3}$$

In this section we recall some known existence and comparison theorems and the list of the assumptions which we will use in the proof of our main results.
Let us start with the list of the following assumptions.

(A_0) (i) For each $i,j = 1,\dots,m$, $a_{i,j}, b_j \in C^{\frac{\overline{\alpha}}{2},\overline{\alpha}}[\overline{Q}_T, R]$ and $\mathcal{L}$ is strictly uniformly parabolic in $\overline{Q}_T$;

(ii) $\partial\Omega$ belongs to the class $C^{2+\overline{\alpha}}$;

(*iii*) $f \in C^{\frac{\overline{\alpha}}{2},\overline{\alpha}}[[0,T]\times\overline{\Omega}\times R, R]$, $g \in C^{\frac{\overline{\alpha}}{2},\overline{\alpha}}[[0,T]\times\overline{\Omega}\times R^2, R]$ that is $f(t,x,u)$, $g(t,x,u)$ are Hölder continuous in t and (x,u) with exponents $\frac{\overline{\alpha}}{2}$ and $\overline{\alpha}$, respectively;

(*iv*) $\Phi \in C^{1+\frac{\overline{\alpha}}{2},2+\overline{\alpha}}[[0,T]\times\partial\Omega, R]$ and $u_0(x) \in C^{2+\overline{\alpha}}[\overline{\Omega}, R]$;

(*v*) $u_0(x) = \Phi(0,x)$, $\quad \Phi_t = Lu_0 + f(0,x,u_0)$ for $t=0$ and $x \in \partial\Omega$.

Next we introduce the following definition.

Definition 2.1. The functions α_0 , $\beta_0 \in C^{1,2}[\overline{Q}_T, R]$ with $g(t,x,u)$ nondecreasing in u are said to be lower and upper solutions of (2.1), respectively, if

$$\begin{aligned} \mathcal{L}\alpha_0 &\le f(t,x,\alpha_0(t,x)) + \int_0^t g(t,x,s,\alpha_0(s,x))ds \quad in \quad Q_T, \\ \alpha_0(t,x) &\le \Phi(t,x), \quad x \in \partial\Omega, \\ \alpha_0(0,x) &\le u_0(x), \ x \in \Omega, \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}\beta_0 &\ge f(t,x,\beta_0(t,x)) + \int_0^t g(t,x,s,\beta_0(s,x))ds \quad in \quad Q_T, \\ \beta_0(t,x) &\ge \Phi(t,x), \quad x \in \partial\Omega, \\ \beta_0(0,x) &\ge u_0(x), \ x \in \Omega. \end{aligned}$$

Furthermore we state a known existence theorem relative to the equation (2.1).

Theorem 2.1. *(*see [5].*) Assume that (A_0) holds. Then (2.1) has a unique smooth solution $u(t,x) \in C^{1+\frac{\overline{\alpha}}{2},2+\overline{\alpha}}[\overline{Q}_T, R]$.*

In addition, we recall the positivity and comparison theorems in order to prove the monotonicity and the order of convergence in our main results.

Theorem 2.2. *(*see [13].*) Let $u(t,x) \in C^{\frac{1+\overline{\alpha}}{2},1+\overline{\alpha}}[\overline{Q}_T, R]$ be such that*

$$\begin{aligned} \mathcal{L}u + cu &\ge 0 \quad in \quad Q_T, \\ u(t,x) &\ge 0, \quad x \in \partial\Omega, \\ u(0,x) &\ge 0, \quad x \in \Omega, \end{aligned}$$

and $c \equiv c(t,x)$ is a bounded function in Q_T. Then $u(t,x) \ge 0$ in $\overline{Q}_T$.

Theorem 2.3. *(*see [14].*) Assume that*

(*i*) *$f_u(t,x,u)$ and $g_u(t,x,s,u)$ are bounded functions with $g(t,x,s,u)$ nondecreasing in u on $\overline{Q}_T$.*

(*ii*) *$\alpha(t,x)$ and $\beta(t,x)$ satisfy*

$$\begin{aligned} \mathcal{L}\alpha &\le f(t,x,\alpha(t,x)) + \int_0^t g(t,x,s,\alpha(s,x))ds \quad in \quad Q_T, \\ \mathcal{L}\beta &\ge f(t,x,\beta(t,x)) + \int_0^t g(t,x,s,\beta(s,x))ds \quad in \quad Q_T, \end{aligned}$$

with

$$\begin{aligned} \alpha(t,x) &\le \beta(t,x), \quad x \in \partial\Omega, \\ \alpha(0,x) &\le \beta(0,x), \quad x \in \Omega. \end{aligned}$$

Then $\alpha(t,x) \le \beta(t,x)$ on $\overline{Q}_T$.

We also need the following comparison theorem which is a special case of Lemma 6.2 in [3].

Theorem 2.4. *Suppose that*

(*i*) $g(t,x,s,u)$ *is monotone nondecreasing in* u *for each fixed point (t,x,s),*

(*ii*) $\alpha(t,x)$ *satisfies*

$$\begin{aligned} \mathcal{L}\alpha &\le f(t,x,\alpha(t,x)) + \int_0^t g(t,x,s,\alpha(s,x))ds \quad in \quad Q_T, \\ \alpha(t,x) &= 0, \ x \in \partial\Omega, \\ \alpha(0,x) &= u_0(x), \quad x \in \Omega, \end{aligned}$$

(*iii*) $r(t)$ *is the solution of the following ordinary integro-differential equation*

$$r' = h_1(t,r) + \int_0^t h_2(t,s,r))ds,$$
$$r(0) = \max\{\max_{x\in\Omega} u_0(x), 0\},$$

where

$$h_1(t,r) \ge \max_{x\in\Omega} f(t,x,r) \ and \ h_2(t,r) \ge \max_{x\in\Omega} g(t,x,s,r).$$

Then $\alpha(t,x) \le r(t)$ *on* $\overline{Q}_T$.

3. MAIN RESULTS

In this section we extend the method of generalized quasilinearization to (2.1) with higher order of convergence m ($m > 2$) when the nonlinearity of the iterates is $m-1$. Since the approximation of the solution depends entirely on m being an even or odd number, we assume at first that m is an odd number, say $m = 2k+1$. This is precisely our first main result, which we state below.

Theorem 3.1. *Assume that all of* (A_0) *holds except* (*iii*)*; further assume that*

(*i*) α_0, β_0 *are lower and upper solutions of (2.1) with* $\alpha_0(t,x) \le \beta_0(t,x)$ *on* $\overline{Q}_T$.

(*ii*) $\dfrac{\partial^l f(t,x,u)}{\partial u^l}$, $\dfrac{\partial^l g(t,x,s,u)}{\partial u^l}$ *exist and are bounded functions on* $\overline{Q}_T$ *for* $l = 0,1,\ldots,2k$ *such that* $\dfrac{\partial f^l(t,x,u)}{\partial u^l}, \dfrac{\partial^l g(t,x,s,u)}{\partial u^l} \in C^{\frac{\alpha}{2},\alpha}[Q_T \times R, R]$.

(*iii*) *Also* $\dfrac{\partial^l g(t,x,s,u)}{\partial u^l}$ *are nondecreasing functions in* u *on* $\overline{Q}_T$ *for* $l = 0,1,\ldots,2k$ *such that*

$$g_u(\alpha_0) \ge g^{2k}(\beta_0)\frac{(\beta_0-\alpha_0)^{2k-1}}{(2k-1)!} \ and$$

$$0 \le \frac{\partial^{2k} f(t,x,\eta_1)}{\partial u^{2k}} - \frac{\partial^{2k} f(t,x,\eta_2)}{\partial u^{2k}} \le M_1(\eta_1-\eta_2) \quad on \ \overline{Q}_T,$$

$$0 \le \frac{\partial^{2k} g(t,x,\xi_1)}{\partial u^{2k}} - \frac{\partial^{2k} g(t,x,\xi_2)}{\partial u^{2k}} \le M_2(\xi_1-\xi_2) \quad on \ \overline{Q}_T,$$

whenever

$$\alpha_0(t,x) \le \eta_2(t,x) \le \eta_1(t,x) \le \beta_0(t,x),$$
$$\alpha_0(t,x) \le \xi_2(t,x) \le \xi_1(t,x) \le \beta_0(t,x).$$

Then there exist monotone sequences $\{\alpha_n(t,x)\}$, $\{\beta_n(t,x)\}$, $n \geq 0$ *which converge uniformly and monotonically to the unique solution of (2.1) and the convergence is of order* $2k+1$.

Proof: To prove this we need to consider the following equations:

$$(3.1)\quad \begin{aligned} \mathcal{L}w \; &= \; F_1(t,x,\alpha;w) \; + \textstyle\int_0^t G_1(t,x,s,\alpha(s,x);w(s,x))ds \\ &= \sum_{i=0}^{2k} \frac{\partial^i f(t,x,\alpha)}{\partial u^i}\frac{(w-\alpha)^i}{i!} \\ &\quad + \textstyle\int_0^t \displaystyle\sum_{i=0}^{2k} \frac{\partial^i g(t,x,s,\alpha(s,x))}{\partial u^i}\frac{(w(s,x)-\alpha(s,x))^i}{i!} ds \quad in \quad Q_T, \\ w(t,x) &= \Phi(t,x), \quad x \in \partial\Omega, \\ w(0,x) &= u_0(x), \;\; x \in \Omega, \end{aligned}$$

$$(3.2)\quad \begin{aligned} \mathcal{L}v \; &= \; F_2(t,x,\beta;v) \; + \textstyle\int_0^t G_2(t,x,s,\beta(s,x);v(s,x))ds \\ &= \sum_{i=0}^{2k} \frac{\partial^i f(t,x,\beta)}{\partial u^i}\frac{(v-\beta)^i}{i!} \\ &\quad + \textstyle\int_0^t \displaystyle\sum_{i=0}^{2k} \frac{\partial^i g(t,x,s,\beta(s,x))}{\partial u^i}\frac{(v(s,x)-\beta(s,x))^i}{i!} ds \quad in \quad Q_T, \\ v(t,x) &= \Phi(t,x), \quad x \in \partial\Omega, \\ v(0,x) &= u_0(x), \;\; x \in \Omega, \end{aligned}$$

where $\alpha(t,x) \leq v, w \leq \beta(t,x)$ and $\alpha(0,x) \leq u_0(x) \leq \beta(0,x)$.
Let us show that (α_0, β_0) are lower and upper solutions of (3.1) and (3.2), respectively. By setting $\alpha = \alpha_0$ and $\beta = \beta_0$ in (3.1) we get

$$(3.3)\quad \begin{aligned} \mathcal{L}\alpha_0 \; &\leq f(t,x,\alpha_0) & &+ \textstyle\int_0^t g(t,x,s,\alpha_0(s,x))ds \\ &= \; F_1(t,x,\alpha_0;\alpha_0) & &+ \textstyle\int_0^t G_1(t,x,s,\alpha_0(s,x);\alpha_0(s,x))ds; \\ \alpha_0(t,x) &\leq \Phi(t,x), & &x \in \partial\Omega, \\ \alpha_0(0,x) &\leq u_0(x), & &x \in \Omega, \end{aligned}$$

$$(3.4)\quad \begin{aligned} \mathcal{L}\beta_0 \; &\geq f(t,x,\beta_0) + \textstyle\int_0^t g(t,x,s,\beta_0(s,x))ds \\ &= \sum_{i=0}^{2k-1} \frac{\partial^i f(t,x,\alpha_0)}{\partial u^i}\frac{(\beta_0-\alpha_0)^i}{i!} + \frac{\partial^{2k} f(t,x,\xi_1)}{\partial u^{2k}}\frac{(\beta_0-\alpha_0)^{2k}}{(2k)!} \\ &+ \textstyle\int_0^t [\displaystyle\sum_{i=0}^{2k-1} \frac{\partial^i g(t,x,s,\alpha_0)}{\partial u^i}\frac{(\beta_0-\alpha_0)^i}{i!} + \frac{\partial^{2k} g(t,x,s,\xi_2)}{\partial u^{2k}}\frac{(\beta_0-\alpha_0)^{2k}}{(2k)!}]ds \\ &\geq \sum_{i=0}^{2k} \frac{\partial^i f(t,x,\alpha_0)}{\partial u^i}\frac{(\beta_0-\alpha_0)^i}{i!} + \textstyle\int_0^t [\displaystyle\sum_{i=0}^{2k} \frac{\partial^i g(t,x,s,\alpha_0)}{\partial u^i}\frac{(\beta_0-\alpha_0)^i}{i!} \\ &= F_1(t,x,\alpha_0;\beta_0) + \int_0^t G_1(t,x,s,\alpha_0;\beta_0)ds, \\ \beta_0(t,x) &\geq \Phi(t,x), \quad x \in \partial\Omega, \end{aligned}$$

$$\beta_0(0,x) \geq u_0(x),\ x \in \Omega,$$

where $\alpha_0 \leq \xi_1, \xi_2 \leq \beta_0$.

Hence α_0 and β_0 are the lower and upper solutions of (3.1). Next we need to apply Theorem 2.1. To do this we will verify (iii) of (A_0) relative to the equation (3.1). For $\eta \in C^{\frac{1+\overline{\alpha}}{2},1+\overline{\alpha}}[\overline{Q_T}, R]$ such that $\alpha_0(x,t) \leq w(x,t), \eta(t,x) \leq \beta_0(t,x)$ on $\overline{Q}_T$ we have

$$\begin{aligned}
F_1(t,x,\eta;w) \ &= \sum_{i=0}^{2k} \frac{\partial^i f(t,x,\eta(t,x))}{\partial u^i} \frac{[w(t,x)-\eta(t,x)]^i}{i!} \\
&= \sum_{i=0}^{2k} \frac{\partial^i f(t,x,\eta(t,x))}{\partial u^i} \frac{\sum_{j=0}^{i}(-1)^j\binom{i}{j} w^{i-j}(t,x)\eta^j(t,x)}{i!} \\
&= \sum_{i=0}^{2k}\sum_{j=0}^{i} \frac{(-1)^j\binom{i}{j}}{i!} \frac{\partial^i f(t,x,\eta(t,x))}{\partial u^i} w^{i-j}(t,x)\eta^j(t,x) \\
&= K_{i,j}\sum_{i=0}^{2k}\sum_{j=0}^{i} d_{i,j}(t,x) w^{i-j}(t,x),
\end{aligned}$$

where $K_{i,j} = \frac{(-1)^j\binom{i}{j}}{i!}$ and $d_{i,j}(t,x) = \frac{\partial^i f(t,x,\eta(t,x))}{\partial u^i}\eta^j(t,x)$. Let us show that $d_{i,j}(t,x)$ belongs to $C^{\frac{\overline{\alpha}}{2},\overline{\alpha}}[\overline{Q_T}, R]$ for $i,j = 0,1,\ldots,2k$ by considering the term $d_{i,j}(t,x)$ when $|\eta| \leq C_1$ and $|\frac{\partial^i f}{\partial u^i}| \leq C_2$.

$$\begin{aligned}
|d_{i,j}(t,x) - d_{i,j}(\overline{t},x)| \ &= |\frac{\partial^i f(t,x,\eta(t,x))}{\partial u^i}\eta^j(t,x) - \frac{\partial^i f(\overline{t},x,\eta(\overline{t},x))}{\partial u^i}\eta^j(\overline{t},x)| \\
&\leq |\frac{\partial^i f(t,x,\eta(t,x))}{\partial u^i}\eta^j(t,x) - \frac{\partial^i f(\overline{t},x,\eta(\overline{t},x))}{\partial u^i}\eta^j(t,x)| \\
&+|\frac{\partial^i f(\overline{t},x,\eta(\overline{t},x))}{\partial u^i}\eta^j(t,x) - \frac{\partial^i f(\overline{t},x,\eta(\overline{t},x))}{\partial u^i}\eta^j(\overline{t},x)| \\
&= |\eta^j(t,x)||\frac{\partial^i f(t,x,\eta(t,x))}{\partial u^i} - \frac{\partial^i f(\overline{t},x,\eta(\overline{t},x))}{\partial u^i}| \\
&+|\frac{\partial^i f(\overline{t},x,\eta(\overline{t},x))}{\partial u^i}||\eta(t,x)-\eta(\overline{t},x)||\sum_{l=0}^{j-1}\eta^{j-l-1}(t,x)\eta^l(\overline{t},x)| \\
&\leq C_1^j C_t(\frac{\partial^i f}{\partial u^i})(|t-\overline{t}|^{\frac{\overline{\alpha}}{2}} + C_t(\eta)|t-\overline{t}|^{\frac{1+\overline{\alpha}}{2}}) \\
&+jC_1^2C_2C_t(\eta)|t-\overline{t}|^{\frac{1+\overline{\alpha}}{2}} \leq C_t(F_1)|t-\overline{t}|^{\frac{\overline{\alpha}}{2}},
\end{aligned}$$

where $C_t(F_1)$ depends on C_1, C_2, $C_t(\frac{\partial^i f}{\partial u^i})$, $C_t(\eta)$, and T. Thus $F_1(t,x,\alpha;w)$ is Hölder continuous in t with exponent $\frac{\overline{\alpha}}{2}$. Similarly, we can prove that $F_1(t,x,\alpha;w)$ is Hölder continuous in (x,w) with exponent $\overline{\alpha}$. That is:

$$\begin{aligned}
|d_{i,j}(t,x) - d_{i,j}(t,\overline{x})| \ &= |\frac{\partial^i f(t,x,\eta(t,x))}{\partial u^i}\eta^j(t,x) - \frac{\partial^i f(t,\overline{x},\eta(t,\overline{x}))}{\partial u^i}\eta^j(t,\overline{x})| \\
&\leq C_1^j C_x(\tfrac{\partial^i f}{\partial u^i})(\|x-\overline{x}\|^{\overline{\alpha}} + C_x(\eta)\|x-\overline{x}\|^{1+\overline{\alpha}}) \\
&+jC_1^2C_2C_x(\eta)\|x-\overline{x}\|^{1+\overline{\alpha}} \leq C_{x,w}(F_1)|x-\overline{x}|^{\overline{\alpha}},
\end{aligned}$$

where $C_{x,w}(F_1)$ depends on C_1, C_2, $C_x(\frac{\partial^i f}{\partial u^i})$, and $C_x(\eta)$. Hence we can conclude that $F_1(t,x,\alpha;w)$ is Hölder continuous in t and (x,w) with exponents $\frac{\overline{\alpha}}{2}$ and $\overline{\alpha}$, respectively. The proof that $G_1(t,x,\alpha;w)$ is Hölder continuous in t and (x,w) with exponents $\frac{\overline{\alpha}}{2}$ and $\overline{\alpha}$, respectively, follows on the same lines. Similar conclusions hold for $F_2(t,x,\beta;v)$ and $G_2(t,x,\beta;v)$. It follows by Theorem 2.1 that there exists a unique solution α_1 of (3.1). Now we prove that $\alpha_0 \le \alpha_1 \le \beta_0$. Letting $\mu = \alpha_1 - \alpha_0$ we get

$$\begin{aligned}
\mathcal{L}\mu &= \mathcal{L}(\alpha_1-\alpha_0)\\
&\ge F_1(t,x,\alpha_0;\alpha_1) + \int_0^t G_1(t,x,s,\alpha_0;\alpha_1)ds\\
&\quad -F_1(t,x,\alpha_0;\alpha_0) - \int_0^t G_1(t,x,s,\alpha_0;\alpha_0)ds\\
&= F_{1u}(t,x,\alpha_0;\xi_1)\mu + \int_0^t G_{1u}(t,x,s,\alpha_0;\xi_2)\mu ds,\\
\mu(t,x) &\ge 0,\ x\in\partial\Omega,\\
\mu(0,x) &\ge 0,\ x\in\Omega.
\end{aligned}$$

Applying Theorem 2.2 one can obtain that $\mu \ge 0$ or $\alpha_0 \le \alpha_1$. Next set $\mu = \beta_0 - \alpha_1$.

$$\begin{aligned}
\mathcal{L}\mu &= \mathcal{L}(\beta_0-\alpha_1)\\
&\ge F_1(t,x,\alpha_0;\beta_0) + \int_0^t G_1(t,x,s,\alpha_0;\beta_0)ds\\
&\quad -F_1(t,x,\alpha_0;\alpha_1) - \int_0^t G_1(t,x,s,\alpha_0;\alpha_1)ds\\
&= F_{1u}(t,x,\alpha_0;\xi_1)\mu + \int_0^t G_{1u}(t,x,s,\alpha_0;\xi_2)\mu ds,\\
\mu(t,x) &\ge 0,\ x\in\partial\Omega,\\
\mu(0,x) &\ge 0,\ x\in\Omega.
\end{aligned}$$

Again using Theorem 2.2 one can conclude that $\mu \ge 0$ or $\alpha_1 \le \beta_0$. Thus $\alpha_0 \le \alpha_1 \le \beta_0$. Similarly we will prove that (α_0, β_0) are lower and upper solutions of (3.2). Setting $\alpha = \alpha_0$ and $\beta = \beta_0$ in (3.2) we get

$$\begin{aligned}
\mathcal{L}\beta_0 &\ge f(t,x,\beta_0) + \int_0^t g(t,x,s,\beta_0(s,x))ds\\
&= F_2(t,x,\beta_0;\beta_0) + \int_0^t G_2(t,x,s,\beta_0(s,x);\beta_0(s,x))ds;\\
\beta_0(t,x) &\ge \Phi(t,x),\quad x\in\partial\Omega,\\
\beta_0(0,x) &\ge u_0(x),\ x\in\Omega,
\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}
\mathcal{L}\alpha_0 &\le f(t,x,\alpha_0) + \int_0^t g(t,x,s,\alpha_0(s,x))ds\\
&= \sum_{i=0}^{2k-1}\frac{\partial^i f(t,x,\beta_0)}{\partial u^i}\frac{(\alpha_0-\beta_0)^i}{i!} + \frac{\partial^{2k} f(t,x,\ \xi_1)}{\partial u^{2k}}\frac{(\alpha_0-\beta_0)^{2k}}{(2k)!}\\
&\quad + \int_0^t[\sum_{i=0}^{2k-1}\frac{\partial^i g(t,x,s,\beta_0)}{\partial u^i}\frac{(\alpha_0-\beta_0)^i}{i!} + \frac{\partial^{2k} g(t,x,s,\ \xi_2)}{\partial u^{2k}}\frac{(\alpha_0-\beta_0)^{2k}}{(2k)!}]ds\\
&\le \sum_{i=0}^{2k}\frac{\partial^i f(t,x,\beta_0)}{\partial u^i}\frac{(\alpha_0-\beta_0)^i}{i!} + \int_0^t\sum_{i=0}^{2k}\frac{\partial^i g(t,x,s,\beta_0)}{\partial u^i}\frac{(\alpha_0-\beta_0)^i}{i!}ds\\
&= F_2(t,x,\beta_0;\alpha_0) + \int_0^t G_2(t,x,s,\beta_0;\alpha_0)ds,\\
&\alpha_0(t,x) \le \Phi(t,x),\quad x\in\partial\Omega,\\
&\alpha_0(0,x) \le u_0(x),\ x\in\Omega.
\end{aligned}\tag{3.6}$$

where $\alpha_0 \le \xi_1, \xi_2 \le \beta_0$.

One can conclude that α_0 and β_0 are the lower and upper solutions of (3.2) and by Theorem 2.1 there exists a unique solution β_1 of (3.2). Next we show that $\alpha_0 \le \beta_1 \le \beta_0$ by setting $\mu = \beta_0 - \beta_1$ and $\mu = \beta_1 - \alpha_0$, respectively.

$$\begin{aligned}
\mathcal{L}\mu &= \mathcal{L}(\beta_0 - \beta_1) \\
&\ge F_2(t, x, \beta_0; \beta_0) \quad + \int_0^t G_2(t, x, s, \beta_0; \beta_0) ds \\
&\quad - F_2(t, x, \beta_0; \beta_1) \quad - \int_0^t G_2(t, x, s, \beta_0; \beta_1) ds \\
&= F_{2_u}(t, x, \beta_0; \xi_1)\mu \quad + \int_0^t G_{2_u}(t, x, s, \beta_0; \xi_2)\mu ds, \\
\mu(t, x) &\ge 0, \ x \in \partial\Omega, \\
\mu(0, x) &\ge 0, \ x \in \Omega.
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}\mu &= \mathcal{L}(\beta_1 - \alpha_0) \\
&\ge F_2(t, x, \beta_0; \beta_1) \quad + \int_0^t G_2(t, x, s, \beta_0; \beta_1) ds \\
&\quad - F_2(t, x, \beta_0; \alpha_0) \quad - \int_0^t G_2(t, x, s, \beta_0; \alpha_0) ds \\
&= F_{2_u}(t, x, \beta_0; \xi_1)\mu \quad + \int_0^t G_{2_u}(t, x, s, \beta_0; \xi_2)\mu ds, \\
\mu(t, x) &\ge 0, \ x \in \partial\Omega, \\
\mu(0, x) &\ge 0, \ x \in \Omega.
\end{aligned}$$

Applying Theorem 2.2 we obtain that $\mu \ge 0$ or $\beta_0 \ge \beta_1$ and $\beta_1 \ge \alpha_0$. Hence $\alpha_0 \le \beta_1 \le \beta_0$. In addition, we prove that $\beta_1 \ge \alpha_1$. For that purpose we observe that

$$\begin{aligned}
f(t, x, \alpha_1) &+ \int_0^t g(t, x, s, \alpha_1(s, x)) ds \qquad = \sum_{i=0}^{2k-1} \frac{\partial^i f(t, x, \alpha_0)}{\partial u^i} \frac{(\alpha_1 - \alpha_0)^i}{i!} \\
&+ \frac{\partial^{2k} f(t, x, \xi_1)}{\partial u^{2k}} \frac{(\alpha_1 - \alpha_0)^{2k}}{(2k)!} \quad + \int_0^t [\sum_{i=0}^{2k-1} \frac{\partial^i g(t, x, s, \alpha_0)}{\partial u^i} \frac{(\alpha_1 - \alpha_0)^i}{i!} \\
&+ \frac{\partial^{2k} g(t, x, s, \xi_2)}{\partial u^{2k}} \frac{(\alpha_1 - \alpha_0)^{2k}}{(2k)!}] ds \ge \sum_{i=0}^{2k} \frac{\partial^i f(t, x, \alpha_0)}{\partial u^i} \frac{(\alpha_1 - \alpha_0)^i}{i!} \\
&+ \int_0^t [\sum_{i=0}^{2k} \frac{\partial^i g(t, x, s, \alpha_0)}{\partial u^i} \frac{(\alpha_1 - \alpha_0)^i}{i!}] ds \\
&= F_1(t, x, \alpha_0; \alpha_1) + \int_0^t G_1(t, x, s, \alpha_0; \alpha_1) = \mathcal{L}\alpha_1, \\
\alpha_1(t, x) &= \Phi(t, x), \quad x \in \partial\Omega, \\
\alpha_1(0, x) &= u_0(x), \ x \in \Omega;
\end{aligned} \tag{3.7}$$

and

$$
\begin{aligned}
f(t,x,\beta_1) &+ \textstyle\int_0^t g(t,x,s,\beta_1(s,x))ds \qquad = \sum_{i=0}^{2k-1} \frac{\partial^i f(t,x,\beta_0)}{\partial u^i}\frac{(\beta_1-\beta_0)^i}{i!} \\
&+ \frac{\partial^{2k} f(t,x,\xi_1)}{\partial u^{2k}}\frac{(\beta_1-\beta_0)^{2k}}{(2k)!} \quad + \textstyle\int_0^t [\sum_{i=0}^{2k-1} \frac{\partial^i g(t,x,s,\beta_0)}{\partial u^i}\frac{(\beta_1-\beta_0)^i}{i!} \\
&+ \frac{\partial^{2k} g(t,x,s,\xi_2)}{\partial u^{2k}}\frac{(\beta_1-\beta_0)^{2k}}{(2k)!}]ds \le \sum_{i=0}^{2k} \frac{\partial^i f(t,x,\beta_0)}{\partial u^i}\frac{(\beta_1-\beta_0)^i}{i!} \\
&+ \textstyle\int_0^t [\sum_{i=0}^{2k} \frac{\partial^i g(t,x,s,\beta_0)}{\partial u^i}\frac{(\beta_1-\beta_0)^i}{i!}]ds \\
&= F_2(t,x,\beta_0;\beta_1) + \textstyle\int_0^t G_2(t,x,s,\beta_0;\beta_1) = \mathcal{L}\beta_1, \\
\beta_1(t,x) &= \Phi(t,x), \quad x \in \partial\Omega, \\
\beta_1(0,x) &= u_0(x), \ x \in \Omega.
\end{aligned} \tag{3.8}
$$

By (3.7) and (3.8) together with Theorem 2.3 one can conclude that $\beta_1 \ge \alpha_1$. Hence we have $\alpha_0 \le \alpha_1 \le \beta_1 \le \beta_0$. Using the method of mathematical induction and the last inequality, one can show that

$$
\alpha_0 \le \alpha_1 \le \cdots \le \alpha_n \le \beta_n \le \cdots \le \beta_1 \le \beta_0 \ for \ all \ n.
$$

Let now u be any solution of (2.1) such that $\alpha_0 \le u \le \beta_0$ with $\alpha_0(0) \le u_0 \le \beta_0(0)$ on $\overline{Q}_T$ and suppose that for some u , we have $\alpha_n \le u \le \beta_n$ on $\overline{Q}_T$. Then set $\Phi_1 = u - \alpha_{n+1}$ and $\Phi_2 = \beta_{n+1} - u$, respectively.

$$
\begin{aligned}
\mathcal{L}\Phi_1 \quad &= \mathcal{L}u - \mathcal{L}\alpha_{n+1} = f(t,x,u) + \textstyle\int_0^t g(t,x,s,u(s,x))ds \\
&- \sum_{i=0}^{2k} \frac{\partial^i f(t,x,\alpha_n)}{\partial u^i}\frac{(\alpha_{n+1}-\alpha_n)^i}{i!} - \ \textstyle\int_0^t \sum_{i=0}^{2k} \frac{\partial^i g(t,x,s,\alpha_n(s,x))}{\partial u^i}\frac{(\alpha_{n+1}-\alpha_n)^i}{i!} ds \\
&\ge f(t,x,u) - \ f(t,x,\alpha_{n+1}) + \textstyle\int_0^t [g(t,x,s,u) - \ g(t,x,s,\alpha_{n+1}(s,x))]ds \\
&\ge f_u(t,x,\eta_1)\Phi_1 + \textstyle\int_0^t [g_u(t,x,s,\eta_2)\Phi_1]ds \\
\Phi_1(t,x) \quad &= 0, \ x \in \partial\Omega, \\
\Phi_1(0,x) \quad &= 0, \ x \in \Omega;
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}\Phi_2 \quad &= \mathcal{L}\beta_{n+1} - \mathcal{L}u = -f(t,x,u) - \textstyle\int_0^t g(t,x,s,u(s,x))ds \\
&+ \sum_{i=0}^{2k} \frac{\partial^i f(t,x,\beta_n)}{\partial u^i}\frac{(\beta_{n+1}-\beta_n)^i}{i!} + \ \textstyle\int_0^t \sum_{i=0}^{2k} \frac{\partial^i g(t,x,s,\beta_n(s,x))}{\partial u^i}\frac{(\beta_{n+1}-\beta_n)^i}{i!} ds \\
&\ge -f(t,x,u) + \ f(t,x,\beta_{n+1}) + \textstyle\int_0^t [-g(t,x,s,u(s,x)) + \ g(t,x,s,\beta_{n+1}(s,x))]ds \\
&\ge f_u(t,x,\eta_3)\Phi_2 + \textstyle\int_0^t [g_u(t,x,s,\eta_4)\Phi_2]ds \\
\Phi_2(t,x) \quad &= 0, \ x \in \partial\Omega, \\
\Phi_2(0,x) \quad &= 0, \ x \in \Omega,
\end{aligned}
$$

where $\eta_1,\ \eta_2$ are between u and α_{n+1}, and $\eta_3,\ \eta_4$ are between u and β_{n+1}. Thus $\alpha_{n+1} \le u \le \beta_{n+1}$ by Theorem 2.2. Initially $\alpha_0 \le u \le \beta_0$. By the method of mathematical induction $\alpha_n \le u \le \beta_n$ for all n. Hence

$$\alpha_0 \le \alpha_1 \le \cdots \le \alpha_n \le u \le \beta_n \le \cdots \le \beta_1 \le \beta_0.$$

Since $\{\alpha_n(t,x)\}$ and $\{\beta_n(t,x)\}$ are in $C^{1+\frac{\overline{\alpha}}{2},2+\overline{\alpha}}[\overline{Q}_T, R]$, one can show that these sequences converge to (ρ, r) using the same technique as in [13]. That is

$$\lim_{n\to\infty} \alpha_n(t,x) = \rho(t,x) \le u \le r(t,x) \lim_{n\to\infty} \beta_n(t,x).$$

Next we show that $\rho(t,x) \ge r(t,x)$. It follows by equations (3.1) and (3.2) that

$$\begin{aligned}
\mathcal{L}\rho(t,x) &= F_1(t,x,\rho;\rho) \ + \textstyle\int_0^t G_1(t,x,s,\rho(s,x);\rho(s,x))ds \\
&= f(t,x,\rho) + \textstyle\int_0^t g(t,x,s,\rho(s,x))ds, \\
\rho(t,x) &= \Phi(t,x),\ \ x \in \partial\Omega, \\
\rho(0,x) &= u_0(x),\ \ x \in \Omega
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}r(t,x) &= F_2(t,x,r;r) \ + \textstyle\int_0^t G_2(t,x,s,r(s,x);r(s,x))ds \\
&= f(t,x,r) + \textstyle\int_0^t g(t,x,s,r(s,x))ds, \\
r(t,x) &= \Phi(t,x),\ \ x \in \partial\Omega, \\
r(0,x) &= u_0(x),\ \ x \in \Omega.
\end{aligned}$$

Let us set $\Theta = r(t,x) - \rho(t,x)$ and apply (iii).

$$\begin{aligned}
\mathcal{L}\Theta = \mathcal{L}r - \mathcal{L}\rho &= f(t,x,r) + \int_0^t g(t,x,s,r(s,x))ds \\
&\quad - f(t,x,\rho) - \int_0^t g(t,x,s,\rho(s,x))ds \\
&\le L_1(r-\rho) + \int_0^t L_2(r-\rho)ds \le L_1\Theta + \int_0^t L_2\Theta ds,\ \ L_1,\ \ L_2 \ge 0, \\
\Theta(t,x) = 0, &\qquad x \in \partial\Omega, \\
\Theta(0,x) = 0, &\qquad x \in \Omega.
\end{aligned}$$

Now applying Theorem 2.2 we have $r(t,x) \le \rho(t,x)$. This proves $r(t,x) = \rho(t,x) = u(t,x)$ is the unique solution of (2.1). Hence $\{\alpha_n(t,x)\}$ and $\{\beta_n(t,x)\}$ converge uniformly and monotonically to the unique solution of (2.1).

Let us consider the order of convergence of $\{\alpha_n(t,x)\}$ and $\{\beta_n(t,x)\}$ to the unique solution $u(t,x)$ of (2.1). Set at first

$$p_n(t,x) = u(t,x) - \alpha_n(t,x) \ge 0,$$

$$q_n(t,x) = \beta_n(t,x) - u(t,x) \ge 0.$$

Using the definitions for α_n, β_n, the Taylor expansion with Lagrange remainder, and the mean value theorem, we obtain

$$\begin{aligned}
\mathcal{L}p_{n+1} &= \mathcal{L}u - \mathcal{L}\alpha_{n+1} \\
&= f(t,x,u) + \int_0^t g(t,x,s,u(s,x))ds \\
&- \sum_{i=0}^{2k} \frac{\partial^i f(t,x,\alpha_n)}{\partial u^i}\frac{(\alpha_{n+1}-\alpha_n)^i}{i!} - \int_0^t \sum_{i=0}^{2k} \frac{\partial^i g(t,x,s,\alpha_n(s,x))}{\partial u^i}\frac{(\alpha_{n+1}-\alpha_n)^i}{i!}ds \\
&= f(t,x,u) - \ f(t,x,\alpha_{n+1}) + \frac{\partial^{2k} f(t,x,\xi_1)}{\partial u^{2k}}\frac{(\alpha_{n+1}-\alpha_n)^{2k}}{(2k)!} \\
&- \frac{\partial^{2k} f(t,x,\alpha_n)}{\partial u^{2k}}\frac{(\alpha_{n+1}-\alpha_n)^{2k}}{(2k)!} + \int_0^t [g(t,x,s,u) - \ g(t,x,s,\alpha_{n+1}(s,x)) \\
&+ \frac{\partial^{2k} g(t,x,s,\xi_2(s,x))}{\partial u^{2k}}\frac{(\alpha_{n+1}-\alpha_n)^{2k}}{(2k)!} - \frac{\partial^{2k} g(t,x,s,\alpha_n(s,x))}{\partial u^{2k}}\frac{(\alpha_{n+1}-\alpha_n)^{2k}}{(2k)!}]ds \\
&\le f_u(t,x,\eta_1)(u-\alpha_{n+1}) + \frac{M_1}{(2k)!}(\xi_1-\alpha_n)(\alpha_{n+1}-\alpha_n)^{2k} \\
&+ \int_0^t [g_u(t,x,s,\eta_2)(u-\alpha_{n+1}) + \frac{M_2}{(2k)!}(\xi_2-\alpha_n)(\alpha_{n+1}-\alpha_n)^{2k}]ds \\
&\le K_1 p_{n+1} + K_2 p_n^{2k+1} + \int_0^t [K_3 p_{n+1} + K_4 p_n^{2k+1}]ds,
\end{aligned}$$

$$\begin{aligned}
p_{n+1}(t,x) &= 0, \qquad x \in \partial\Omega, \\
p_{n+1}(0,x) &= 0, \qquad x \in \Omega,
\end{aligned}$$

where $\alpha_n \le \xi_1, \xi_2 \le \alpha_{n+1},\ \alpha_{n+1} \le \eta_1, \eta_2 \le u,\ |f_u| \le K_1,\ \frac{M_1}{(2k)!} = K_2,\ |g_u| \le K_3$, and $\frac{M_2}{(2k)!} = K_4$. Let $r(t)$ be the solution of the following ordinary integro-differential equation.

$$r'(t) = K_1 r(t) + K_3 \int_0^t r(s)ds + (K_2 + K_4 T)\max_{\Omega} p_n^{2k+1},\ r(0) = 0.$$

By computing the solution of the above equation, we get

$$r(t) \le \frac{2exp(\sqrt{K_1^2 + 4K_3}\ T)}{\sqrt{K_1^2 + 4K_3}}[(K_2 + K_4 T)\max_{\Omega} p_n^{2k+1}].$$

One can see easily that

$$\int_0^t K_4 p_n^3 ds \le K_4 T \max_{\Omega} p_n^{2k+1}.$$

It follows that $p_{n+1}(t,x) \le r(t)$ by Theorem 2.4. Hence

$$\max_{\overline{Q}_T} |p_{n+1}(t,x)| \le [(K_2 + K_4 T)]\Big[\frac{2exp(\sqrt{K_1^2 + 4K_3}\ T)}{\sqrt{K_1^2 + 4K_3}}\Big] \max_{\overline{Q}_T} |p_n^{2k+1}(t,x)|.$$

Similarly, one can prove that

$$\max_{\overline{Q}_T} |q_{n+1}(t,x)| \le [(K_2 + K_4T)]\Big[\frac{2exp(\sqrt{K_1^2+4K_3}\ T)}{\sqrt{K_1^2+4K_3}}\Big]\max_{\overline{Q}_T}|q_n^{2k+1}(t,x)|.$$

Hence the order of convergence of the sequences $\{\alpha_n(t,x)\}$, $\{\beta_n(t,x)\}$ is $2k+1$. That completes the proof of Theorem 3.1.

Assume now that m is an even number, say $m = 2k$. Next we state our second main theorem.

Theorem 3.2. *Assume that all of* (A_0) *holds except* (iii)*; further assume that*

(i) α_0, β_0 *are lower and upper solutions of (2.1) with* $\alpha_0(t,x) \le \beta_0(t,x)$ *on* $\overline{Q}_T$.

(ii) $\dfrac{\partial^l f(t,x,u)}{\partial u^l}$, $\dfrac{\partial^l g(t,x,s,u)}{\partial u^l}$ *exist and are bounded functions on* $\overline{Q}_T$ *for* $l = 0,1,2,\ldots,2k-1$ *such that* $\dfrac{\partial f^l(t,x,u)}{\partial u^l}, \dfrac{\partial^l g(t,x,s,u)}{\partial u^l} \in C^{\frac{\overline{\alpha}}{2},\overline{\alpha}}[Q_T \times R, R]$.

(iii) Also g, g_u, g_{uu} *are nondecreasing functions in* u *on* $\overline{Q}_T$ *and*

$$\begin{aligned} g_u(\alpha_0) &\ge [\frac{\partial^{2k-2} g(\beta_0)}{\partial u^{2k-2}} - \frac{\partial^{2k-2} g(\alpha_0)}{\partial u^{2k-2}}]\frac{(\beta_0-\alpha_0)^{2k-3}}{(2k-3)!}, \\ 0 &\le \frac{\partial^{2k-1} f(t,x,\eta_1)}{\partial u^{2k-1}} - \frac{\partial^{2k-1} f(t,x,\eta_2)}{\partial u^{2k-1}} \le M_1(\eta_1-\eta_2) \text{ on } \overline{Q}_T, \\ 0 &\le \frac{\partial^{2k-1} g(t,x,\xi_1)}{\partial u^{2k-1}} - \frac{\partial^{2k-1} g(t,x,\xi_2)}{\partial u^{2k-1}} \le M_2(\xi_1-\xi_2) \text{ on } \overline{Q}_T, \end{aligned}$$

such that

$$\alpha_0(t,x) \le \eta_2(t,x) \le \eta_1(t,x) \le \beta_0(t,x),$$
$$\alpha_0(t,x) \le \xi_2(t,x) \le \xi_1(t,x) \le \beta_0(t,x).$$

Then there exist monotone sequences $\{\alpha_n(t,x)\}$, $\{\beta_n(t,x)\}$, $m \ge 0$ *which converge uniformly and monotonically to the unique solution of (2.1) and the convergence is of order* $2k$.

Proof: In order to construct monotone sequences $\{\alpha_n(t,x)\}$ and $\{\beta_n(t,x)\}$, $n \ge 0$ which converge uniformly and monotonically to the unique solution of (2.1) when $m = 2k$ is an even number, we need to consider the following nonlinear parabolic integro-differential equations for $n = 1,2,\ldots$

$$(3.9)\qquad \begin{aligned} \mathcal{L}\alpha_{n+1} &= F_1(t,x,\alpha_n;\alpha_{n+1}) + \textstyle\int_0^t G_1(t,x,s,\alpha_n(s,x);\alpha_{n+1}(s,x))ds \\ &= \sum_{i=0}^{2k-1} \frac{\partial^i f(t,x,\alpha_n)}{\partial u^i}\frac{(\alpha_{n+1}-\alpha_n)^i}{i!} \\ &\quad + \int_0^t \sum_{i=0}^{2k-1} \frac{\partial^i g(t,x,s,\alpha_n(s,x))}{\partial u^i}\frac{(\alpha_{n+1}(s,x)-\alpha_n(s,x))^i}{i!} ds \text{ in } Q_T, \\ \alpha_{n+1}(t,x) &= \Phi(t,x), \quad x \in \partial\Omega, \\ \alpha_{n+1}(0,x) &= u_0(x),\ x \in \Omega, \end{aligned}$$

and

$$
\begin{aligned}
\mathcal{L}\beta_{n+1} &= F_2(t,x,\alpha_n,\beta_n;\beta_{n+1}) + \textstyle\int_0^t G_2(t,x,s,\alpha_n(s,x),\beta_n(s,x);\beta_{n+1}(s,x))ds \\
&= \sum_{i=0}^{2k-2} \frac{\partial^i f(t,x,\beta_n)}{\partial u^i}\frac{(\beta_{n+1}-\beta_n)^i}{i!} + \frac{\partial^{2k-1} f(t,x,\alpha_n)}{\partial u^{2k-1}}\frac{(\beta_{n+1}-\beta_n)^{2k-1}}{(2k-1)!} \\
&\qquad + \textstyle\int_0^t [\displaystyle\sum_{i=0}^{2k-2} \frac{\partial^i g(t,x,s,\beta_n(s,x))}{\partial u^i}\frac{(\beta_{n+1}(s,x)-\beta_n(s,x))^i}{i!} \\
&\qquad + \frac{\partial^{2k-1} g(t,x,\alpha_n(s,x))}{\partial u^{2k-1}}\frac{(\beta_{n+1}(s,x)-\beta_n(s,x))^{2k-1}}{(2k-1)!}]ds \quad in \quad Q_T, \\
\beta_{n+1}(t,x) &= \Phi(t,x), \quad x \in \partial\Omega, \\
\beta_{n+1}(0,x) &= u_0(x), \; x \in \Omega,
\end{aligned}
\tag{3.10}
$$

where $\alpha_n(t,x) \le \alpha_{n+1}(t,x), \beta_{n+1}(t,x) \le \beta_n(t,x)$ and $\alpha(0,x) \le u_0(x) \le \beta(0,x)$. We omit the details of the proof, since it follows on the same lines as in Theorem 3.1.

4. NUMERICAL RESULTS

In this section we demonstrate the application of the main results which we have developed in Section 3. Let us consider the following nonlinear parabolic integro-differential equation:

$$
\begin{aligned}
& u_t - u_{xx} = u^4 - 9u + \sin^2 t + \textstyle\int_0^t [0.5u^4(s,x) + 6u(s,x)]ds, \quad 0 \le x, t \le 1 \\
& u(0,t) = 1, \quad u(1,t) = 0, \qquad 0 \le t \le 1 \\
& u(0,x) = \cos(0.5\pi x), \qquad 0 \le x \le 1.
\end{aligned}
\tag{4.1}
$$

If we choose $\alpha_0(t,x) \equiv 0$ and $\beta_0(t,x) \equiv 1$ with $0 \le t, x \le 1$, we have

$$
\begin{aligned}
0 &\le \sin^2 t, \quad 0 \le t \le 1, \\
0 &\ge 1 - 15 + \sin^2 t + 13t, \quad 0 \le t \le 1, \\
0 &\le 1, \qquad 0 \le t \le 1, \\
0 &\le \cos(0.5\pi x) \le 1, \quad 0 \le x \le 1.
\end{aligned}
$$

Hence $\alpha_0(t,x) \equiv 0$ and $\beta_0(t,x) \equiv 1$ are natural lower and upper solutions for (4.1) respectively. Denote

$$
\begin{aligned}
f(t,x,u) &= u^4(t,x) - 9u(t,x) + \sin^2 t, \\
g(t,x,u) &= 0.5u^4(t,x) + 6u(t,x).
\end{aligned}
$$

It is true that

$$g_u(0) = 2(0)^3 + 6 \ge [g_{uu}(1) - g_{uu}(0)](1-0) = 6(1-0),$$

$$0 \le f_{uuu}(t,x,u_1) - f_{uuu}(t,x,u_2) \le 24(u_1 - u_2), \; u_1 \ge u_2,$$

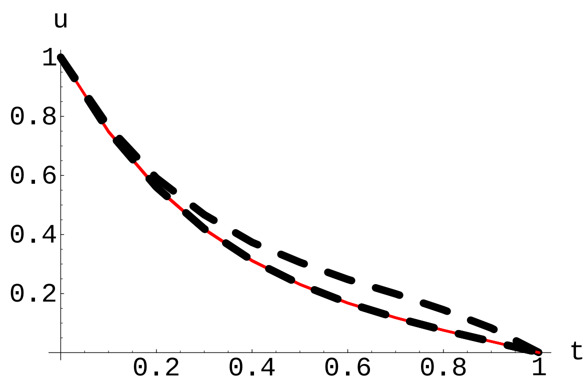
$$0 \le g_{uuu}(t,x,u_1) - g_{uuu}(t,x,u_2) \le 12(u_1 - u_2), \; u_1 \ge u_2.$$

Thus we can apply the iterates of Theorem 3.2 with the Lipschitzian constants $M_1 = 24$ and $M_2 = 12$ to find the approximate solution of the equation (4.1). After only three iterates of α and β we can derive the approximate solution u of (4.1) as shown in the following table for $t = 0.5$:

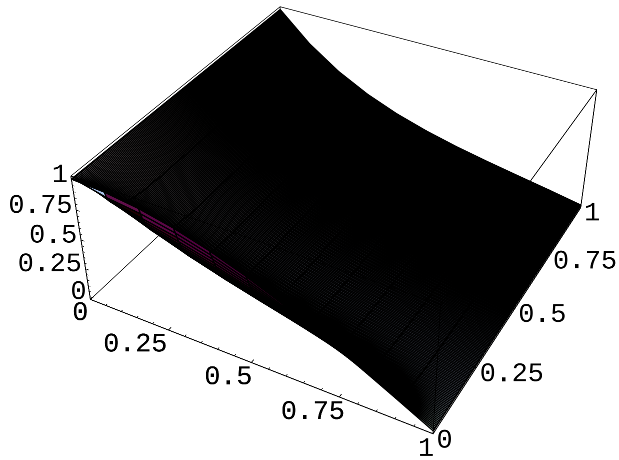
Table of Three $\alpha, \beta -$ Iterates and the Solution

x	$\alpha_1(t)$	$\alpha_2(t)$	$\alpha_3(t)$	u	$\beta_3(t)$	$\beta_2(t)$	$\beta_1(t)$
0.1	0.7464340	0.7494670	0.749467	0.749467	0.749467	0.7494670	0.7650150
0.1	0.5573120	0.5605830	0.560583	0.560583	0.560583	0.5605850	0.5914700
0.3	0.4166630	0.4193020	0.419302	0.419302	0.419302	0.4193040	0.4671700
0.4	0.3101140	0.3122690	0.312269	0.312269	0.312269	0.3122730	0.3735150
0.5	0.2301470	0.2316430	0.231643	0.231643	0.231643	0.2316470	0.3059930
0.6	0.1671680	0.1683730	0.168373	0.168373	0.168373	0.1683780	0.2478720
0.7	0.1177880	0.1185300	0.118530	0.118530	0.118530	0.1185350	0.1991020
0.8	0.0753628	0.0759111	0.075911	0.075911	0.075911	0.0759147	0.1449800
0.9	0.0376411	0.0378713	0.037871	0.037871	0.037871	0.0378728	0.0832674

On the follow figure we also can see the α-iterates (with unbroken line) and the β-iterates (with broken line) for $t = 0.5$ again.



In addition the graph in next figure shows the approximate solution of (4.1) using the finite-difference method and Mathematica for each iterate.



Since the convergence of the iterates is of order 4 we obtained the approximate solution very fast, in three steps only.

Remark: The above result can be extended to include the situation when $f(t,x,u) = f_1(t,x,u) + f_2(t,x,u)$ where $f_1(t,x,u)$ satisfies the hypothesis of the theorem whereas $f_2(t,x,u)$ satisfies

$$0 \geq \frac{\partial^{2k-1} f_2(t,x,\zeta_1)}{\partial u^{2k-1}} - \frac{\partial^{2k-1} f_2(t,x,\zeta_2)}{\partial u^{2k-1}} \geq M_1(\zeta_1 - \zeta_2) \text{ on } \overline{Q}_T$$

for $\alpha_0(t,x) \leq \zeta_2(t,x) \leq \zeta_1(t,x) \leq \beta_0(t,x)$.

Conclusion: In the above theorems we assumed that the $m-1$-th derivative of the functions $f(t,x,u)$ and $g(t,x,u)$ are nondecreasing and one-sided Lipschitzian with respect to u. We have developed iterates of nonlinearity of order $m-1$ which converge rapidly (order m) to the unique solution of nonlinear integro-differential equations of parabolic type. We demonstrate the application of the theoretical results with numerical example.

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