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A NOTE ON FACTORIZATION OF BOUNDED LINEAR OPERATORS

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ABSTRACT: We give some conditions for simultaneous factorization of a finite family of bounded linear operators with values in a normed space such that the collection of its closed balls has the binary intersection property.

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1. INTRODUCTION AND PRELIMINARIES

Throughout this paper we shall assume that all normed spaces are real.

A family of sets is said to be *chained* if every pair of sets of the family has a nonempty intersection. We shall describe a normed space W as an $\mathcal{M}$-*type space* if the collection of its closed balls has the *binary intersection property*, that is, every chained family of closed balls of W has a nonempty intersection. An example of an $\mathcal{M}$-type space is given by the space of all real bounded functions on a set Ω, endowed with the norm $\|f\| = sup\{|f(x)| : x \in \Omega\}$. In particular, the real line is an $\mathcal{M}$-type space. For further information see Nachbin [3] and Kantorovich and Akilov [2].

Let U and V be normed spaces. We shall denote by $L(U, V)$ the space of all bounded linear operators on U into V and by $L(U)$ when $U = V$. The null space and the range of a linear operator $T \in L(U, V)$ will be denoted by $\mathcal{N}(T)$ and $\mathcal{R}(T)$, respectively. For $T \in L(U, V)$, we associate the usual adjoint $T^* \in L(V^*, U^*)$, where U^* and V^* are the dual spaces of U and V, respectively.

Example 1. Let $F_b(\Omega)$ be the space of all real bounded functions on the nonempty set Ω, endowed with the supremum norm. Consider the space $F_b(\Omega) \times F_b(\Omega)$ with

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the norm $\|(f_1, f_2)\| = \|f_1\| + \|f_2\|$. Given $P, Q, B \in L(F_b(\Omega))$, consider the linear operators:

$$\begin{aligned} T_1 : F_b(\Omega) \times F_b(\Omega) &\to F_b(\Omega) \times F_b(\Omega)\,, \\ (f_1, f_2) &\mapsto (0, Pf_1)\,, \end{aligned}$$

$$\begin{aligned} T_2 : F_b(\Omega) \times F_b(\Omega) &\to F_b(\Omega) \times F_b(\Omega)\,, \\ (f_1, f_2) &\mapsto (Qf_2 - Bf_1, 0)\,, \end{aligned}$$

$$\begin{aligned} S_1 : F_b(\Omega) \times F_b(\Omega) &\to F_b(\Omega)\,,\,, \\ (f_1, f_2) &\mapsto Pf_1\,, \end{aligned}$$

$$\begin{aligned} S_2 : F_b(\Omega) \times F_b(\Omega) &\to F_b(\Omega)\,, \\ (f_1, f_2) &\mapsto Qf_2 - Bf_1\,. \end{aligned}$$

These operators satisfy the following conditions:

(i) $\|S_1(f_1, f_2) + S_2(g_1, g_2)\| \leq \|T_1(f_1, f_2) + T_2(g_1, g_2)\|$ for all

$$(f_1, f_2), (g_1, g_2) \in F_b(\Omega) \times F_b(\Omega).$$

(ii) There exists $A \in L(F_b(\Omega) \times F_b(\Omega), F_b(\Omega))$ such that $AT_i = S_i$ for $i = 1, 2$.

(iii) $R(S_i^*) \subset R(T_i^*)$ for $i = 1, 2$.

Indeed,

(i) We have

$$\begin{aligned} \|S_1(f_1, f_2) + S_2(g_1, g_2)\| &= \|Pf_1 + Qg_2 - Bg_1\| \leq \|Pf_1\| + \|Qg_2 - Bg_1\| \\ &= \|(Qg_2 - Bg_1, Pf_1)\| = \|(0, Pf_1) + (Qg_2 - Bg_1, 0)\| \\ &= \|T_1(f_1, f_2) + T_2(g_1, g_2)\|. \end{aligned}$$

(ii) Take

$$\begin{aligned} A : F_b(\Omega) \times F_b(\Omega) &\to F_b(\Omega)\,, \\ (f_1, f_2) &\mapsto f_1 + f_2. \end{aligned}$$

$AT_1(f_1, f_2) = A(0, Pf_1) = Pf_1 = S_1(f_1, f_2)$.
$AT_2(f_1, f_2) = A(Qf_2 - Bf_1, 0) = Qf_2 - Bf_1 = S_2(f_1, f_2)$.

(iii) For each $\phi \in F_b(\Omega)^*$, consider the linear functional

$$\begin{aligned} \psi_\phi : F_b(\Omega) \times F_b(\Omega) &\to \mathbb{R}\,, \\ (f_1, f_2) &\mapsto \phi(f_1 + f_2). \end{aligned}$$

We have

$$\begin{aligned} S_1^*(\phi)(f_1, f_2) &= \phi \circ S_1(f_1, f_2) = \phi(Pf_1) = \phi(0 + Pf_1) \\ &= \psi_\phi(0, Pf_1) = \psi_\phi(T_1(f_1, f_2)) = T_1^*(\psi_\phi)(f_1, f_2) \end{aligned}$$

for any $(f_1, f_2) \in F_b(\Omega) \times F_b(\Omega)$. Hence $R(S_1^*) \subset R(T_1^*)$. Similarly, we can show that $R(S_2^*) \subset R(T_2^*)$.

Motivated by Example 1 and Theorem 2.8 in Jo et al [1], we state our result.

Theorem 1. *Let U and V be normed spaces and W be an $\mathcal{M}$-type space. Let $S_1, ..., S_n \in L(U, W)$ and $T_1, ..., T_n \in L(U, V)$. If*

$$\|T_k f_k\| \leq \|\sum_{i=1}^{n} T_i f_i\| \quad \text{for each} \quad f_k \in U,$$

$k = 1, ..., n$, then the following statements are equivalent:

(a) *There exists a constant $C > 0$ such that*

$$\|\sum_{i=1}^{n} S_i f_i\| \leq C \|\sum_{i=1}^{n} T_i f_i\|$$

for all finite collections of vectors $\{f_1, ..., f_n\}$ in U.

(b) *There exists $A \in L(V, W)$ such that $AT_i = S_i$ for $i = 1, ..., n$.*

(c) *$\mathcal{R}(S_i^*) \subset \mathcal{R}(T_i^*)$ for $i = 1, ..., n$.*

2. PROOF OF THE THEOREM

We need the following linear extension result showed by Nachbin Nachbin [3].

Lemma 1. *Let V be a normed space and W be an $\mathcal{M}$-type space. Further, let E be a vector subspace of V and $A_o : E \to W$ be a bounded linear operator. Then there exists a bounded linear operator $A : V \to W$ such that $Ax = A_o x$ for all $x \in E$ and $\|A\| = \|A_o\|$.*

Proof of Theorem 1. Assume that (a) holds. Note that

$$E := \{\sum_{i=1}^{n} T_i f_i : f_1, ..., f_n \in U\}$$

is a vector subspace of V. Let $A_0 : E \to W$ be defined by

$$A_0(\sum_{i=1}^{n} T_i f_i) \quad = \quad \sum_{i=1}^{n} S_i f_i$$

for every $f_1, ..., f_n \in U$. Let us verify that A_0 is well defined. If $\sum_{i=1}^{n} T_i f_i = \sum_{i=1}^{n} T_i g_i$ for $f_i, g_i \in U$, $i = 1, ..., n$, then

$$0 = \|\sum_{i=1}^{n} T_i f_i - \sum_{i=1}^{n} T_i g_i\| = \|\sum_{i=1}^{n} T_i (f_i - g_i)\| . \tag{1}$$

It follows from (a) that there exists a constant $C > 0$ such that

$$\|\sum_{i=1}^{n} S_i(f_i - g_i)\| \leq C\|\sum_{i=1}^{n} T_i(f_i - g_i)\|.$$

Hence, by (1)

$$\sum_{i=1}^{n} S_i f_i = \sum_{i=1}^{n} S_i g_i.$$

Note that A_0 is a bounded linear operator since (a) holds. Thus, by Lemma 1, there exists a bounded linear extension $A : V \to W$ of A_0 and we conclude that $AT_i = S_i$ for $i = 1, ..., n$.

The statement (c) follows from (b) since $S_i^* = T_i^* A^*$ for $i = 1, ..., n$.

To prove that (c) implies (a), let us assume that $\mathcal{R}(S_i^*) \subset \mathcal{R}(T_i^*)$ for $i = 1, ..., n$. Then for every $\phi \in W^*$, there exist $\psi_i \in V^*$ such that $S_i^*\phi = T_i^*\psi_i$, for $i = 1, ..., n$. Let $f_1, ..., f_n$ be arbitrary vectors in U such that $\|\sum_{i=1}^{n} T_i f_i\| \neq 0$. We have

$$\begin{aligned} |\phi(\sum_{i=1}^{n} S_i f_i)| &= |\sum_{i=1}^{n} \phi(S_i f_i)| = |\sum_{i=1}^{n} (S_i^*\phi) f_i| = |\sum_{i=1}^{n} (T_i^*\psi_i) f_i| \\ &= |\sum_{i=1}^{n} \psi_i(T_i f_i)| \leq \sum_{i=1}^{n} |\psi_i(T_i f_i)| \leq \sum_{i=1}^{n} \|\psi_i\|\|T_i f_i\| \\ &\leq K \sum_{i=1}^{n} \|T_i f_i\|, \end{aligned}$$

where $K = max\{\|\psi_i\| : i = 1, ..., n\}$. Hence

$$|\phi(\sum_{i=1}^{n} S_i f_i / \|\sum_{i=1}^{n} T_i f_i\|)| \leq K \sum_{i=1}^{n} (\|T_i f_i\| / \|\sum_{i=1}^{n} T_i f_i\|) \leq Kn.$$

Therefore, it follows from the Principle of Uniform Boundedness that the set

$$\{\sum_{i=1}^{n} S_i f_i / \|\sum_{i=1}^{n} T_i f_i\| : \|\sum_{i=1}^{n} T_i f_i\| \neq 0; f_1, ..., f_n \in U\}$$

is bounded. Hence there exists $C > 0$ such that

$$\|\sum_{i=1}^{n} S_i f_i\| \leq C\|\sum_{i=1}^{n} T_i f_i\|$$

for all $f_1, ..., f_n \in U$ such that $\|\sum_{i=1}^{n} T_i f_i\| \neq 0$.

On the other hand, if $\|\sum_{i=1}^{n} T_i f_i\| = 0$ for some collection of vectors $\{f_1, ..., f_n\}$ in U then by hypothesis $\|T_i f_i\| = 0$ for $i = 1, ..., n$. We claim that $\sum_{i=1}^{n} S_i f_i = 0$. Indeed, if $\sum_{i=1}^{n} S_i f_i \neq 0$, by the Hahn-Banach Theorem there exists $\psi \in W^*$ such that $\psi(\sum_{i=1}^{n} S_i f_i) = \|\sum_{i=1}^{n} S_i f_i\|$. Since $R(S_i^*) \subset R(T_i^*)$ there exists $\varphi_i \in V^*$ such that $(S_i^*\psi)f_i = (T_i^*\varphi_i)f_i$ for $i = 1, ..., n$. Thus,

$$\begin{aligned}\|\sum_{i=1}^n S_i f_i\| &= \psi(\sum_{i=1}^n S_i f_i) = \sum_{i=1}^n \psi(S_i f_i) = \sum_{i=1}^n (S_i^*\psi)f_i \\ &= \sum_{i=1}^n (T_i^*\varphi_i)f_i = \sum_{i=1}^n \varphi_i(T_i f_i) \leq \sum_{i=1}^n |\varphi_i(T_i f_i)| \\ &\leq \sum_{i=1}^n \|\varphi_i\|\|(T_i f_i)\| \leq M \sum_{i=1}^n \|(T_i f_i)\| = 0,\end{aligned}$$

where $M = \max\{\|\varphi_i\| : i = 1, ..., n\}$. Hence we obtain a contradiction.

Since $\|\sum_{i=1}^n T_i f_i\| = 0$ and $\sum_{i=1}^n S_i f_i = 0$, it follows that

$$\|\sum_{i=1}^n S_i f_i\| = C\|\sum_{i=1}^n T_i f_i\|.$$

Therefore, we have proved that there exists a constant $C > 0$ such that

$$\|\sum_{i=1}^n S_i f_i\| \leq C\|\sum_{i=1}^n T_i f_i\|$$

for all finite collections of vectors $\{f_1, ..., f_n\}$ in U. □

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