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A NEW BOUNDEDNESS RESULT TO NONLINEAR DIFFERENTIAL EQUATIONS OF THIRD ORDER WITH FINITE LAG

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ABSTRACT. Criteria for boundedness of solutions to the nonlinear third order delay differential equation

$$\begin{aligned} x'''(t) + \varphi(x(t), x'(t), x''(t))x''(t) + \psi(x(t-r(t)), x'(t-r(t))) + h(x(t-r(t))) \\ = p(t, x(t), x(t-r(t)), x'(t), x'(t-r(t)), x''(t)) \end{aligned}$$

are obtained by Lyapunov's second method. By introducing a Lyapunov functional, sufficient conditions are established that guarantee that all solutions of this equation are bounded. An example is also given to illustrate the importance of result obtained. Our findings improve a result existing in the literature to boundedness of solutions for this delay differential equation.

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1. Introduction

The area of differential equations has played a central role in the development of mathematics and its applications since 1660s, yet it still displays an unabated vitality. At the end of 19^{th} century, a Russian mathematician, Aleksandr Mikhailovich Lyapunov, laid the foundation for modern stability theory of differential equations in a lengthy monograph published in Russian. In his work published in 1892, Lyapunov dealt with stability by two distinct methods. His so-called first method presupposes an explicit solution known and is only applicable to some restricted but important cases. As against this, the second or direct method of Lyapunov [11] is of great generality and power and above all does not require the great knowledge of the solutions themselves. Briefly, the central idea of Lyapunov's second (or direct) method [11] is to detect stability, in addition, boundedness of solutions for a differential system by means of properties of a Lyapunov function or functional and to do this, not directly from a knowledge of solutions, but indirectly from the differential system under consideration. Meanwhile, it is worth mentioning that, beginning in the

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1960s, the stability and boundedness of solutions to delay differential systems has been considered in the literature, intensively. Up to now, many books and papers deal with stability and boundedness of solutions to delay differential equations, and many good results on the stability and boundedness of solutions of these equations have been obtained. See, for example, Burton [1], Èl'sgol'ts [2], Èl'sgol'ts and Norkin [3], Gopalsamy [4], Hale ([6], [7]), Kolmanovskii and Myshkis [8], Kolmanovskii and Nosov [9], Krasovskii [10], Sadek [14], Tunç ([15], [16], [17]), Yoshizawa [18] and the references citied in these sources. It should be noted that delay differential equations, or more generally functional differential equations, are used as models to describe many physical and biological systems. Hereby, in fact, many actual systems have the property of aftereffect, i.e., the future states depend not only on the present, but also on the past history. Aftereffect is believed to occur in mechanics, control theory, physics, chemistry, biology, medicine, economics, atomic energy, information theory, etc. This wide appearance of aftereffect is reason to regard it as a universal property of the surrounding world. In particular, for a comprehensive treatment of the subject we refer the reader to the book by Kolmanovskii and Myshkis [8] and those mentioned above. Moreover, if the solutions of a differential equation describing a dynamical system or of any differential equation under consideration are known in closed form, one can determine the boundedness properties of system or the solutions of differential equation, appealing directly the definitions of boundedness. But, it is well-known in general, it is not possible to find the solution of all linear and nonlinear differential equations, except numerically. This case is also very difficult and some times become impossible for delay differential equations. Therefore, it is very important to interpret the qualitative behaviors of solutions without solving differential equations. This fact shows the importance and applicability of Lyapunov's second (or direct) method [11]. Now, after a literature survey about nonlinear equations of third order with bounded delay, one can conclude that there are not so many results on the boundedness of solutions of higher order nonlinear delay differential equations. At the same time, we should recognize that some significant theoretical results concerning boundedness of solutions of nonlinear third order differential equations with delay have been achieved; see, for example, the paper of Tunç ([15], [16], [17]) and the references citied in that papers. At the same time, it should be noted that, in 1969, Palusinski et al. [12] applied an energy metric algorithm for the generation of a Lyapunov function to third order ordinary nonlinear differential equation without delay:

$$x''' + a_1 x'' + f_2(x')x' + a_3 x = 0.$$

They found conditions for the stability of zero solution of this equation as follows:

$$a_1 > 0, \quad a_3 > 0, \quad f_2(x') > \frac{a_3}{a_1}.$$

Later, in 2007, based on the result of Palusinski et al. [12], Tunç [17] improved the result established by Palusinski et al [12] to the third order nonlinear delay differential equation

$$x'''(t) + \varphi(x(t), x'(t))x''(t) + \psi(x(t - r(t)), x'(t - r(t))) + h(x(t - r(t))) = 0$$

and proved a result related to the asymptotic stability of trivial solution of this equation. In this paper, we will be concerned with the following third order nonlinear differential equation with finite lag

$$\begin{aligned} x'''(t) &+ \varphi(x(t), x'(t), x''(t))x''(t) + \psi(x(t - r(t)), x'(t - r(t))) + h(x(t - r(t))) \\ &= p(t, x(t), x(t - r(t)), x'(t), x'(t - r(t)), x''(t)) \qquad (1.1) \end{aligned}$$

whose associated system is

$$x'(t) = y(t), \quad y'(t) = z(t),$$

$$\begin{aligned} z'(t) = &-\varphi(x(t), y(t), z(t))z(t) - \psi(x(t), y(t)) - h(x(t)) \\ &+ \int_{t-r(t)}^{t} \psi_x(x(s), y(s))y(s)ds + \int_{t-r(t)}^{t} \psi_y(x(s), y(s))z(s)ds \\ &+ \int_{t-r(t)}^{t} h'(x(s))y(s)ds + p(t, x(t), x(t - r(t)), y(t), y(t - r(t)), z(t)), \quad (1.2) \end{aligned}$$

where r is a bounded delay, $0 \le r(t) \le \gamma$, $r'(t) \le \sigma$, $0 < \sigma < 1$, γ and σ are some positive constants and γ will be determined later; the functions φ, ψ, h and $p$depend only on the arguments displayed explicitly and the primes in equation (1.1) denote differentiation with respect to $t \in \Re^+$, $\Re^+ = [0, \infty)$. It is principally assumed that the functions φ and ψ, h and p are continuous for all values their respective arguments on $\Re^3$, $\Re^2$, $\Re$ and $\Re^+ \times \Re^5$, respectively. Besides, it is also supposed that $\psi(x, 0) = h(0) = 0$, and the derivatives $\varphi_x(x, y, z) \equiv \frac{\partial}{\partial x}\varphi(x, y, z)$, $\varphi_z(x, y, z) \equiv \frac{\partial}{\partial z}\varphi(x, y, z)$, $\psi_x(x, y) \equiv \frac{\partial}{\partial x}\psi(x, y)$, $\psi_y(x, y) \equiv \frac{\partial}{\partial y}\psi(x, y)$ and $h'(x) \equiv \frac{dh}{dx}$ exist and are continuous; throughout this paper $x(t)$, $y(t)$ and $z(t)$ are, respectively, abbreviated as x, y and z. All solutions considered are also assumed to be real valued.

2. Main result

We prove the following theorem, which is our main result.

Theorem 2.1. *In addition to the basic assumptions imposed on the functions φ, ψ, h and p appearing in equation* (1.1)*, we assume that there are positive constants a_1, a_2, a_3, λ, α, β, ε, μ, γ, L and M, such that the following conditions hold for all x, y and z:*

(i) $a_1a_2 - a_3 \geq \varepsilon > 0$.

(ii) $\varphi(x,y,z) \geq a_1 + 2\lambda$, $y\varphi_x(x,y,0) \leq 0$ *and* $y\varphi_z(x,y,z) \geq 0$.

(iii) $\psi(x,0) = 0$, $\frac{\psi(x,y)}{y} \geq a_2 + 2\mu$, $(y \neq 0)$, $-L \leq \psi_x(x,y) \leq 0$ *and* $|\psi_y(x,y)| \leq M$.

(iv) $h(0) = 0$ *and* $0 < h'(x) \leq a_3$.

(v) $|p(t,x(t),x(t-r(t)),y(t),y(t-r(t)),z(t))| \leq q(t)$ *for all* t, $x(t)$, $x(t-r(t))$, $y(t)$, $y(t-r(t))$ *and* $z(t)$, *where* $\max q(t) < \infty$ *and* $q \in L^1(0,\infty)$, $L^1(0,\infty)$ *is space of integrable Lebesgue functions.*

Then, there exists a finite positive constant K *such that the solution* $x(t)$ *of equation* (1.1) *defined by the initial functions*

$$x(t) = \phi(t), \quad x'(t) = \phi'(t), \quad x''(t) = \phi''(t)$$

satisfies the inequalities

$$|x(t)| \leq K, \quad |x'(t)| \leq K, \quad |x''(t)| \leq K$$

for all $t \geq t_0$, *where* $\phi \in C^2([t_0 - r, t_0], \Re)$, *provided that*

$$\gamma < \min\left\{\frac{4\mu a_3}{2\alpha + a_3L + a_3M + a_3^2}, \frac{2\varepsilon + 4\lambda a_2}{a_2(L+M) + a_2a_3 + 2\beta}\right\}.$$

Remark 2.2. When $r(t) = 0$ in (1.1), then equation (1.1) reduces to the following third order ordinary nonlinear differential equation without delay:

$$x'''(t) + \varphi(x(t),x'(t),x''(t))x''(t) + \psi(x(t),x'(t)) + h(x(t)) = p(t,x(t),x'(t),x''(t))$$

This equation includes some third order ordinary differential equation discussed in Greguš [5] and Reissig et al. [13]. Thus, our result is more general than that obtained in Greguš [5] and Reissig et al. [13] on the subject.

Proof. To prove this theorem, the following differentiable Lyapunov functional $V = V(x_t, y_t, z_t)$ is introduced:

$$\begin{aligned} V(x_t,y_t,z_t) &= a_3\int_0^x h(\eta)d\eta + a_2yh(x) + \frac{1}{2}a_2z^2 + a_3yz + a_2\int_0^y \psi(x,\xi)d\xi \\ &\quad + a_3\int_0^y \varphi(x,\xi,0)\xi d\xi + \alpha\int_{-r(t)}^0\int_{t+s}^t y^2(\theta)d\theta ds + \beta\int_{-r(t)}^0\int_{t+s}^t z^2(\theta)d\theta ds, \end{aligned} \tag{2.1}$$

where a_2, a_3, α and β are some positive constants, α and β will be determined later in the proof. The assumptions $h(0) = 0$ and $0 < h'(x) \leq a_3$ imply that

$$h^2(x) = 2\int_0^x h(\eta)h'(\eta)d\eta \leq 2a_3\int_0^x h(\eta)d\eta.$$

Hence,

$$a_2 y h(x) \geq -\sqrt{2a_2^2 a_3 \int_0^x h(\eta)d\eta}\, |y| \,.$$

Now, making use of the last inequality, one can recast the Lyapunov functional $V = V(x_t, y_t, z_t)$, which is defined by (2.1), as:

$$\begin{aligned} V(x_t, y_t, z_t) &\geq \frac{1}{2}a_2\left(z + \frac{a_3}{a_2}y\right)^2 + \frac{1}{2}\left(\sqrt{2a_3\int\limits_0^x h(\eta)d\eta} - a_2|y|\right)^2 \\ &+ \int\limits_0^y \left[a_3\varphi(x,\xi,0) - a_2^2 - \frac{a_3^2}{a_2} + a_2\frac{\psi(x,\xi)}{\xi}\right]\xi d\xi \\ &+ \alpha\int\limits_{-r(t)}^0\int\limits_{t+s}^t y^2(\theta)d\theta ds + \beta\int\limits_{-r(t)}^0\int\limits_{t+s}^t z^2(\theta)d\theta ds. \qquad (2.2) \end{aligned}$$

The assumptions $\varphi(x,y,z) \geq a_1 + 2\lambda$ and $\frac{\psi(x,y)}{y} \geq a_2 + 2\mu$ yield that

$$\begin{aligned} &\int\limits_0^y \left[a_3\varphi(x,\xi,0) - a_2^2 - \frac{a_3^2}{a_2} + a_2\frac{\psi(x,\xi)}{\xi}\right]\xi d\xi \\ &\geq \int\limits_0^y \left[a_1 a_3 - \frac{a_3^2}{a_2} + 2(a_3\lambda + a_2\mu)\right]\xi d\xi \\ &= \left(\frac{a_1a_2a_3 - a_3^2 + 2a_2(a_3\lambda + a_2\mu)}{2a_2}\right)y^2 > 0. \qquad (2.3) \end{aligned}$$

Now, taking into account (2.2) and (2.3) together, it follows that

$$\begin{aligned} V(x_t, y_t, z_t) &\geq \frac{1}{2}a_2\left(z + \frac{a_3}{a_2}y\right)^2 + \frac{1}{2}\left(\sqrt{2a_3\int\limits_0^x h(\eta)d\eta} - a_2|y|\right)^2 \\ &+ \left(\frac{a_1a_2a_3 - a_3^2 + 2a_2(a_3\lambda + a_2\mu)}{2a_2}\right)y^2 \\ &+ \alpha\int\limits_{-r(t)}^0\int\limits_{t+s}^t y^2(\theta)d\theta ds + \beta\int\limits_{-r(t)}^0\int\limits_{t+s}^t z^2(\theta)d\theta ds. \end{aligned}$$

By noting this inequality, it can be easily obtained that there exist sufficiently small positive constants D_i, $(i = 1, 2, 3)$, such that

$$\begin{aligned} V(x_t, y_t, z_t) &\geq D_1x^2 + D_2y^2 + D_3z^2 + \alpha\int\limits_{-r(t)}^0\int\limits_{t+s}^t y^2(\theta)d\theta ds + \beta\int\limits_{-r(t)}^0\int\limits_{t+s}^t z^2(\theta)d\theta ds \\ &\geq D_1x^2 + D_2y^2 + D_3z^2 \geq D_4(x^2 + y^2 + z^2), \qquad (2.4) \end{aligned}$$

where $D_4 = \min\{D_1, D_2, D_3\}$, since the integrals $\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds$ and $\int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds$ are non-negative. Let $(x(t), y(t), z(t))$ be a solution of system (1.2). Differentiating the functional $V(x_t, y_t, z_t)$ along this solution, we obtain

$$
\begin{aligned}
\frac{d}{dt}V(x_t,y_t,z_t) &= -a_3\left(\frac{\psi(x,y)}{y}-a_2\right)y^2-(a_1a_2-a_3)z^2+a_2y\int_0^y\psi_x(x,\xi)d\xi \\
&+(a_2z+a_3y)p(t,x(t),x(t-r(t)),y(t),y(t-r(t)),z(t)) \\
&-a_3(\varphi(x,y,z)-\varphi(x,y,0))yz \\
&+a_3y\int_{t-r(t)}^{t}\psi_x(x(s),y(s))y(s)ds+a_3y\int_{t-r(t)}^{t}\psi_y(x(s),y(s))z(s)ds \\
&+a_2z\int_{t-r(t)}^{t}\psi_x(x(s),y(s))y(s)ds+a_2z\int_{t-r(t)}^{t}\psi_y(x(s),y(s))z(s)ds \\
&+\rho r(t)y^2-\rho(1-r'(t))\int_{t-r(t)}^{t}y^2(s)ds+\mu r(t)z^2-\mu(1-r'(t))\int_{t-r(t)}^{t}z^2(s)ds \\
&=-\left(a_3\frac{\psi(x,y)}{y}-a_2a_3-\rho r(t)\right)y^2-(a_1a_2-a_3-\mu r(t))z^2 \\
&+a_2y\int_0^y\psi_x(x,\xi)d\xi-a_3(\varphi(x,y,z)-\varphi(x,y,0))yz \\
&+(a_2z+a_3y)p(t,x(t),x(t-r(t)),y(t),y(t-r(t)),z(t)) \\
&+a_3y\int_{t-r(t)}^{t}\psi_x(x(s),y(s))y(s)ds+a_3y\int_{t-r(t)}^{t}\psi_y(x(s),y(s))z(s)ds \\
&+a_2z\int_{t-r(t)}^{t}\psi_x(x(s),y(s))y(s)ds+a_2z\int_{t-r(t)}^{t}\psi_y(x(s),y(s))z(s)ds \\
&-\rho(1-r'(t))\int_{t-r(t)}^{t}y^2(s)ds-\mu(1-r'(t))\int_{t-r(t)}^{t}z^2(s)ds. \qquad (2.5)
\end{aligned}
$$

In the light of the assumptions of the theorem and the inequality $2|uv| \le u^2 + v^2$, an argument similar to the as in Tunç [17], one can easily conclude from (2.5) that, for some positive constants α and σ,

$$
\begin{aligned}
\frac{d}{dt}V(x_t,y_t,z_t) \le& -\alpha y^2-\sigma z^2-a_3(\varphi(x,y,z)-\varphi(x,y,0))yz \\
&+(a_2z+a_3y)p(t,x(t),x(t-r(t)),y(t),y(t-r(t)),z(t)) \qquad (2.6)
\end{aligned}
$$

provided

$$\gamma < \min\left\{\frac{4\mu a_3}{2\alpha + a_3L + a_3M + a_3^2}, \frac{2\varepsilon + 4\lambda a_2}{a_2(L+M) + a_2a_3 + 2\beta}\right\}.$$

We now consider the terms

$$a_3\left(\varphi(x,y,z) - \varphi(x,y,0)\right)yz$$

and

$$(a_2z + a_3y)p(t, x(t), \quad x(t-r(t)), \quad y(t), \quad y(t-r(t)), \quad z(t)),$$

which are contained in (2.6). By using the mean value theorem (for derivatives), we have

$$\begin{aligned} a_3\left(\varphi(x,y,z) - \varphi(x,y,0)\right)yz &= a_3\left[\frac{\varphi(x,y,z) - \varphi(x,z,0)}{z}\right]yz^2 \\ &= a_3yz^2\varphi_z(x,y,\theta z), 0 \le \theta \le 1. \end{aligned}$$

Making use of assumption (ii), it also follows that

$$a_3yz^2\varphi_z(x,y,\theta z) \ge 0, \quad 0 \le \theta \le 1.$$

Next, assumption (v) implies that

$$\begin{aligned} &(a_2z + a_3y)p(t, x(t), x(t-r(t)), y(t), y(t-r(t)), z(t)) \\ &\quad \le |a_2z + a_3y|\,|p(t, x(t), x(t-r(t)), y(t), y(t-r(t)), z(t))| \\ &\quad \le D_5\left(|y| + |z|\right)q(t) \le D_5\left(2 + y^2 + z^2\right)q(t) \end{aligned}$$

since $|u| < 1 + u^2$, where $D_5 = \max\{a_2, a_3\}$. In view of the above discussion, it follows from (2.6) that

$$\frac{d}{dt}V(x_t, y_t, z_t) \le D_5\left(2 + y^2 + z^2\right)q(t). \tag{2.7}$$

Inequality (2.4) implies that

$$\left(y^2 + z^2\right) \le D_4^{-1}V(x_t, y_t, z_t).$$

Using this fact into (2.7), we have

$$\begin{aligned} \frac{d}{dt}V(x_t, y_t, z_t) &\le D_5\left(2 + D_4^{-1}V(x_t, y_t, z_t)\right)q(t) \\ &= 2D_5q(t) + D_5D_4^{-1}V(x_t, y_t, z_t)q(t). \end{aligned} \tag{2.8}$$

Now, integrating (2.8) from 0 to t, using the assumption that $q \in L^1(0,\infty)$ and the Gronwall-Reid-Bellman inequality, we obtain

$$
\begin{aligned}
V(x_t, y_t, z_t) &\le V(x_0, y_0, z_0) + 2D_5A + D_5D_4^{-1}\int\limits_0^t (V(x_s, y_s, z_s))q(s)ds \\
&\le (V(x_0, y_0, z_0) + 2D_5A)\exp\left(D_5D_4^{-1}\int\limits_0^t q(s)ds\right) \\
&\le (V(x_0, y_0, z_0) + 2D_5A)\exp\left(D_5D_4^{-1}A\right) = K_1 < \infty, \qquad (2.9)
\end{aligned}
$$

where $K_1 > 0$ is a constant, $K_1 = (V(x_0, y_0, z_0) + 2D_5A)\exp\left(D_5D_4^{-1}A\right)$, and $A = \int\limits_0^\infty q(s)ds$. Now, the inequalities (2.4) and (2.9) together imply that

$$x^2(t) + y^2(t) + z^2(t) \le D_4^{-1}V(x_t, y_t, z_t) \le K,$$

where $K = K_1D_4^{-1}$. Thus, we conclude that

$$|x(t)| \le K, \quad |y(t)| \le K, \quad |z(t)| \le K$$

for all $t \ge t_0$. That is,

$$|x(t)| \le K, \quad |x'(t)| \le K, \quad |x''(t)| \le K$$

for all $t \ge t_0$. The proof of theorem is now complete. □

Example 2.3. We consider the following third order nonlinear delay differential equation

$$
\begin{aligned}
&x'''(t) + \left(4 + \frac{1}{1+(x'^2}\right)x''(t) + 4x'(t-r(t)) + \sin x'(t-r(t)) + 2arctgx(t-r(t)) \\
&= \frac{1}{1+t^2+x^2(t)+x^2(t-r(t))+x'^2(t)+x'^2(t-r(t))+x''^2(t)}, \qquad (2.10)
\end{aligned}
$$

whose associated system is

$$x'(t) = y(t), \quad y'(t) = z(t),$$

$$
\begin{aligned}
z'(t) = &-\left(4 + \frac{1}{1+y^2}\right)z(t) - (4y(t) + \sin y(t)) - 2arctgx(t) \\
&+ 2\int\limits_{t-r(t)}^t \frac{1}{1+(x(s))^2}y(s)ds + \int\limits_{t-r(t)}^t (4+\cos y(s))\,z(s)ds \\
&+ \frac{1}{1+t^2+x^2(t)+x^2(t-r(t))+y^2(t)+y^2(t-r(t))+z^2(t)}. \qquad (2.11)
\end{aligned}
$$

Now, it is clear that

$$\varphi(y) = 4 + \frac{1}{1+y^2} \geq 4 = a_1 + 2\lambda,$$
$$\psi(y) = 4y(t) + \sin y(t), \quad \psi(0) = 0,$$
$$\frac{\psi(y)}{y} = 4 + \frac{\sin y}{y}, \quad (y \neq 0, |y| < \pi),$$
$$4 + \frac{\sin y(t)}{y(t)} \geq 3 = a_2 + 2\mu,$$
$$h(x) = 2arctgx,$$
$$h(0) = 0, \quad h'(x) = \frac{2}{1+x^2}, \quad 0 < h'(x) \leq 2 = a_3,$$
$$a_1 a_2 > 2,$$

$$p(t, x(t), x(t-r(t)), y(t), y(t-r(t)), z(t))$$
$$= \frac{1}{1 + t^2 + x^2(t) + x^2(t-r(t)) + y^2(t) + y^2(t-r(t)) + z^2(t)} \leq \frac{1}{1+t^2}$$

and

$$\int_0^\infty q(s)ds = \int_0^\infty \frac{1}{1+s^2}ds = \frac{\pi}{2} < \infty,$$

that is, $q \in L^1(0,\infty)$. Hence, the above facts show that all the conditions (i) to (v) of theorem are satisfied. We also introduce the following Lyapunov functional

$$\begin{aligned} V_1(x_t, y_t, z_t) = {} & 2a_3 \int_0^x arctg\eta d\eta + 2a_2 yarctgx + \frac{1}{2}a_2 z^2 + a_3 yz \\ & + 2a_2 y^2 + a_2(1-\cos y) + 2a_3 y^2 + \frac{a_3}{2}\ln(1+y^2) \\ & + \alpha \int_{-r(t)}^0 \int_{t+s}^t y^2(\theta)d\theta ds + \beta \int_{-r(t)}^0 \int_{t+s}^t z^2(\theta)d\theta ds. \end{aligned} \tag{2.12}$$

By an elementary calculation from (2.12), it can be shown that all solutions of equation considered are bounded. We omit the details of the related operations.

REFERENCES

[1] T. A. Burton, Stability and Periodic Solutions of Ordinary and Functional-Differential Equations. Mathematics in Science and Engineering, 178. Academic Press, Inc., Orlando, FL, 1985.

[2] L. È. Èl'sgol'ts, Introduction to the Theory of Differential Equations with Deviating Arguments. Translated from the Russian by Robert J. McLaughlin Holden-Day, Inc., San Francisco, Calif.-London-Amsterdam, 1966.

[3] L. È. Èl'sgol'ts, S. B. Norkin, Introduction to the Theory and Application of Differential Equations with Deviating Arguments. Translated from the Russian by John L. Casti. Mathematics in Science and Engineering, Vol. 105. Academic Press [A Subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1973.

[4] K. Gopalsamy, Stability and Oscillations in Delay Differential Equations of Population Dynamics. Mathematics and its Applications, 74. Kluwer Academic Publishers Group, Dordrecht, 1992.

[5] M. Greguš, Third Order Linear Differential Equations. Translated from the Slovak by J. Dravecký. Mathematics and its Applications (East European Series), 22. D. Reidel Publishing Co., Dordrecht, 1987.

[6] J. Hale, Theory of Functional Differential Equations. Springer-Verlag, New York-Heidelberg, 1977.

[7] J. Hale, S. M. Verduyn Lunel, Introduction to Functional-Differential Equations. Applied Mathematical Sciences, 99. Springer-Verlag, New York, 1993.

[8] V. Kolmanovskii, A. Myshkis, Introduction to the Theory and Applications of Functional Differential Equations. Kluwer Academic Publishers, Dordrecht, 1999.

[9] V. B. Kolmanovskii, V. R. Nosov, Stability of Functional-Differential Equations. Mathematics in Science and Engineering, 180. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London, 1986.

[10] N. N. Krasovskii, Stability of Motion. Applications of Lyapunov's Second Method to Differential Systems and Equations with Delay. Translated by J. L. Brenner Stanford University Press, Stanford, Calif. 1963.

[11] A. M. Lyapunov, Stability of Motion. Mathematics in Science and Engineering, Vol. 30 Academic Press, New York-London, 1966.

[12] O. Palusinski, P. Stern, E. Wall , M. Moe, Comments on "An energy metric algorithm for the generation of Liapunov functions". *IEEE Transactions on Automatic Control*, Volume 14, Issue 1, (1969), 110 -111.

[13] R. Reissig, G. Sansone, R. Conti, Non-linear Differential Equations of Higher Order, Translated from the German. Noordhoff International Publishing, Leyden, 1974.

[14] A. I. Sadek, Stability and boundedness of a kind of third-order delay differential system. *Appl. Math. Lett.* 16 (2003), no. 5, 657-662.

[15] C. Tunç, A new boundedness theorem for a class of second order differential equations. *Arab. J. Sci. Eng. Sect. A Sci.* 33 (2008), no. 1, 83–92.

[16] C. Tunç, Stability and boundedness of solutions of nonlinear differential equations of third-order with delay. *Differ. Uravn. Protsessy Upr.* 2007, no. 3, 1–13.

[17] C. Tunç, On asymptotic stability of solutions to third order nonlinear differential equations with retarded argument. *Commun. Appl. Anal.* 11 (2007), no. 3–4, 515–527.

[18] T. Yoshizawa, Stability Theory by Liapunov's Second Method. The Mathematical Society of Japan, Tokyo, 1966.