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CONTROLLABILITY AND PRACTICAL STABILIZATION OF NONLINEAR ITÔ-TYPE STOCHASTIC CONTROLLED SYSTEMS

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ABSTRACT. In this paper, the concept of practical stability is investigated for a class of controlled stochastic systems of the Ito-Doob type. By using vector Lyapunov-like functions and comparison principle, sufficient conditions are established for various types of practical stability criteria in the p-th mean and in probability. This comparison principle allows one to determine the practical stability criteria of a nonlinear stochastic system by testing the practical stability of the corresponding auxiliary deterministic system with random initial condition. Finally, for the stochastic systems we discuss the controllability and study the optimal practical stabilization of controlled stochastic systems via the well known Hamilton-Jacobi-Bellman equation.

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1. INTRODUCTION

Stability analysis of both deterministic and stochastic systems in the Lyapunov sense is well known and is widely used in the real world problems [1,2,3,6,9]. However, sometimes for a practical system, the desired state may be unstable in the Lyapunov sense, but still good enough in the sense that the system is oscillating sufficiently near the state with an acceptable performance [5]. For example, an aircraft or a missile may oscillate around a mathematically unstable course, yet its performance may be acceptable. Problems falling in this category include the traveling of a space vehicle between two points, and the problem in a chemical process, of keeping the temperature within certain bound. To deal with such situations, the notion of practical stability is more useful. This concept was first proposed by Lasalle and Lefschetz [4], and was developed by Laksmikantham et. al [5], among others.

For stochastic systems, practical stability in the p-th mean was introduced by Zhao-Shu et. al [1] and the stability in probability by Allan Tsoi and Bo Zhang [9]. In their study, the stochastic system and the auxiliary equation has deterministic initial conditions. We have generalized this concept to stochastic systems and hence the

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auxiliary systems with random initial conditions [6,7,8]. This is achieved by using a very powerful comparison theorem developed by Bonita and Ladde [3]. In this paper, practical stability in the p-th moment and in probability is extended for a larger class of Itô-type nonlinear stochastic systems via Lyapunov-like functions and comparison principle. We have also obtained sufficient conditions for the instability criteria of the stochastic systems. Furthermore, the stabilizing control law to optimize some performance index such as the cost of the control leads to the optimal stabilization problem which is a very interesting area of research in the engineering sciences. We also present the controllability of the stochastic systems and the optimal stabilization via the standard Hamilton-Jacobi Bellman equation. Numerical examples are given to demonstrate the fruitfulness of the developed theory.

2. PRELIMINARIES

Consider a large-scale system described by a system of stochastic differential equations of Itô-Doob type,

$$dx = f(t,x)dt + \sigma(t,x)d\xi, \quad x(t_0) = x_0 \tag{2.1}$$

Equation (2.1) can be written as:

$$dx = f(t,x)dt + \sum_{r=1}^{m} \sigma_r(t,x)d\xi_r, \tag{2.2}$$

where $x(t_0) = x_0$ is an n-dimensional random vector defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$; $\xi(t) = (\xi_1(t), \ldots, \xi_m(t))$ is an m-dimensional normalized Wiener process that is $\mathcal{F}_t$-measurable for each $t \geq t_0$ and the increment $\xi(t+h) - \xi(t)$ is independent of every event in $\mathcal{F}_t$; $\mathcal{F}_t$ is an increasing family of sub-sigma algebras of $\mathcal{F}$; x_0 and $\xi(t)$ are independent for each $t \geq t_0$, $f \in \mathcal{C}[\mathcal{J} \times \Re^n, \Re^n]$, $\sigma_r \in \mathcal{C}[\mathcal{J} \times \Re^n, \Re^n]$ for $1 \leq r \leq m$; and f and g_r are smooth enough to guarantee the existence of a solution process, $x(t) = x(t, t_0, x_0)$ of (2.2) for $t \geq t_0$.

Let $x(t, t_0, x_0)$ be any solution process of (2.2) with initial data (t_0, x_0). We shall assume that$f(t,0) \equiv 0$, $\sigma_r(t,0) \equiv 0$ so that the system (2.2) possess a trivial solution.

Our objective is to investigate practical stability concepts in the sense of (i) pth mean; and (ii) probability of the solution processes of (2.2) and also the practical stability concepts for some controlled stochastic systems. This is achieved by a powerful comparison theorem [3] in the context of vector Lyapunov-like functions and systems of differential inequalities.

Definition 2.1: System (2.2) is said to be (PSM) practically stable in the p-th mean if, given $(\lambda, \mathcal{A})$ with $0 < \lambda < \mathcal{A}$, we have $E\|x_0\|^p < \lambda$ imply

$$E_{t_0,x_0}\|x(t,t_0,x_0)\|^p < \mathcal{A}, \quad \forall t \geq t_0, \quad \text{for some } t_0 \in \Re_+ \tag{2.3}$$

Definition 2.2: System (2.2) is said to be (UPSM) uniformly practically stable in the p-th mean if it is (PSM) and (2.3) holds for all $t_0 \in \Re_+$.

Definition 2.3: System (2.2) is said to be (PQSM) practically quasi-stable in the p-th mean with $(\lambda, \mathcal{B})$ if, there exists positive numbers $\lambda, \mathcal{B}$ and T, we have $E\|x_0\|^p < \lambda$ imply

$$E_{t_0,x_0}\|x(t, t_0, x_0)\|^p < \mathcal{B}, \quad \forall t \geq t_0 + T, \quad \text{for some } t_0 \in \Re_+ \tag{2.4}$$

Definition 2.4: System (2.2) is said to be (UPQSM) uniformly practically quasi-stable in the p-th mean if it is (PQSM) and (2.4) holds for every $t_0 \in \Re_+$.

Definition 2.5: System (2.2) is said to be (SPSM) strongly practically stable in the p-th mean if (PSM) and (PQSM) hold simultaneously.

Definition 2.6: System (2.2) is said to be (SUPSM) strongly uniformly practically stable in the p-th mean (UPSM) and (UPQSM) hold simultaneously.

Definition 2.7: System (2.2) is said to be (PSP) practically stable in probability if, given $(\lambda, \mathcal{A})$ with $0 < \lambda < \mathcal{A}$, and for any $\epsilon > 0$, such that

$$P\{\omega : \|x_0(\omega)\| > \lambda\} < \epsilon$$

implies

$$P\{\omega : \|x(t, \omega)\| \geq \mathcal{A}\} < \epsilon, \quad \forall t \geq t_0, \quad \text{for some } t_0 \in \Re^+ \tag{2.5}$$

Definition 2.8: System (2.2) is said to be (UPSP) uniformly practically stable in probability if it is (PSP) and (2.5) holds for all $t_0 \in \Re^+$.

Definition 2.9: System (2.2) is said to be (PQSP) with λ, $\mathcal{B}$ if, there exists positive numbers $\lambda, \mathcal{B}$ and T, and for any $\epsilon > 0$, such that

$$P\{\omega : \|x_0(\omega)\| > \lambda\} < \epsilon$$

implies

$$P\{\omega : \|x(t, \omega)\| \geq \mathcal{B}\} < \epsilon, \quad \forall t \geq t_0 + T, \quad \text{for some } t_0 \in \Re^+ \tag{2.6}$$

Definition 2.10: System (2.2) is said to be (UPQSP) if it is (PQSP) and (2.6) holds for every $t_0 \in \Re^+$.

Definition 2.11: System (2.2) is said to be (SPSP) if (PSP) and (PQSP) hold simultaneously.

Definition 2.12: System (2.2) is said to be (SUPSP) if (UPSP) and (UPQSP) hold simultaneously.

Consider now the auxiliary random differential system

$$u' = g(t, u), \qquad u(t_0) = u_0 \tag{2.7}$$

where $g \in \mathcal{C}[J \times \Re^N_+, \Re^N]$, $g(t, u)$ is concave and quasinilpotent nondecreasing in u for each fixed $t \in J$ and u_0 is an N-dimensional random vector.

Let $u(t, t_0, u_0)$ be any solution of (2.7) and $r(t, t_0, u_0)$ be the maximal solution processes of (2.7) through (t_0, u_0).

We need the following corresponding definitions of practical stability of the auxiliary system (2.7).

Definition 2.13: System (2.7) is said to be partial practically stable (PPS^*), if given $(\lambda, \mathcal{A})$ with $0 < \lambda < \mathcal{A}$, we have $E[\sum_{i=1}^{N} u_{i0}] < \lambda$ imply

$$\sum_{i=1}^{N} r_i(t, t_0, Eu_0) < \mathcal{A}, \quad \forall t \geq t_0, \quad \text{for some } t_0 \in \Re_+ \tag{2.8}$$

Definition 2.14: System (2.7) is said to be partial practically stable in probability ($PPSP^*$) if, given $(\lambda, \mathcal{A})$ with $0 < \lambda < \mathcal{A}$, and for any $\epsilon > 0$ such that

$$P\left\{\omega : \sum_{i=1}^{N} u_{i0}(\omega) > \lambda\right\} < \epsilon$$

implies

$$P\left\{\omega : \sum_{i=1}^{N} r_i(t, t_0, Eu_0(\omega)) \geq \mathcal{A}\right\} < \epsilon, \quad \forall t \geq t_0, \quad \text{for some } t_0 \in \Re^+ \tag{2.9}$$

The notions of $(UPPSP^*), (PPQSP^*), (SPPSP^*), (SUPPSP^*)$ can be defined similarly to the corresponding ones in the above definitions.

3. PRACTICAL STABILITY CRITERIA

In this section, by employing the Lyapunov-like functions and the basic comparison principle of stochastic systems, we give results for various types of practical stability in the pth mean and in probability of system (2.1).

Lemma 3.1: [Theorem 3.1, [3]] Assume that there exist functions $V(t, x)$, and $g(t, u)$ satisfying the following conditions:

(i) $V(t, x) \in \mathcal{C}[\mathcal{J} \times \Re^n, \Re_+^N]$, $\frac{\partial V(t,x)}{\partial t}$, $\frac{\partial V(t,x)}{\partial x}$, and $\frac{\partial^2 V(t,x)}{\partial x^2}$ exist and are continuous for $(t, x) \in \mathcal{J} \times \Re^n$ and

$$\mathcal{L}V(t, x) \leq g\left(t, V(t, x)\right), \tag{3.1}$$

for $(t, x) \in \mathcal{J} \times \Re^n$, where

$$\mathcal{L} = \frac{\partial}{\partial t} + f(t, x). \frac{\partial}{\partial x} + \frac{1}{2} tr\left(\mathcal{A}(t, x)\frac{\partial^2}{\partial x^2}\right)$$

$$\mathcal{A}(t, x) = (a_{ij}(t, x) = \sum_{r=1}^{m} (\sigma_r(t, x)\sigma_r^T(t, x))$$

(ii) $g \in \mathcal{C}[\mathcal{J} \times \Re_+^N, \Re^N]$; $g(t, 0) \equiv 0$, $g(t, u)$ is concave and quasi-monotone nondecreasing function in u, for each $t \in \mathcal{J}$;

(iii) $r(t) = r(t, t_0, u_0)$ is the maximal solution of the auxiliary differential system

$$u' = g(t, u), \qquad u(t_0) = u_0 \tag{3.2}$$

existing for $t \geq t_0$; where u_0 is an N-dimensional random vector;

(iv) for the solution process $x(t) = x(t, t_0, x_0)$ of (2.2), $E[V(t, x(t))]$ exist for $t \geq t_0$. Then,

$$E[V(t, x(t))|x(t_0) = x_0] \leq r(t, t_0, u_0), \quad t \geq t_0 \tag{3.3}$$

whenever

$$V(t_0, x_0) \leq u_0$$

Theorem 3.1: Assume that

(i) there exist functions $V(t, x)$, and $g(t, u)$ satisfying the following conditions: $V(t, x) \in \mathcal{C}[\mathcal{J} \times \Re^n, \Re^N_+]$, $\frac{\partial V(t,x)}{\partial t}$, $\frac{\partial V(t,x)}{\partial x}$, and $\frac{\partial^2 V(t,x)}{\partial x^2}$ exist and are continuous on $\mathcal{J} \times \Re^n$, and for $(t, x) \in \mathcal{J} \times \Re^n$ and

$$\mathcal{L}V(t, x) \leq g\left(t, V(t, x)\right), \tag{3.4}$$

where $g \in \mathcal{C}[\mathcal{J} \times \Re^N_+, \Re^N]$; $g(t, 0) \equiv 0$, $g(t, u)$ is concave and quasi-monotone non-decreasing function in u, for each fixed $t \in \mathcal{J}$;

(ii) there exists a matrix function $\mathcal{T}(t)$ such that $\mathcal{T} \in \mathcal{C}[\Re_+, \Re^{N^2}]$, $\mathcal{T}(t) = (\tau_{ij}(t))$, $\tau_{ij}(t) \geq 0$, and for $u, v \in \Re^N$, $u \geq v$, $g(t, u) - g(t, v) \geq -\mathcal{T}(t)(u - v)$;

(iii) $r(t) = r(t, t_0, u_0)$ is the maximal solution of the auxiliary differential system (3.2) existing for $t \geq t_0$ and $E[r(t, t_0, u_0)]$ exists;

(iv) for $(t, x) \in \mathcal{J} \times \Re^N$,

$$b(\|x\|^p) \leq \sum_{i=1}^{n} V_i(t, x) \leq a(t, \|x\|^p) \tag{3.5}$$

where $b \in \mathcal{VK}[\Re_+, \Re_+]$, $\mathcal{VK}$ is the collection of all continuous convex and increasing functions defined on $\Re_+$ into itself with b(0)=0 and $a \in \mathcal{CK}[\Re_+ \times \Re_+, \Re_+]$, $\mathcal{CK}$ is the collection $a(t, u)$ of all continuous concave and increasing functions in u for each $t \in \Re_+$ defined on $\Re_+ \times \Re_+$ into $\Re_+$ with $a(t, 0) = 0$;

(v) λ and $\mathcal{A}$ are given such that $0 < \lambda < \mathcal{A}$ and $a(t_0, \lambda) < b(\mathcal{A})$.

Then, the partial practical stability (PPS^*) of (2.7) implies that the system (2.2) is practically stable in the p-th mean (PSM).

Proof: Let $x(t) = x(t, t_0, x_0)$ be any solution process of (2.2). By hypothesis (iv),

$$0 \leq E[b(\|x\|^p)] \leq E\left[\sum_{i=1}^{N} V_i(t, x(t))\right] \leq a(t, E\|x(t)\|^p) \tag{3.6}$$

Hence, we have by Lemma 3.1, that $V(t_0, x_0) \leq u_0$ implying

$$E[V(t, x(t))|x(t_0) = x_0] \leq r(t, t_0, u_0)$$

so that

$$\sum_{i=1}^{N} E[V_i(t,x(t))|x(t_0)=x_0] \le \sum_{i=1}^{N} r_i(t,t_0,u_0), \quad \forall t \ge t_0 \tag{3.7}$$

where $r(t,t_0,u_0)$ is the maximal solution of (2.7) existing on $[t_0,\infty)$.

Taking expected value of (3.7) both sides and using Lemma 3.2 [3],

$$\sum_{i=1}^{N} E[V_i(t,x(t))] \le \sum_{i=1}^{N} E[r_i(t,t_0,u_0)] \le \sum_{i=1}^{N} r_i(t,t_0,Eu_0) \tag{3.8}$$

Since $r(t,t_0,u_0)$ of (2.7) is partially practically stable; we have, $\sum_{i=1}^{N} E[u_{i0}] < a(t_0,\lambda)$ implies that

$$\sum_{i=1}^{N} r_i(t,t_0,Eu_0) < b(\mathcal{A}), \quad \forall t \ge t_0 \tag{3.9}$$

Now, we claim that $E\|x_0\|^p < \lambda$ implies $E\|x(t)\|^p < \mathcal{A}$, $t \ge t_0$, where $x(t,t_0,x_0)$ is any solution of (2.2) with $E\|x_0\|^p < \lambda$.

Suppose this claim is not true, then there exists a $t_1 > t_0$ and a solution $x(t) = x(t,t_0,x_0)$ of (2.2) with $E\|x_0\|^p < \lambda$ such that

$$E\|x(t_1)\|^p = \mathcal{A} \quad \textit{and} \quad E\|x(t)\|^p < \mathcal{A} \quad \textit{for} \quad t_0 \le t < t_1 \tag{3.10}$$

By hypothesis (iv), we have

$$\sum_{i=1}^{N} E[V_i(t_1,x(t_1))] \ge b(E\|x(t_1)\|^p) = b(\mathcal{A}) \tag{3.11}$$

Let us choose u_0 such that $V(t_0,x_0) = u_0$ and $\sum_{i=1}^{N} E[u_{i0}] = a(t_0, E\|x_0\|^p)$; and by the previous estimate (3.8)

$$\sum_{i=1}^{N} E[V_i(t_1,x(t_1))] \le \sum_{i=1}^{N} E[r_i(t_1,t_0,u_0)] < \sum_{i=1}^{N} r_i(t_1,t_0,Eu_0) \tag{3.12}$$

The relations, (3.9), (3.11), and (3.12) lead to the contradiction

$$b(\mathcal{A}) \le \sum_{i=1}^{N} E[V_i(t_1,x(t_1))] < b(\mathcal{A}).$$

This completes the proof.

Theorem 3.2: Assume that the conditions in Theorem 3.1 are satisfied by replacing $a \in \mathcal{CK}$ by $a \in \mathcal{K}$ and $a(\lambda) < b(\mathcal{A})$. Then system (2.7) is $(UPPS^*)$ with $(a(\lambda), b(\mathcal{A}))$ implies system (2.2) is (UPSM).

Proof: It is not difficult to see, since a does not depend on $t_0 \in \Re_+$. We thus complete the proof.

Theorem 3.3: Assume that the conditions of Theorem 3.1 are satisfied. Then, for some positive numbers $(\lambda, \mathcal{B}, T)$, that system (2.7) is $(PPQS^*)$ with $(a(t_0, \lambda), b(\mathcal{B}), T)$ implies that system (2.2) is (PQSM) with $(\lambda, \mathcal{B}, T)$.

Proof: From the practical-quasi-stability with $(a(t_0, \lambda), b(\mathcal{B}), T)$ of system (2.7), we know that for some $t_0^* \in \Re_+$, we have $\sum_{i=1}^N E[u_{i0}] < a(t_0, \lambda)$ implies that

$$\sum_{i=1}^N r_i(t, t_0^*, Eu_0) < b(\mathcal{B}), \quad \forall t \geq t_0^* + T. \tag{3.13}$$

Now, we claim that $E\|x_0\|^p < \lambda$ implies $E_{t_0^*, x_0}\|x(t)\|^p < \mathcal{B}$, $t \geq t_0^* + T$, where $x(t, t_0, x_0)$ is any solution of (2.2) with $E\|x_0\|^p < \lambda$.

Suppose this claim is not true, then there would exists a $t_1 > t_0^* + T$ and a solution $x(t) = x(t, t_0^*, x_0)$ of (2.2) with $E\|x_0\|^p < \lambda$ such that

$$E\|x(t_1)\|^p = \mathcal{B} \quad \textit{and} \quad E\|x(t)\|^p < \mathcal{B} \quad \textit{for} \quad t_0^* + T \leq t < t_1 \tag{3.14}$$

By hypothesis (iv), we have

$$\sum_{i=1}^N E_{t_0^*, x_0}[V_i(t_1, x(t_1))] \geq b(E\|x(t_1)\|^p) = b(\mathcal{B}) \tag{3.15}$$

Let us choose u_0 such that $V(t_0, x_0) = u_0$ and $\sum_{i=1}^N E[u_{i0}] = a(t_0, E\|x_0\|^p)$. Hence, we have by Lemma 3.1, that $V(t_0, x_0) \leq u_0$ implying

$$E_{t_0^*, x_0}[V(t, x(t)) | x(t_0^*) = x_0] \leq r(t, t_0^*, u_0)$$

so that

$$\sum_{i=1}^N E_{t_0^*, x_0}[V_i(t, x(t)) | x(t_0^*) = x_0] \leq \sum_{i=1}^N r_i(t, t_0^*, u_0), \quad \forall t \geq t_0 \tag{3.16}$$

where $r(t, t_0^*, u_0)$ is the maximal solution of (2.7) existing on $[t_0^*, \infty)$.

Taking expected value of (3.16) both sides and using Lemma 3.2 [3],

$$\sum_{i=1}^N E_{t_0^*, x_0}[V_i(t, x(t))] \leq \sum_{i=1}^N E[r_i(t, t_0^*, u_0)] \leq \sum_{i=1}^N r_i(t, t_0^*, Eu_0) \tag{3.17}$$

The relations, (3.13), (3.15), and (3.17) lead to the contradiction

$$b(\mathcal{B}) \leq \sum_{i=1}^N E_{t_0^*, x_0}[V_i(t_1, x(t_1))] < b(\mathcal{B})$$

This completes the proof.

Theorem 3.4: Assume that the conditions of Theorem 3.2 are satisfied. If system (2.7) is $(UPPQS^*)$, then system (2.2) is (UPQM).

Proof: We observe that the function a in Theorem 3.2 does not depend on $t_0 \in \Re_+$, so the conclusion of Theorem 3.2 holds for all $t_0 \in \Re_+$ which complete the proof.

From the definitions of (SPSM) and (SUPSM), and using Theorems 3.1-3.4, it is easy to see that we have the following Corollaries:

Corollary 3.1: Assume that the conditions in Theorem 3.1 are satisfied. If system (2.7) is ($SPPS^*$), then system (2.2) is (SPSM).

Corollary 3.2: Assume that the conditions in Theorem 3.2 are satisfied. If system (2.7) is ($SUPPS^*$), then system (2.2) is (SUPSM).

The following Theorem illustrates the sufficient conditions for practical instability of a stochastic system given by equation (2.2).

Theorem 3.5: Assume that there exists $V(t,x)$ and $g(t,u)$ satisfying the following conditions:

(i) for $(t,x) \in \mathcal{J} \times \Re^N$,

$$f_1(\|x\|^p) \leq \sum_{i=1}^{s} V_i(t,x) \leq f_2(\|x\|^p) \tag{3.18}$$

where $f_1 \in \mathcal{VK}[\Re_+, \Re_+]$, $\mathcal{VK}$ is the collection of all continuous convex and increasing functions defined on $\Re_+$ into itself with $f_1(0) = 0$ and $f_2 \in \mathcal{CK}[\Re_+ \times \Re_+, \Re_+]$, $\mathcal{CK}$ is the collection $f_2(u)$ of all continuous concave and increasing functions in u for each $t \in \Re_+$ defined on $\Re_+ \times \Re_+$ into $\Re_+$ with $f_1(0) = 0$;

(ii) $V(t,x) \in \mathcal{C}[\mathcal{J} \times \Re^n, \Re_+^N]$, $\frac{\partial V(t,x)}{\partial t}$, $\frac{\partial V(t,x)}{\partial x}$, and $\frac{\partial^2 V(t,x)}{\partial x^2}$ exist and are continuous for on $\mathcal{J} \times \Re^n$, and for $(t,x) \in \mathcal{J} \times \Re^n$ and

$$\mathcal{L}V(x) \geq g\left(t, V(x)\right), \tag{3.19}$$

where the operator $\mathcal{L}$ is defined as (3.10).

(iii) $g \in \mathcal{C}[\Re^+ \times \mathcal{B}(\mathcal{J}), \mathcal{B}(\mathcal{J})]$; $g(t,0) \equiv 0$, $g(t,u)$ is convex and quasi-monotone non-decreasing function in u, for each $t \in \Re^+$;

(iv) $\rho(t) = \rho(t, t_0, u_0)$ is the minimal solution of the auxiliary differential system

$$u' = g(t,u), u(t_0) = u_0 \tag{3.20}$$

existing for $t \geq t_0$; where $V(t_0, x_0) \geq u_0$ and u_0 is a d-dimensional random vector;

(v) $E[\rho(t, t_0, u_0)] \to \infty$ as $t \to \infty$;

Then, the stochastic process X_t given by system (2.2) is practically unstable in the p-th mean.

Proof: By imitating the proof of Theorem 3.1, it can be easily shown that

$$E[V(X_t)|x_0] \geq \rho(t, t_0, u_0), \tag{3.21}$$

whenever $V(x_0) \geq u_0$.

Taking the expected value of the above expression both sides and let $t \to \infty$ we can show that

$$E[V(X_t)] \to \infty$$

Using the hypothesis (i), and the fact that f_2 is concave, we have deduce that as $t \to \infty$

$$E[\|X_t\|^p] \to \infty.$$

Hence, the stochastic process given by the system (2.2) is practically unstable. This completes the proof of the theorem.

Next we will discuss the partial practical stability in probability of system (2.2).

Theorem 3.6: Assume that the conditions in Theorem 3.1 are satisfied. If system (2.7) is $(PPSP^*)$ with $(a(t_0, \lambda), b(\mathcal{A}))$, then system (2.2) is (PSP) with $(\lambda, \mathcal{A})$.

Proof: Let $x(t) = x(t, t_0, x_0)$ be a solution process of (2.2). By the condition (iv) in Theorem 3.1,

$$0 \leq b(\|x\|^p) \leq \sum_{i=1}^{N} V_i(t, x) \leq a(t, \|x\|^p) \tag{3.22}$$

By Lemma 3.2 [3], that $V(t_0, x_0) \leq u_0$ implying

$$E_{t_0,x_0}[V(t, x(t))|x(t_0) = x_0] \leq r(t, t_0^*, u_0)$$

so that

$$\sum_{i=1}^{N} E_{t_0,x_0}[V_i(t, x(t))] \leq \sum_{i=1}^{N} r_i(t, t_0, Eu_0), \quad \forall t \geq t_0 \tag{3.23}$$

where $r(t, t_0, u_0)$ is the maximal solution of (2.7) existing on $[t_0, \infty)$.

From the practical stability of (2.7) in probability, we know that for any $\epsilon > 0$

$$P\left\{\omega : \sum_{i=1}^{N} u_{i0}(\omega) > a(t_0, \lambda)\right\} < \epsilon$$

implies

$$P_{t_0^*,u_0}\left\{\omega : \sum_{i=1}^{N} r_i(t, t_0^*, Eu_0(\omega)) \geq b(\mathcal{A})\right\} < \epsilon, \quad \forall t \geq t_0^*, \quad \text{for some } t_0^* \in \Re^+ \tag{3.24}$$

Let us choose u_0 so that $u_0 \geq V(t_0, x_0)$ and $\sum_{i=1}^{m} u_{i0} = a(t_0, \|x_0\|)$.

Since $a(t_0, \omega) \in \mathcal{K}$ we have

$$P\{\omega : a(t_0, \|x_0\|) > a(t_0, \lambda)\} = P\{\omega : \|x_0\| > \lambda\} \tag{3.25}$$

Now, we claim that system (2.2) is (PSP) with $(\lambda, \mathcal{A})$. If this claim is not true, there would exists an $\epsilon_0 > 0$, a solution $x(t) = x(t, t_0^*, x_0)$, with

$$P\{\omega : \|x_0(\omega)\| > \lambda\} < \epsilon_0 \tag{3.26}$$

and a $t_1 > t_0^*$ such that

$$P_{t_0^*,x_0}\{\omega : \|x(t_1, t_0^*, \omega)\| \geq \mathcal{A}\} = \epsilon_0 \tag{3.27}$$

However from (3.22), we have

$$P_{t_0^*,x_0}\left\{\omega : \sum_{i=1}^{N} EV_i(t_1, x(t_1, t_0^*, \omega) \geq b(\mathcal{A})\right\} = \epsilon_0 \tag{3.28}$$

Choose an $\epsilon > 0$ such that $1 - \epsilon > \frac{\epsilon_0}{2}$, using (3.23) and (3.24), we will get

$$P_{t_0^*,x_0}\left\{\omega : \sum_{i=1}^{m} E[V_i(t_1, x(t_1))] \leq \sum_{i=1}^{N} r_i(t_1, t_0^*, Eu_0) < b(\mathcal{A})\right\} > \frac{\epsilon_0}{2} \tag{3.29}$$

by (3.28) and (3.29), we get the following contradiction

$$P_{t_0^*,x_0}\left\{\omega : b(\mathcal{A}) \leq \sum_{i=1}^{N} E[V_i(t_1, x(t_1))] \leq \sum_{i=1}^{N} r_i(t_1, t_0^*, Eu_0) < b(\mathcal{A})\right\} > \frac{\epsilon_0}{2} \tag{3.30}$$

This completes the proof.

Corollary 3.3: Assume that the conditions in Theorem 3.5 are satisfied by replacing $a \in \mathcal{CK}$ by $a \in \mathcal{K}$. If system (2.7) is $(UPPSP^*)$, then system (2.2) is (UPSP).

Theorem 3.6: Assume that the conditions in Theorem 3.5 are satisfied. If system (2.7) is $(PPQSP^*)$, then system (2.2) is (PQSP).

Proof: We will leave the proof of this theorem to the reader.

Example 3.1: [2] Consider the system of stochastic differential equations

$$dx = f(t, x(t))x(t)dt + \sum_{r=1}^{2} \sigma_r(t, x(t))d\xi_r, \tag{3.31}$$

where $x \in \Re^2$, $f \in \mathcal{C}[\mathcal{J} \times \Re^2, \Re^2]$, $\sigma_r \in \mathcal{C}[\mathcal{J} \times \Re^2, \Re^2]$ for $1 \leq r \leq 2$. We assume that $f(t, 0, 0) \equiv 0$, $\sigma_r(t, 0, 0) \equiv 0$

$$[\sigma_1(t, x) + \sigma_2(t, x)]^2 \leq (x_1 + x_2)^2 \lambda(t)$$
$$[\sigma_1(t, x) - \sigma_2(t, x)]^2 \leq (x_1 - x_2)^2 \lambda(t)$$

where $\lambda \in \mathcal{C}[\Re_+ \times \Re_+] \cap L_1[0, \infty)$, and

$$F(t, x) = \begin{bmatrix} e^{-t} - f_0(t, x) & \sin t \\ \sin t & e^{-t} - f_0(t, x) \end{bmatrix}$$

with $f_0 \in \mathcal{C}[\Re_+ \times \Re^2, \Re]$, $f_0(t, x) \geq 0$ on the ball $B(\rho)$ with radius ρ and center at $0 \in \Re^2$ for all $t \in \Re_+$ and $f_0(t, 0) \equiv 0$. Consider the function:

$$V(t, x) = \begin{bmatrix} (x_1 + x_2)^2 \\ (x_1 + x_2)^2 \end{bmatrix}$$

The components of V, V_1 and V_2 satisfy the hypotheses of Lemma 3.4 with $N = 2$.

$$(x_1^2 + x_2^2) \leq \sum_{i=1}^{2} V_i(t, x) \leq 2(x_1^2 + x_2^2)$$

If we take $b(s) = s$ and $a(s) = 2s$, then the inequality in Theorem 3.2 is satisfied with positive numbers λ and A with $2\lambda < A$. Also, the inequality

$$LV(x) \leq g(t, V(x))$$

is satisfied in $\Re_+ \times \Re^2$ with

$$g(t, u) = \begin{bmatrix} (2e^{-t} + 2\sin t + \lambda(t))u_1 \\ (2e^{-t} + 2\sin t + \lambda(t))u_2 \end{bmatrix}$$

It is easy to observe that $g(t, u)$ is concave and quasi-monotone nondecreasing in u for each fixed t and the auxiliary system (2.7) is $(UPPS^*)$. Therefore, system (3.40) is (UPSM).

4. STABILIZATION OF CONTROLLED STOCHASTIC SYSTEMS

We consider a controlled stochastic systems of the type:

$$dx = f(t, x, u)dt + \sum_{r=1}^{m} \sigma_r(t, x, u)d\xi_r, \tag{4.1}$$

$f \in \mathcal{C}[\mathcal{J} \times \Re^n \times \Re^N, \Re^n]$, $\sigma_r \in \mathcal{C}[\mathcal{J} \times \Re^n \times \Re^N, \Re^n]$ for $1 \leq r \leq m$. We assume that $f(t, 0, 0) \equiv 0$, $\sigma_r(t, 0, 0) \equiv 0$. Here u is a control parameter with values in a given Borel set $E \subset \Re^N$ described by

$$E = \{u \in \Re^N : U(t, u) \leq v_0(t), t \geq t_0\}$$

where $U \in \mathcal{C}[\Re_+ \times \Re_N, \Re_+]$ and $v_0(t) \in \mathcal{C}[\Re_+, \Re_+]$ is a given function.

Assume that the control u in (4.1) is a function of t and $x(t)$, that is $u = u(t, x(t))$. Then the process described by (4.1) is Markovian.

A control $u = u(t, x)$ is said to be *admissible* if the coefficients in (4.41), $f(t, x, u(t, x))$ and $\sigma_r(t, x, u(t, x))$ are continuous and have continuous partial derivatives with respect to x which are bounded uniformly in $t > 0$ and $u(t, 0) \equiv 0$.

We denote by $\mathcal{U}$ the class of admissible controls. Each function $u \in \mathcal{U}$ can be associated with a Markov process $x(t, t_0, x_0, u)$ which is a solution to equation (4.1) with initial condition $x(t_0, t_0, x_0, u) = x_0$.

Next we will discuss some practical stability results on control system (4.1). For this purpose, consider the auxiliary system

$$w' = g(t, w, u), w(t_0) = w_0 \tag{4.2}$$

where $g \in \mathcal{C}[J \times \Re_+^N \times \Re^N, \Re^N]$, $g(t, w, u)$ is a concave and quasi-monotone nondecreasing in w for each fixed $(t, u) \in J \times E$, nondecreasing in u, and w_0 is an N-dimensional random vector.

Let $w(t, t_0, w_0)$ be any solution of (4.2) and $r(t, t_0, w_0)$ be the maximal solution process of (4.2) through (t_0, w_0).

Theorem 4.1: Assume that the hypotheses (ii) and (iii) of Theorem 3.1 are satisfied and moreover

(i) there exist functions $V(t,x)$, and $g(t,u)$ satisfying the following conditions: $V(t,x) \in \mathcal{C}[\mathcal{J} \times \Re^n, \Re^N_+]$, $\frac{\partial V(t,x)}{\partial t}$, $\frac{\partial V(t,x)}{\partial x}$, and $\frac{\partial^2 V(t,x)}{\partial x^2}$ exist and are continuous on $\mathcal{J} \times \Re^n$, and for $(t,x) \in \mathcal{J} \times \Re^n$ and

$$\mathcal{L}V(t,x) \leq g\left(t, V(t,x), U(t,u)\right), \tag{4.3}$$

where $g \in \mathcal{C}[\mathcal{J} \times \Re^N_+ \times \Re^N, \Re^N]$; $g(t,0,0) \equiv 0$, $g(t,w,v)$ is concave and quasi-monotone non-decreasing function in w and nondecreasing in v, for each fixed $t \in \mathcal{J}$;

(ii) for $(t,x) \in \mathcal{J} \times \Re^n$,

$$b(\|x\|^p) \leq \sum_{i=1}^{N} V_i(t,x) \leq a(\|x\|^p) \tag{4.4}$$

where $b \in \mathcal{VK}[\Re_+, \Re_+]$, $\mathcal{VK}$ is the collection of all continuous convex and increasing functions defined on $\Re_+$ into itself with b(0)=0 and $a \in \mathcal{K}[\Re_+, \Re_+]$, $\mathcal{K}$ is the collection $a(u)$ of all continuous concave and increasing functions in u with $a(0) = 0$;

(iii) λ and $\mathcal{A}$ are given such that $0 < \lambda < \mathcal{A}$ and $a(\lambda) < b(\mathcal{A})$;

(iv) there exists $u^* \in \mathcal{C}[\Re_+, \Re^N_+]$ such that the maximal solution $r(t) = r(t, t_0, w_0, u^*)$ of the auxiliary system (4.42) is partial practically stable with $(a(\lambda), b(\mathcal{A}))$.

Then, $u^* = u^*(t) \in \mathcal{U}$ gives system (4.41) practically stable in the p-th mean (PSM).

Proof: Let $\mathcal{U}(t, u^*(t)) \leq v_0(t)$ for $t \geq t_0$, and $X^{u^*}(t) = X(t, t_0, x_0, u^*)$ be any solution process of (4.1).

By hypothesis (ii),

$$0 \leq E[b(\|X^{u^*}(t)\|^p)] \leq E[\sum_{i=1}^{N} V_i(t, x^{u^*}(t))] \leq a(E\|X^{u^*}(t)\|^p) \tag{4.5}$$

Hence, we have by Lemma 3.1, that $V(t_0, x_0) \leq w_0$ implying

$$E[V(t, X^{u^*}(t)) | X^{u^*}(t_0) = x_0] \leq r(t, t_0, w_0)$$

so that

$$\sum_{i=1}^{N} E[V_i(t, X^{u^*}(t)) | X^{u^*}(t_0) = x_0] \leq \sum_{i=1}^{N} r_i(t, t_0, u_0), \forall t \geq t_0 \tag{4.6}$$

where $r(t, t_0, w_0)$ is the maximal solution of (4.2) existing on $[t_0, \infty)$.

Taking expected value of (4.6) both sides and using Lemma 3.2[3],

$$E[b(\|X^{u^*}(t)\|^p)] \leq \sum_{i=1}^{N} E[V_i(t, X^{u^*}(t))] \leq \sum_{i=1}^{N} E[r_i(t, t_0, w_0)] \leq \sum_{i=1}^{N} r_i(t, t_0, Ew_0) \tag{4.7}$$

Since $r(t, t_0, w_0, u^*)$ of (4.2) is partially practically stable with $(a(\lambda), b(\mathcal{A}))$; we have, $\sum_{i=1}^{N} E[w_{i0}] < a(\lambda)$ implies that

$$\sum_{i=1}^{N} r_i(t, t_0, Ew_0) < b(\mathcal{A}), \quad \forall t \geq t_0 \tag{4.8}$$

Now, we claim that $E\|x_0\|^p < \lambda$ implies $E\|X^{u^*}(t)\|^p < \mathcal{A}$, $t \geq t_0$, where $X^{u^*}(t, t_0, x_0, u^*)$ is any solution of (4.1) with $E\|x_0\|^p < \lambda$.

Suppose this claim is not true, then there exists a $t_1 > t_0$ and a solution $X^{u^*}(t) = X^{u^*}(t, t_0, x_0, u^*)$ of (4.1) with $E\|x_0\|^p < \lambda$ such that

$$E\|X^{u^*}(t_1)\|^p = \mathcal{A} \quad and \quad E\|X^{u^*}(t)\|^p < \mathcal{A} \quad for \quad t_0 \leq t < t_1 \tag{4.9}$$

By assumption (ii), we have

$$\sum_{i=1}^{N} E[V_i(t_1, X^{u^*}(t_1))] \geq b(E\|X^{u^*}(t_1)\|^p) = b(\mathcal{A}) \tag{4.10}$$

Let us choose w_0 such that $V(t_0, x_0) = w_0$ and $\sum_{i=1}^{N} E[w_{i0}] = a(E\|x_0\|^p)$; and by the previous estimate (4.7)

$$\sum_{i=1}^{N} E[V_i(t_1, X^{u^*}(t_1))] \leq \sum_{i=1}^{N} E[r_i(t_1, t_0, w_0)] < \sum_{i=1}^{N} r_i(t_1, t_0, Ew_0) \tag{4.11}$$

The relations, (4.7), (4.10), and (4.11) lead to the contradiction

$$b(\mathcal{A}) \leq \sum_{i=1}^{m} E[V_i(t_1, x(t_1))] < b(\mathcal{A})$$

This completes the proof.

Remark 4.1: In condition (i) of the theorem, function g depends on t, V, and U. It can be easily shown that the Lemma 3.2 is still valid for this function g.

Remark 4.2: The function a does not depend on t_0, so that uniform partial practical stability of system (4.2) implies the uniform practical stability of system (4.1).

Corollary 4.1: Assume that the conditions of Theorem 4.1 are satisfied and moreover (v) there exists some $T = T(t_0, w_0)$ such that

$$\sum_{i=1}^{m} r_i(t_0 + T, t_0, w_0, v) \leq b(\beta)$$

where $r(t) = r(t, t_0, w_0, v)$ is the maximal solution of (4.2) through (t_0, w_0).

Then system (4.1) is controllable in the p-th mean. That is all the solutions $X^u(t) = X(t, t_0, x_0, u)$ starting in $\{x \in \Re^n : \|x_0\|^p < \lambda\}$ enter into the bounded region $\{x \in \Re^n : \|x(\omega)\|^p < \beta\}$. That is , $E\|x_0\|^p < \lambda$ imply $E\|x(t_0 + T)\|^p < \beta$.

Proof: From (4.7) in the proof of the Theorem 4.1, we have by (v) that

$$b(E[(\|X(t_0+T)\|^p)]) \le \sum_{i=1}^{N} E[r_i(t_0+T, t_0, w_0, v)] \le \sum_{i=1}^{N} r_i(t, t_0, Ew_0, v) \le b(\beta), \tag{4.12}$$

where $r(t, t_0, w_0, v)$ is the maximal solution of (4.2). Therefore,

$$E\|X(t_0+T)\|^p \le \beta$$

The proof is complete.

Now, we will seek an optimal control function, i.e. a control function $u_0 \in U$ which minimizes the cost functional:

$$J^{t_0,x_0}(u) = E\left\{\int_{t_0}^{\infty} g\left(t, V(t, x(t, t_0, x_0, u), x(t, t_0, x_0, u), u(t, x(t, t_0, x_0, u)\right)dt\right\}$$

where $g(t, v, x, u) > 0$ for all $t \in J$, $v \in \Re^N$, $x \in \Re^n$ and $u \in \Re$. Moreover, we will show that the system (4.1) with $u = u_0(t, x)$ is practically stable in p-th mean.

Theorem 4.2: Assume that the hypotheses (ii) and (iii) of Theorem 3.1 are satisfied and moreover,

(i) there exist functions $V(t, x)$, and $g(t, u)$ satisfying the following conditions: $V(t,x) \in \mathcal{C}[\mathcal{J} \times \Re^n, \Re^N_+]$, $\frac{\partial V(t,x)}{\partial t}$, $\frac{\partial V(t,x)}{\partial x}$, and $\frac{\partial^2 V(t,x)}{\partial x^2}$ exist and are continuous for on $\mathcal{J} \times \Re^n$, and for $(t,x) \in \mathcal{J} \times \Re^n$ and

$$\mathcal{L}_u V(t,x) + g\left(t, V(t,x), x, u)\right) \ge 0, \quad \forall u \in U \tag{4.13}$$

where $g \in \mathcal{C}[\mathcal{J} \times \Re^N_+ \times \Re^N, \Re^N_+]$; $g(t, 0, 0) \equiv 0$, $g(t, w, v)$ is concave and quasi-monotone non-decreasing function in w and nondecreasing in v, for each $t \in \mathcal{J}$;

(ii) the set E is a convex compact set, For $u^0(t,x) \in \mathcal{U}$ system (4.1) admits a unique solution for $t \ge t_0$ and for $(t,x) \in \mathcal{J} \times \Re^n$, and

$$\mathcal{L}_{u^0} + g\left(t, V(t,x), x, u)\right) \equiv 0, \quad \forall u \in U \tag{4.14}$$

(iii) for $(t,x) \in \mathcal{J} \times \Re^N$,

$$b(\|x\|^p) \le \sum_{i=1}^{N} V_i(t,x) \le a(\|x\|^p) \tag{4.15}$$

where $b \in \mathcal{VK}[\Re_+, \Re_+]$, $\mathcal{VK}$ is the collection of all continuous convex and increasing functions defined on $\Re_+$ into itself with $b(0) = 0$ and $a \in \mathcal{K}[\Re_+, \Re_+]$, $\mathcal{K}$ is the collection $a(u)$ of all continuous concave and increasing functions in u with $a(0) = 0$;

(iv) λ and $\mathcal{A}$ are given such that $0 < \lambda < \mathcal{A}$ and $a(\lambda) < b(\mathcal{A})$.

(v) the maximal solution $r(t) = r(t, t_0, w_0, u^0)$ of

$$w' = g(t, w, x, u^0(t,x)), w(t_0) = w_0 \tag{4.16}$$

exists on $[t_0, \infty)$ and this system is partial practically stable with $(a(\lambda), b(\mathcal{A}))$ and $\lim_{t\to\infty} r(t, t_0, Ew_0, u^0) = 0$.

Then the control $u^0(t, x)$ is a solution to the optimal stabilization for the control system (4.1) in the sense of minimizing the cost,

$$J^{t_0,x_0}(u^0) = \min_{u\in\mathcal{U}} J^{t_0,x_0}(u) \tag{4.17}$$

and $u^0(t, x)$ makes the system (4.1) practically stable in the p-th mean and $J^{t_0,x_0}(u^0) = V(t_0, x_0)$.

Remark 4.2: The conditions (i) and (ii) can be combined into one equation

$$\min_{u\in E}[\mathcal{L}V(x) + g(t, V(x), x, u)] = 0$$

This is the well known Hamilton-Bellman-Jacobi equation corresponding to our stochastic System (4.1).

Proof of Theorem 4.2: Let $X^0(t) = X(t, t_0, x_0, u^0)$ be the solution of (4.1) corresponding to the control $u^0(t, x) \in \mathcal{U}$. Then by following the proof of Theorem 4.1 and Corollary 4.1 we can prove that system (4.1) is practically stable in the p-th mean. We also have

$$\sum_{i=1}^{N} EV_i(t, X^0(t)) \le \sum_{i=1}^{N} r_i(t, t_0, Ew_0, u^0) \to 0 \tag{4.18}$$

Hence by condition (iii), $b(E\|X^0(t)\|^p) \to 0$, and therefore we get $E\|X^0(t)\|^p \to 0$, as $t \to \infty$. Let $u = u(t, x)$ be any admissible control and $X(t) = X(t, t_0, x_0, u)$ be the corresponding solution. By applying Itô formula to $V(t, X(t))$ we obtain

$$dV(t, X(t)) = \mathcal{L}_u V(t, X(t)) + V_x(t, X(t)).\sum_{r=1}^{m} g_r(t, X(t))d\xi_r(t) \tag{4.19}$$

Integrating (4.19) from t_0 to t, we obtain

$$\begin{aligned} V(t, X(t)) \quad - \quad V(t_0, x_0) &= \int_{t_0}^{t} \mathcal{L}_u V(s, X(s))ds, \\ &+ \int_{t_0}^{t} V_x(s, X(s)) \sum_{r=1}^{m} g_r(s, X(s))d\xi_r(t) \end{aligned} \tag{4.20}$$

Taking the expected value of (4.20) and letting $u = u^0(t, x)$

$$\begin{aligned} EV(t, X^0(t)) \quad - \quad EV(t_0, x_0) &= E[\int_{t_0}^{t} \mathcal{L}_u V(s, X^0(s))ds], \\ &= E[\int_{t_0}^{t} g(s, V(s, X^0(s)), X^0(s), u^0(s, X^0(s)))ds] \end{aligned} \tag{4.21}$$

Letting $t \to \infty$ and using (4.16) we get

$$J^{t_0,x_0}(u^0) = E[V(t_0, x_0)] \tag{4.22}$$

Now if $u(t,x)$ is any admissible control and $X(t) = X(t,t_0,x_0,u)$ then using similar argument we can get

$$\begin{aligned} EV(t,X(t)) \quad - \quad & EV(t_0,x_0) = E[\int_{t_0}^{t} \mathcal{L}_u V(s,X(s))ds], \\ & \geq -E[\int_{t_0}^{t} g(s,V(s,X(s)),X(s),u^0(s,X(s)))ds] \end{aligned} \tag{4.23}$$

Letting $t \to \infty$, we have

$$0 \geq EV(t_0,x_0) - E[\int_{t_0}^{\infty} g(s,V(s,X(s)),X(s),u(s,X(s)))ds]$$

That is $J^{t_0,x_0}(u) \geq E[V(t_0,x_0)]$ which with (4.22) implies

$$J^{t_0,x_0}(u^0) = \min_{u \in \mathcal{U}} J^{t_0,x_0}(u)$$

Hence the proof is complete.

The following example will illustrate the fruitfulness of the obtained results.

Example 4.1 Consider the following nonlinear time-varying hybrid system

$$dx = [f(t,x) + B(t,x)u]dt + \sum_{r=1}^{m} g(t,x)w_r(t) \quad x(t_0) = x_0,\ \eta(t_0) = i_0 \tag{4.24}$$

where $x \in \Re^n$, $x(t_0) = x_0$ is an n-dimensional random vector. Let $f(t,x) \in \mathcal{C}[\mathcal{J} \times \Re^n, \Re^n]$, $B(t,x) : \mathcal{J} \times \Re^n \to \Re^n \times \Re^m$ is a continuous $n \times m$ matrix and $u \in \Re^m (m \leq n)$, $w(t) = (w_1, w_2, \ldots, w_m)$ is a m-dimensional normalized Wiener process. The corresponding uncontrolled stochastic system is given by

$$dx = f(t,x)dt + \sum_{r=1}^{m} g(t,x)w_r(t), \quad x(t_0) = x_0$$

Let us use the following performance index:

$$\mathcal{J}_{t_0,x_0} = E\int_{t_0}^{\infty} \left[Q_1(t,x) + u^T Q_2 u | x_0, t_0\right] dt \tag{4.25}$$

where $G(t,V,x,u) = Q_1(t,x) + u^T Q_2 u$, $Q_1(t,x) \in \mathcal{C}[\mathcal{J} \times \Re^n, \Re]$, Q_2 are real symmetric positive definite matrices of dimension $m \times m$. The optimal index $V(t,x) : \mathcal{J} \times \Re^n \to \Re^+$, which belongs to the class C^2 in both arguments, satisfies the following:

$$\begin{aligned} \mathcal{L}_u V(t,x,k) &= \frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} \cdot [f(t,x,k) + B(t,x,k)u] + \frac{1}{2} tr \left(\mathcal{A}(t,x)\frac{\partial^2}{\partial x^2}\right) \\ &= W(t,V(t,x,k),k) + \frac{\partial V(t,x)}{\partial x} \cdot B(t,x,k)u \end{aligned} \tag{4.26}$$

where

$$\mathcal{A}(t,x) = (a_{ij}(t,x) = \sum_{r=1}^{m} (g(t,x)g^T(t,x))$$

and $W(t, V(t,x,k), U(t,u))$ be defined as

$$W(t,x,k) = \frac{\partial V(t,x,k)}{\partial t} + \frac{\partial V(t,x,k)}{\partial x} \cdot f(t,x,k) + \frac{1}{2} tr \left(\mathcal{A}(t,x) \frac{\partial^2}{\partial x^2} \right).$$

Assume that the uncontrolled stochastic system is practically stable. This ensures the existence of a positive definite $V(t,x)$ such that $W(t,x) \leq 0$.

Now, let us define

$$\begin{aligned} \mathcal{L}V(t,x) + \hat{g}(t,V,x,u) &= L(t,V,x,u) \\ &= W(t,x) + Q_1(t,x) + u^T Q_2 u + \frac{\partial V(t,x)}{\partial x} \cdot B(t,x)u. \end{aligned}$$

Hence, the Hamilton-Jacobi-Bellman equation using the vector-Lyapunov function $V(t,x)$, gives the following set of conditions to find the control u^0:

$$L(V(t,x), t, x, u) = 0, \text{ at } u = u^0 \tag{4.27}$$

$$\frac{\partial}{\partial u} L(V(t,x), t, x, u) = 0, \text{ at } u = u^0. \tag{4.28}$$

From these two set of conditions it follows that we obtain the following relationship

$$\begin{aligned} W(t,x) + Q_1(t,x) + u^T Q_2 u + \frac{\partial V(t,x)}{\partial x} &= 0 \\ B(t,x) V_x(t,x) + 2Q_2 u^0 &= 0 \end{aligned} \tag{4.29}$$

Thus, we get the control u^0 that satisfies the above set of two conditions

$$u^0 = -\frac{1}{2} Q_2^{-1} B(t,x) V_x(t,x). \tag{4.30}$$

To discuss the problem of minimization of the functional (4.25), we look at the following expression

$$\begin{aligned} V_x(t,x) B(t,x) u + u^T Q_2 u &= -2 \left[u^0\right]^T Q_2 \left[u^0\right] + u^T Q_2 u \\ &= \left(u - u^0\right)^T Q_2 \left(u - u^0\right) - \left[u^0\right]^T Q_2 u^0. \end{aligned} \tag{4.31}$$

Using the condition (4.27) and (4.28), we will get

$$W(t,x) + Q_{1k}(t,x) - \left[u^0\right]^T Q_2 u^0 = 0. \tag{4.32}$$

Thus, we obtain

$$Q_1(t,x) = -W(t,x) + \left[u^0\right]^T Q_2 u^0. \tag{4.33}$$

Now, we need to prove that $\lim_{t \to \infty} w(t,x) = 0$ where $w(t,x)$ is the solution of the comparison system

$$\begin{aligned} w' &= -g\left(t, w, x, u^0\right) = -Q_1(t,x) - \left[u^0\right]^T Q_2 u^0 \\ &= W(t,x) - 2\left[u^0\right]^T Q_2 u^0 \\ &\leq 0 \end{aligned} \tag{4.34}$$

Thus, $\lim_{t \to \infty} w(t,x) = 0$.

Therefore, under the control law $u^0 = -\frac{1}{2}Q_2^{-1}B(t,x)V_x(t,x)$, we obtain the optimal stabilization of system (4.24). Hence, we have the optimal performance index

$$J^0(t_0, x_0) = E[V(t_0, x_0, i_0)].$$

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REFERENCES

[1] Z. S. Feng, Y. Q. Liu and F. W. Guo, Criteria for Practical Stability in the p-th Mean of Nonlinear Stochastic Systems, Appl. Math. Comput. 49, 251–260, 1992.

[2] G. S. Ladde and V. Laksmikantham, Random Differential Inequalities, Acedemic Press, New York (1980).

[3] G. S. Ladde and B. A. Lawrence, Stability and Convergence of Large-Scale Stochastic Approximation Procedures, Int. J. System. Sc., Vol. 26, 595–618, 1995.

[4] J. P. Lasalle and S. Lefschetz, Stability by Lyapunov's Direct Method with Applications, Academic Press, New York, 1961.

[5] V. Laksmikantham, S. Leela, and A. A. Martynyuk, Practical Stability of Nonlinear Systems, World Scientific, Singapore, 1990.

[6] S. Sathananthan and S. Suthaharan, Practical Stability Criteria For Large-Scale Nonlinear Stochastic Systems By Decomposition and Aggregation, Dynamics of Continuous, Discrete and Impulsive Systems, Vol. 8, pp. 227–248, 2001

[7] S. Sathananthan and L. H. Keel, Connective Practical Stabilization of Large-Scale Decentralized Stochastic Control Systems, Dynamics of Continuous, Discrete and Impulsive Systems, Vol. 11, pp. 569–588, 2004

[8] S. Sathananthan and L.H. Keel, Optimal Practical Stabilization and Controllability of Systems with Markovian Jumps, Nonlinear Analysis, Theory, Methods and Applications, Vol. 54, pp. 1011–1027, 2003

[9] A. H. Tsoi and B. Zhang, Practical Stabilities of Ito Type Nonlinear Stochastic Differential Systems and Related Control Problems, Dynamic Systems and App. Vol. 6, 107–124, 1997.