Communications in Applied Analysis 11 (2007), 429-440

PERTURBED FUNCTIONAL DIFFERENTIAL EQUATIONS WITH FRACTIONAL ORDER

ABDELKADER BELARBI, MOUFFAK BENCHOHRA, SAMIRA HAMANI, AND SOTIRIS K. NTOUYAS

Laboratoire de Mathématiques, Université de Sidi Bel-Abbès
B.P. 89, 22000, Sidi Bel-Abbès, ALGERIE aek_belarbi@yahoo.fr
Laboratoire de Mathématiques, Université de Sidi Bel-Abbès, B.P. 89, 22000, Si di Bel-Abbès, ALGERIE benchohra@univ-sba.dz
Laboratoire de Mathématiques, Université de Sidi Bel-Abbès, B.P. 89, 22000, Sidi Bel-Abbès, ALGERIE hamani_samira@yahoo.fr
Department of Mathematics, University of Ioannina, 451 10 Ioannina GREECE
sntouyas@cc.uoi.gr

ABSTRACT. In this paper, we shall establish sufficient conditions for the existence of solutions, as well as extremal solutions, for perturbed fractional functional differential equations.

AMS (MOS) Subject Classification. 26A33, 34K05

1. INTRODUCTION

This paper deals with the existence of solutions as well as the existence of extremal solutions for initial value problems (IVP for short) for perturbed fractional order functional differential equations

$$D^\alpha y(t) = f(t, y_t) + g(t, y_t), \text{ for each } t \in J = [0, b], \quad 0 < \alpha < 1, \tag{1.1}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{1.2}$$

where D^α is the Riemman-Liouville fractional derivative, $f, g : J \times C([-r, 0], \mathbb{R}) \to \mathbb{R}$ are given functions and $\phi \in C([-r, 0], \mathbb{R})$ with $\phi(0) = 0$. For any continuous function y defined on $[-r, b]$ and any $t \in J$, we denote by y_t the element of $C([-r, 0], \mathbb{R})$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0].$$

Here $y_t(\cdot)$ represents the history of the state from time $t - r$ up to the present time t.

Functional differential equations arise in a variety of areas of biological, physical, and engineering applications, see, for example, the books of Kolmanovskii and Myshkis [20], and Hale and Verduyn Lunel [14] and the references therein. Differential equations of fractional order have recently proved to be valuable tools in the

Received July 2, 2007

modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [4, 11, 13, 17, 21, 22, 26]). For three noteworthy papers dealing with the integral operator and the arbitrary fractional order differential operator, see [7, 8, 10]. There has been a significant development in fractional differential equations in recent years; see the monographs of Kilbas *et al.* [19], Miller and Ross [23], Podlubny [27], Samko *et al.* [29] and the papers of Delbosco and Rodino [2], Diethelm *et al.* [4, 5, 6], El-Sayed [9], Kilbas and Marzan [18], Mainardi [21], Momani and Hadid [24], Momani *et al.* [25], Podlubny *et al.* [28], Yu and Gao [30] and the references therein.

In this paper, we shall prove the existence of solutions, as well as, the existence of extremal solutions for the problem (1.1)- (1.2). Our approach is based, for the existence of solutions, on a new fixed point theorem of Burton and Kirk BuKi1, and for the existence of extremal solutions, on the concept of upper and lower solutions combined with a fixed point theorem on ordered Banach spaces established recently by Dhage and Henderson [3]. These results can be considered as a contribution to this emerging field.

2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\}.$$

Also, $C([-r,0],\mathbb{R})$ is endowed with the norm $\|\cdot\|_C$ defined by

$$\|\phi\|_C := \sup\{|\phi(\theta)| : -r \le \theta \le 0\}.$$

Definition 2.1. The fractional primitive of order $\alpha > 0$ of a function $h : (0,b] \to \mathbb{R}$ is defined by

$$I_0^\alpha h(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

provided the right side is pointwise defined on $(0,b]$ and Γ is the gamma function.

For instance, $I^\alpha h$ exists for all $\alpha > 0$ when $h \in C((0,b],\mathbb{R}) \cap L^1((0,b],\mathbb{R})$; note also that when $h \in C([0,b],\mathbb{R})$ then $I^\alpha h \in C([0,b],\mathbb{R})$ and moreover $I^\alpha h(0) = 0$.

Definition 2.2. The fractional derivative of order $\alpha > 0$ of a continuous function $h : (0, b] \to \mathbb{R}$ is given by

$$\begin{aligned} \frac{d^\alpha h(t)}{dt^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha} h(s) ds \\ &= \frac{d}{dt} I_a^{1-\alpha} h(t). \end{aligned}$$

More details on fractional integrals and fractional derivatives can be found in [19, 23, 27, 29].

3. EXISTENCE OF SOLUTIONS

Let us start by defining what we mean by a solution of the problem (1.1)–(1.2).

Definition 3.1. A function $y \in C([-r, b], \mathbb{R})$ is said to be a solution of (1.1)–(1.2) if y satisfies the equation $D^\alpha y(t) = f(t, y_t) + g(t, y_t)$ on J and the condition $y(t) = \phi(t)$ on $[-r, 0]$.

For existence results for the problem (1.1)-(1.2) we need the following auxiliary lemma.

Lemma 3.2. ([2]) *Let $0 < \alpha < 1$ and let $h : (0, b] \to \mathbb{R}$ be continuous and $\lim_{t\to 0^+} h(t) = h(0^+) \in \mathbb{R}$. Then y is a solution of the fractional integral equation*

$$y(t) = \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds,$$

if and only if, y is a solution of the initial value problem for the fractional differential equation

$$D^\alpha y(t) = h(t), \quad t \in (0, b],$$
$$y(0) = 0.$$

We state the following generalization of Gronwall's lemma for singular kernels, whose proof can be found in [16], will be essential for the main result of this section.

Lemma 3.3. *Let $v : [0, b] \to [0, \infty)$ be a real function and $w(\cdot)$ is a nonnegative, locally integrable function on $[0, b]$ and let be the constants $a > 0$ and $0 < \alpha < 1$ such that*

$$v(t) \le w(t) + a \int_0^t \frac{v(s)}{(t-s)^\alpha} ds.$$

Then, there exists a constant $K = K(\alpha)$ such that

$$v(t) \le w(t) + Ka \int_0^t \frac{w(s)}{(t-s)^\alpha} ds,$$

for every $t \in [0, b]$.

Our first existence result for the IVP (1.1)–(1.2) is based on the following fixed point theorem due to Burton and Kirk [1]:

Theorem 3.4. *Let X be a Banach space, and A, B two operators satisfying:*

(i) *A is a contraction, and*
(ii) *B is completely continuous.*

Then either

(a) *the operator equation $y = A(y) + B(y)$ has a solution, or*
(b) *the set $\mathcal{E} = \left\{u \in X : \lambda A\left(\frac{u}{\lambda}\right) + \lambda B(u) = u\right\}$ is unbounded for $\lambda \in (0,1)$.*

Let us introduce the following hypotheses witch are assumed hereafter:

(H1) The function $f : J \times C([-r,0],\mathbb{R}) \to \mathbb{R}$ is continuous.
(H2) There exists a constant $k > 0$ such that

$$|g(t,u) - g(t,\overline{u})| \leq k\|u - \overline{u}\|_C, \text{ for each } t \in J, \text{ and all } u,\ \overline{u} \in C([-r,0],\mathbb{R}).$$

(H3) There exists a constant $M > 0$ such that

$$|f(t,u)| \leq M \text{ for each } t \in J \text{ and all } u \in C([-r,0],\mathbb{R}).$$

Theorem 3.5. *Assume that hypotheses (H1)-(H3) hold. If*

$$\frac{b^\alpha k}{\Gamma(\alpha+1)} < 1, \tag{3.1}$$

then the IVP (1.1)-(1.2) has at least one solution on $[-r,b]$.

Proof. Consider the operators:

$$F, G : C([-r,b],\mathbb{R}) \to C([-r,b],\mathbb{R})$$

defined by

$$F(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0], \\ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,y_s)ds, & \text{if } t \in [0,b], \end{cases}$$

and

$$G(y)(t) = \begin{cases} 0, & \text{if } t \in [-r,0], \\ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s,y_s)ds, & \text{if } t \in [0,b]. \end{cases}$$

Then the problem of finding the solutions of the IVP (1.1)-(1.2) is reduced to finding the solutions of the operator equation $F(y)(t) + G(y)(t) = y(t), t \in J$. We shall show that the operators F and G satisfy all the conditions of Theorem 3.4. The proof will be given in several steps.

Step 1: *F is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C([-r,b],\mathbb{R})$. Let $\eta > 0$ such that

$$\|y_n\|_\infty \leq \eta.$$

Then

$$\begin{aligned}
|F(y_n)(t) - F(y)(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,y_{ns}) - f(s,y_s)|ds \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sup_{s\in[0,b]}|f(s,y_{ns}) - f(s,y_s)|ds \\
&\leq \frac{\|f(\cdot,y_{n\cdot}) - f(\cdot,y_\cdot)\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\
&\leq \frac{b^\alpha\|f(\cdot,y_{n\cdot}) - f(\cdot,y_\cdot)\|_\infty}{\alpha\Gamma(\alpha)}.
\end{aligned}$$

Since f is a continuous function, then we have

$$\|F(y_n) - F(y)\|_\infty \leq \frac{b^\alpha\|f(\cdot,y_{n\cdot}) - f(\cdot,y_\cdot)\|_\infty}{\Gamma(\alpha+1)} \to 0 \quad as\ n \to \infty.$$

Step 2: *F maps bounded sets into bounded sets in $C([-r,b],\mathbb{R})$.*

Indeed, it is enough to show that for any $\eta^* > 0$, there exists a positive constant ℓ such that for each $y \in B_{\eta^*} = \{y \in C([-r,b],\mathbb{R}) : \|y\|_\infty \leq \eta^*\}$, we have $\|F(y)\|_\infty \leq \ell$. By (H3) we have for each $t \in [0,b]$,

$$\begin{aligned}
|F(y)(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,y_s)|ds \\
&\leq \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\
&\leq \frac{M}{\alpha\Gamma(\alpha)}b^\alpha.
\end{aligned}$$

Thus

$$\|F(y)\|_\infty \leq \frac{M}{\Gamma(\alpha+1)}b^\alpha := \ell.$$

Step 3: *F maps bounded sets into equicontinuous sets of $C([-r,b],\mathbb{R})$.*

Let $t_1, t_2 \in (0,b]$, $t_1 < t_2$, B_{η^*} be a bounded set of $C([-r,b],\mathbb{R})$ as in Step 2, and let $y \in B_{\eta^*}$. Then

$$\begin{aligned}
|F(y)(t_2) - F(y)(t_1)| &= \Big|\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^{t_1}[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]f(s,y_s)ds \\
&\quad + \tfrac{1}{\Gamma(\alpha)}\textstyle\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s,y_s)ds\Big| \\
&\leq \tfrac{M}{\Gamma(\alpha)}\textstyle\int_0^{t_1}[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}]ds \\
&\quad + \tfrac{M}{\Gamma(\alpha)}\textstyle\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds \\
&\leq \tfrac{M}{\Gamma(\alpha+1)}[(t_2-t_1)^\alpha + t_1^\alpha - t_2^\alpha] + \tfrac{M}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha \\
&\leq \tfrac{M}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha + \tfrac{M}{\Gamma(\alpha+1)}(t_1^\alpha - t_2^\alpha).
\end{aligned}$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $t_1 < t_2 \leq 0$ and $t_1 \leq 0 \leq t_2$ is obvious.

As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $F : C([-r,b],\mathbb{R}) \longrightarrow C([-r,b],\mathbb{R})$ is continuous and completely continuous.

Step 4: *G is a contraction.*

Let $x, y \in C([-r,b],\mathbb{R})$. Then, for each $t \in J$ we have

$$\begin{aligned}
|G(x)(t) - G(y)(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|g(s,x_s) - g(s,y_s)|ds \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}k\|x-y\|_\infty ds \\
&\leq \frac{k\|x-y\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\
&\leq \frac{b^\alpha k}{\alpha\Gamma(\alpha)}\|x-y\|_\infty.
\end{aligned}$$

Thus,

$$\|G(x) - G(y)\|_\infty \leq \frac{b^\alpha k}{\Gamma(\alpha+1)}\|x-y\|_\infty,$$

and consequently G is a contraction, since by (3.1), $\frac{b^\alpha k}{\Gamma(\alpha+1)} < 1$.

Step 5: *A priori bounds.*

Now it remains to show that the set

$$\mathcal{E} = \left\{ y \in C(J,\mathbb{R}) : y = \lambda F(y) + \lambda G\left(\frac{y}{\lambda}\right) \quad \text{for some} \quad 0 < \lambda < 1 \right\}$$

is bounded.

Let $y \in \mathcal{E}$; then $y = \lambda F(y) + \lambda G\left(\frac{y}{\lambda}\right)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$ we have

$$y(t) = \lambda\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,y_s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g\left(s,\frac{y_s}{\lambda}\right)ds\right].$$

This implies by (H2) and (H3) that for each $t \in J$ we have

$$\begin{aligned}
|y(t)| &\leq \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\
&\quad +\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|g\left(s,\frac{y_s}{\lambda}\right)-g(s,0)\right|ds \\
&\quad +\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|g(s,0)|ds \\
&\leq \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\
&\quad +\frac{k}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|y_s\|_C ds + \frac{b^\alpha g^*}{\Gamma(\alpha+1)},
\end{aligned}$$

where

$$g^* = \sup_{s\in J}|g(s,0)|.$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : \quad 0 \leq s \leq t\}, \quad 0 \leq t \leq b.$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0,b]$, then by the previous inequality we have for $t \in [0,b]$

$$\mu(t) \leq \frac{b^\alpha(M+g^*)}{\Gamma(\alpha+1)} + \frac{k}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mu(s)ds.$$

If $t^* \in [-r,0]$, then $\mu(t) = \|\phi\|_C$ and the previous inequality holds.

Then, from Lemma 3.3, there exists $\tilde{K} = \tilde{K}(\alpha)$ such that we have

$$\begin{aligned}
\|\mu\|_\infty &\leq \frac{b^\alpha(M+g^*)}{\Gamma(\alpha+1)} + \tilde{K}\frac{k}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Rds \\
&\leq \frac{b^\alpha(M+g^*)}{\Gamma(\alpha+1)} + \tilde{K}\frac{kb^\alpha}{\Gamma(\alpha+1)}R := \tilde{R},
\end{aligned}$$

where

$$R = \frac{b^\alpha(M+g^*)}{\Gamma(\alpha+1)}.$$

Since for every $t \in [0,b]$, $\|y_t\|_\infty \leq \mu(t)$, we have

$$\|y\|_\infty \leq \max(\|\phi\|_C, \tilde{R}) := A.$$

This shows that the set $\mathcal{E}$ is bounded. As a consequence of Theorem 3.4, we deduce that $F(y)+G(y)$ has a fixed point which is a solution of the problem (1.1) − (1.2).

□

4. EXISTENCE OF EXTREMAL SOLUTIONS

In this section, we shall prove the existence of minimal and maximal solutions for the IVP (1.1)-(1.2) under suitable monotonicity conditions on the functions involved.

Definition 4.1. A nonempty closed subset C of a Banach space X is said to be a cone if

(i) $C + C \subset C$,
(ii) $\lambda C \subset C$, and
(iii) $\{-C\} \cap \{C\} = \{0\}$.

A cone C is called normal if the norm $\|\cdot\|$ is semi-monotone on C, i.e., there exists a constant $N > 0$ such that $\|x\| \leq N\ |y\|$, whenever $x \leq y$. We equip the space $X = C([-r, b], \mathbb{R})$ with the order relation $\leq$ induced by a regular cone in E, that is for all $y, \overline{y} \in X : y \leq \overline{y}$ if and only if $\overline{y}(t) - y(t) \geq 0,\ \forall t \in [-r, b]$. Cones and their properties are detailed in [12, 15]. Let $\alpha,\ \beta \in X$ be such that $\alpha \leq \beta$. Then, by an order interval $[\alpha, \beta]$ we mean a set of points in X given by

$$[\alpha, \beta] = \{x \in X \mid \alpha \leq x \leq \beta\}.$$

Definition 4.2. Let X be an ordered Banach space. A mapping $T : X \to X$ is called isotone increasing if $T(x) \leq T(y)$ for any $x, y \in X$ with $x < y$. Similarly, T is called isotone decreasing if $T(x) \geq T(y)$ whenever $x < y$.

Definition 4.3. ([15]) We say that $x \in X$ is the least fixed point of G in X if $x = Gx$ and $x \leq y$ whenever $y \in X$ and $y = Gy$. The greatest fixed point of G in X is defined similarly by reversing the inequality. If both a least and greatest fixed point of G in X exist, we call them extremal fixed points of G in X.

We need the following fixed point theorem in the sequel.

Theorem 4.4. ([3]) *Let $[\alpha, \beta]$ be an order interval in a Banach space and let $B_1, B_2 : [\alpha, \beta] \to X$ be two functions satisfying:*

(a) *B_1 is a contraction,*
(b) *B_2 is completely continuous,*
(c) *B_1 and B_2 are strictly monotone increasing, and*
(d) *$B_1(x) + B_2(x) \in [\alpha, \beta]$, for all $x \in [\alpha, \beta]$.*

Furthermore, if the cone C in X is normal, then the equation $x = B_1(x) + B_2(x)$ has a least fixed point x_* and a greatest fixed point $x^* \in [\alpha, \beta]$. Moreover $x_* = \lim_{n\ to\infty} x_n$ and $x^* = \lim_{n\to\infty} y_n$, where $\{x_n\}$ and $\{y_n\}$ are the sequences in $[\alpha, \beta]$ defined by

$$x_{n+1} = B_1(x_n) + B_2(x_n),\ x_0 = \alpha \ \text{ and } \ y_{n+1} = B_1(y_n) + B_2(y_n),\ y_0 = \beta.$$

We adopt the following definitions.

Definition 4.5. A function $v \in C([-r,b],\mathbb{R})$ is called a lower solution of the IVP (1.1)-(1.2) if $D^\alpha v(t) \leq f(t,v_t) + g(t,v_t),$ for each $t \in J$ and $v(t) \leq \phi(t)$ if $t \in [-r,0]$. Similarly an upper solution w of IVP (1.1)-(1.2) is defined by reversing the order of the above inequalities.

Definition 4.6. A solution x_M of the IVP (1.1)-(1.2) is said to be maximal if for any other solution x of the IVP (1.1)-(1.2) on $[-r,b]$ we have $x(t) \leq x_M(t)$ for each $t \in [-r,b]$.
Similarly a minimal solution of IVP (1.1)-(1.2) is defined by reversing the order of the inequalities.

Definition 4.7. A function $f(t,x)$ is called strictly monotone increasing in x almost everywhere for $t \in J$, if $f(t,x) \leq f(t,y)$ for each $t \in J$ and all $x,y \in C([-r,0],\mathbb{R})$ with $x < y$. Similarly $f(t,x)$ is called strictly monotone decreasing in x almost everywhere for $t \in J$, if $f(t,x) \geq f(t,y)$ a.e. $t \in J$ for all $x,y \in C([-r,0],\mathbb{R})$ with $x < y$.

We need the following assumptions in the sequel.

(H4) The functions $f(t,y)$ and $g(t,y)$ are strictly monotone nondecreasing in y for each $t \in J$.

(H5) The IVP (1.1)-(1.2) has a lower solution v and an upper solution w with $v \leq w$.

Theorem 4.8. *Assume that assumptions (H1)-(H5) hold. Then the IVP (1.1)-(1.2) has minimal and maximal solutions on* $[-r,b]$.

Proof. It can be shown, as in the proof of Theorem 3.5, that F is completely continuous and G is a contraction on $[v,w]$. We shall show that F and G are isotone increasing on $[v,w]$. Let $y,\overline{y} \in [v,w]$ be such that $y \leq \overline{y}$, $y \neq \overline{y}$. Then by (H4), we have for each $t \in J$

$$\begin{aligned} F(y)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y_s)ds \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,\overline{y}_s)ds \\ &= F(\overline{y})(t). \end{aligned}$$

Similarly, $G(y) \leq G(\overline{y})$. Therefore F and G are isotone increasing on $[v,w]$. Finally, let $x \in [v,w]$ be any element. By (H5), we deduce that

$$v \leq F(v) + G(v) \leq F(x) + G(x) \leq F(w) + G(w) \leq w,$$

which shows that $F(x)+G(x) \in [v,w]$ for all $x \in [v,w]$. Thus, the functions F and G satisfy all the conditions of Theorem 4.4, and hence the IVP (1.1)-(1.2) has minimal and maximal solutions on $[-r,b]$. This completes the proof.

□

5. AN EXAMPLE

In this section we give an example to illustrate the usefulness of our main results. Let us consider the fractional functional differential equation,

$$D^{(\alpha)}y(t) = \frac{ce^t\|y_t\|}{(e^t+e^{-t})(1+\|y_t\|)} + \frac{e^{-t}}{1+\|y_t\|}, \quad t \in J := [0,b], \quad \alpha \in (0,1), \tag{5.1}$$

$$y(t) = \phi(t), \quad t \in [-r,0], \tag{5.2}$$

where $\phi(0)=0,\ c\Gamma(\alpha) = c_0\int_0^b s^{\alpha-1}e^{-s}ds$, and $c_0 > 1$ fixed. Set

$$f(t,x) = \frac{e^t x}{c(e^t+e^{-t})(1+x)}, \quad (t,x) \in J \times [0,\infty),$$

and

$$g(t,x) = \frac{e^{-t}}{1+x}, \quad (t,x) \in J \times [0,\infty).$$

Let $x, y \in [0,\infty)$ and $t \in J$. Then we have

$$\begin{aligned} |f(t,x)-f(t,y)| &= \frac{e^t}{c(e^t+e^{-t}))}\left|\frac{x}{1+x} - \frac{y}{1+y}\right| \\ &= \frac{e^t|x-y|}{c(e^t+e^{-t})(1+x)(1+y)} \\ &\leq \frac{e^t}{c(e^t+e^{-t})}|x-y| \\ &\leq \frac{1}{c}|x-y|, \end{aligned}$$

Hence the condition $(H2)$ holds. Also, the function g is continuous on $J\times[0,\infty)$ and

$$|g(t,x)| \leq 1, \text{ for each } (t,x) \in J \times [0,\infty).$$

Thus conditions (H1) and (H3) hold. Assume that $\frac{b^\alpha}{c\Gamma(\alpha+1)} < 1$, then by Theorem 3.4 the problem (5.1)-(5.2) has at least one solution on $[-r,b]$.

REFERENCES

[1] T. A. Burton and C. Kirk, A fixed point theorem of Krasnoselskii-Schaefer type, *Math. Nachr.* **189** (1998), 23–31.

[2] D. Delbosco and L. Rodino, Existence and uniqueness for a nonlinear fractional differential equation, *J. Math. Anal. Appl.* **204** (1996), 609–625.

[3] B. C. Dhage and J. Henderson, Existence theory for nonlinear functional boundary value problems, *Electron. J. Qual. Theory Differ. Equ.* 2004, No. 1, 15 pp.

[4] K. Diethelm and A.D. Freed, On the solution of nonlinear fractional order differential equations used in the modeling of viscoplasticity, in "Scientifice Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties" (F. Keil, W. Mackens, H. Voss, and J. Werther, Eds), pp 217–224, Springer-Verlag, Heidelberg, 1999.

[5] K. Diethelm and N. J. Ford, Analysis of fractional differential equations, *J. Math. Anal. Appl.* **265** (2002), 229–248.

[6] K. Diethelm and G. Walz, Numerical solution of fractional order differential equations by extrapolation, *Numer. Algorithms* **16** (1997), 231–253.

[7] A. M. A. El-Sayed, Fractional order evolution equations, *J. Fract. Calc.* **7** (1995), 89–100.

[8] A. M. A. El-Sayed, Fractional order diffusion-wave equations, *Intern. J. Theo. Physics* **35** (1996), 311–322.

[9] A. M. A. El-Sayed, Nonlinear functional differential equations of arbitrary orders, *Nonlinear Anal.* **33** (1998), 181–186.

[10] A. M. A. El-Sayed and A. G. Ibrahim, Multivalued fractional differential equations, *Appl. Math. Comput.* **68** (1995), 15–25.

[11] L. Gaul, P. Klein and S. Kempfle, Damping description involving fractional operators, *Mech. Systems Signal Processing* **5** (1991), 81–88.

[12] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, 1988.

[13] W. G. Glockle and T. F. Nonnenmacher, A fractional calculus approach of self-similar protein dynamics, *Biophys. J.* **68** (1995), 46–53.

[14] J. Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Applied Mathematical Sciences, 99, Springer-Verlag, New York, 1993.

[15] S. Heikkila and V. Lakshmikantham, *Monotone Iterative Technique for Nonlinear Discontinuous Differential Equations*, Marcel Dekker Inc., New York, 1994.

[16] D. Henry, *Geometric Theory of Semilinear Parabolic Partial Differential Equations*, Springer-Verlag, Berlin/New York, 1989.

[17] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[18] A. A. Kilbas and S. A. Marzan, Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions, *Differential Equations* **41** (2005), 84–89.

[19] A. A. Kilbas, Hari M. Srivastava, and Juan J. Trujillo, *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[20] V. Kolmanovskii, and A. Myshkis, *Introduction to the Theory and Applications of Functional-Differential Equations*. Mathematics and its Applications, 463. Kluwer Academic Publishers, Dordrecht, 1999.

[21] F. Mainardi, Fractional calculus: Some basic problems in continuum and statistical mechanics, in "Fractals and Fractional Calculus in Continuum Mechanics" (A. Carpinteri and F. Mainardi, Eds), pp. 291–348, Springer-Verlag, Wien, 1997.

[22] F. Metzler, W. Schick, H. G. Kilian and T. F. Nonnenmacher, Relaxation in filled polymers: A fractional calculus approach, *J. Chem. Phys.* **103** (1995), 7180–7186.

[23] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[24] S. M. Momani and S. B. Hadid, Some comparison results for integro-fractional differential inequalities. *J. Fract. Calc.* **24** (2003), 37–44.

[25] S. M. Momani, S. B. Hadid and Z. M. Alawenh, Some analytical properties of solutions of differential equations of noninteger order, *Int. J. Math. Math. Sci.* **2004** (2004), 697–701.

[26] K. B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, London, 1974.

[27] I. Podlubny, *Fractional Differential Equations*, Academic Press, Singapore, 1999.

[28] I. Podlubny, I. Petraš, B. M. Vinagre, P. O'Leary and L. Dorčak, Analogue realizations of fractional-order controllers. Fractional order calculus and its applications, *Nonlinear Dynam.* **29** (2002), 281–296.

[29] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[30] C. Yu and G. Gao, Existence of fractional differential equations, *J. Math. Anal. Appl.* **310** (2005), 26–29.