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NUMERICAL ANALYSIS OF REFLECTING BROWNIAN MOTION AND A NEW MODEL OF SEMI-REFLECTING BROWNIAN MOTION WITH SOME DOMAINS

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Dedicated to the memory of Dr. Masasuke Tanaka, Ph.D.

ABSTRACT. We investigate the error of the Euler-Maruyama approximate solution of the multidimensional reflecting Brownian motion and semi-reflecting Brownian motion using the penalty method and show their numerical simulations. Furthermore we consider a new model of semi-reflecting Brownian motion with some domains.

AMS (MOS) Subject Classification. 60G50, 60H10, 65C30

1. PRELIMINARIES

Recently, from the viewpoint of applications for financial mathematics, statistical physics and so on, one of important problems in stochastic analysis is to consider stochastic differential equations with reflecting boundaries on multi-dimensional domains (so-called Skorohod SDE). For example, in the case of mathematical finance, when a stock continues to climb or drop investors often suppose that its ceiling height or bottom, respectively. Therefore such fluctuations are often considered to have a kind of reflecting barrier. In this paper we investigate the behavior of reflecting Brownian motion with some conditions for reflecting boundaries on multidimensional domains using numerical simulation.

We also investigate sample paths of multidimensional reflecting Brownian motion with a semi-reflecting boundary, i.e. any path of the Brownian motion does not reflect at the boundary immediately but is absorbed for a short period according to the speed of the path getting out of the boundary. Such situation is observed when electromagnetic wave reaches the ionosphere of the earth. Furthermore, in the case of mathematical finance, the fluctuations of some financial assets, e.g. stocks, bonds etc., are often considered to have a kind of reflecting barrier which seems to have a kind of flexibility, not perfect elasticity.

In the last section we propose a new model of semi-reflecting Brownian motion with semi-reflecting domains in which the semi-reflecting Brownian motion moves

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from a domain to another domain, one after another. We realize a semi-reflecting Brownian motion by a modification of the penalty function which defines an usual reflecting Brownian motion.

For observing numerical data by computer simulation of Skorohod SDE (stochastic differential equation) we define a discretized approximate solution of the SDE and obtain its mean square error estimation. There are two approaches to define an approximate solution of such SDE. [19], and [11] and [15] constructed Skorohod equations using the projection π on the boundary. Their method is called *projection scheme*. Roughly speaking, the reflecting path $x(t)$ is defined for given function $w(t)$ by the following manner: Define a step function $w_n(t)$ by discretization of $w(t)$ and construct the reflecting step function $x_n(t)$ for $w_n(t)$ as follows. For a domain D, if $x_n(t_0) \in \overline{D}$ and $y_n(s) \equiv x_n(t_0) + w_n(s) - w_n(t_0) \in \overline{D}$, $t_0 \le s \le t$ for some t_0 and t, put $x_n(s) = y_n(s)$, $t_0 \le s \le t$. On the other hand, if $y_n(s) \in \overline{D}$, $t_0 \le s < t$ and $y_n(t) \notin \overline{D}$, put $x_n(t) = \pi(y_n(t))$. We can show x_n tends to x uniformly in t. For non-convex domains, this projection can be defined in r_0 neighborhood of the domain for some finite constant r_0.

A second one is the *penalty method* type, that is, we approximate Skorohod equations by equations with coefficients of gradient type. For the purpose of obtaining numerical data by computer simulation of Skorohod SDE's we need discretized approximate solutions of them. In this paper, we focus our attention on reflecting Brownian motion on multi-dimensional domains and give a new penalty method type approximation. Our scheme is separated into two steps: we first approximate reflecting Brownian motions by standard Brownian motions with a drift term whose coefficients are gradient type (penalty method) in the pathwise sense and secondly we approximate the Brownian motion with the drift term by the Euler-Maruyama scheme. Here we do not need to assume boundedness and convexity for domains. We make use of a slight modification of the result in [2] to obtain the rate of convergence. As for approximation of solutions of general SDE, see e.g. [13], [18], and [8].

2. STRONG APPROXIMATION OF MULTIDIMENSIONAL REFLECTING BROWNIAN MOTION

2.1. **Skorohod equation.** In the section we consider a strong approximation of multidimensional reflecting Brownian motion according to [9] and [5] in which stochastic differential equations with some boundary conditions on multidimensional domains (so-called Skorohod SDE) are considered using a penalty method. It had been known that the Skorohod SDE with smooth, convex and bounded domain has a unique solution until Saisho [15] extended the result to prove the existence of it for non-convex and non-smooth domain (see Figure 1). Now we show some results which are obtained in [17] and [15]. We first consider two essential conditions for a domain

$D \subset \mathbb{R}^d$. Define some sets of inward normal vectors at $x \in \partial D$ by

$$\mathcal{N}_{x,r} = \{\boldsymbol{n} : |\boldsymbol{n}| = 1,\ B\left(x - r\boldsymbol{n}, r\right) \cap D = \emptyset\}, \quad r > 0,$$
$$\mathcal{N}_x = \bigcup_{r>0} \mathcal{N}_{x,r}, \quad x \in \partial D,$$

where $B(y, r) = \{z \in \mathbb{R}^d : |z - y| < r\}$, $y \in \mathbb{R}^d$, $r > 0$.

Condition (A) There exists a constant $r_0 > 0$ such that for all $x \in \partial D$

$$\mathcal{N}_x = \mathcal{N}_{x,r_0} \neq \emptyset.$$

Condition (B) There exist constants $\delta > 0$ and $\beta \in [1, \infty)$ such that the following holds: for any $x \in \partial D$ there is a unit vector ℓ_x with

$$\langle \ell_x, \boldsymbol{n} \rangle \geq 1/\beta \quad \text{forany} \quad \boldsymbol{n} \in \bigcup_{\mathrm{y} \in \mathrm{B}(\mathrm{x},\delta) \cap \partial \mathrm{D}} \mathcal{N}_{\mathrm{y}},$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^d$.

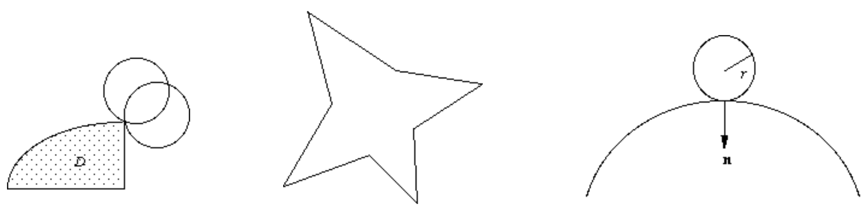

FIGURE 1. Non-smooth domain, Star shaped domain, Normal vector

Remark 2.1. A star shape domain in Fig. 1 does not satisfy Condition (A).

Remark 2.2. For any convex domain D, Condition (A) is satisfied for all $r_0 > 0$.

Condition(A) means that for any $x \in \partial D$ there exists a ball with the radius larger than r_0 which touches D at x but is disjoint with D as in the cases of Figures 1-2 and 7-12. On the other hand, from Fig. 1, at the roots of hands of the star there does not exist such ball. Therefore the star shaped domain does not satisfy Condition (A). If D satisfies Condition (A) and $\operatorname{dist}(x, \overline{D}) < r_0$ there exists a unique $\pi(x) \equiv \overline{x} \in \overline{D}$ with $\operatorname{dist}(x, \overline{D}) = |x - \overline{x}|$. Especially, we have $\overline{x} = x$ for $x \in D$ and if $x \notin D$ we have $\overline{x} \in \partial D$.

Skorohod problem/equation Skorohod problem is means to find a pair (x, φ) satisfying the equation (called *Skorohod equation*)

$$\begin{cases} x\left(t\right) = w\left(t\right) + \varphi\left(t\right), \quad t \geq 0 \\ w\left(0\right) \in \bar{D} \end{cases} \tag{2.1}$$

under conditions

· $x\left(t\right) \in \overline{D}$, iscontinuous,

- $\cdot\ \varphi(t):$ iscontinuousandboundedvariation, $\quad \varphi(0)=0,$
- $\cdot\ \varphi(t)=\int_0^t \boldsymbol{n}(s)d\,|\varphi|_s\,, \quad |\varphi|_t=\int_0^t I_{\{x(s)\in\partial D\}}(s)d\,|\varphi|_s,$
- $\cdot\ \boldsymbol{n}(s)\in\mathcal{N}_{x(s)} \quad \text{if x (s)} \in \partial\text{D}$

for given continuous function w. Here, $|\varphi|_t$ is the total variation of φ on $[0,t]$ and

$$I_A(s)=\begin{cases}1, & s\in A\\ 0, & s\notin A\end{cases}.$$

Existence of the unique solution of the Skorohod equation (2.1) is guaranteed by the following theorem.

Theorem 1. [15] Suppose that D satisfies Conditions (A) and (B). Then for any continuous w, there exists a unique solution (x,φ) of the Skorohod problem.

2.2. **Euler-Maruyama scheme.** Let $\sigma:\mathbb{R}^d\to\mathbb{R}^d\otimes\mathbb{R}^d, \quad b:\mathbb{R}^d\to\mathbb{R}^d$ be bounded Lipschitz continuous functions and $\triangle_n$ be a partition of the interval $[0,T]$:

$$\triangle_n:\ 0=t_0<t_1<t_2<\cdots<t_n=T$$

and put

$$|\triangle_n|=\max_{1\le k\le n}|t_k-t_{k-1}|, \quad \triangle B(t_k)=B(t_k)-B(t_{k-1}), \quad \triangle t_k=t_k-t_{k-1}.$$

We note that $\{B(t)\}$ is a standard Brownian motion. According to Maruyama [13] the Euler-Maruyama scheme for the SDE

$$\begin{cases}X(0)=x\\ X(t)=\int_0^t\sigma(X(s))\,dB(s)+\int_0^t b(X(s))\,ds, \quad 0<t\le T\end{cases}$$

is defined by

$$\begin{cases}X^n(t)=x\in\mathbb{R}^d, \quad 0\le t<t_1,\\ X^n(t)=X^n(t_{k-1})+\sigma(X^n(t_{k-1}))\triangle B(t_k)\\ \qquad\qquad +b(X^n(t_{k-1}))\triangle t_k, \quad t_k\le t<t_{k+1}.\end{cases} \tag{2.2}$$

2.3. **Penalty method.** It is known that if a domain $D\,(\subset\mathbb{R}^d)$ satisfies Condition (A), there is a function $U:\mathbb{R}^d\to\mathbb{R}_+$ satisfying the following conditions ([17]):

- $\cdot\ U\in C^1(\mathbb{R}^d)$
- $\cdot\ U(x)=|x-\overline{x}|^2 \quad \text{if } \operatorname{dist}(x,\overline{D})\le r_0/2$
- $\cdot\ \nabla U$ is bounded and Lipschitz continuous with the Lipschitz constant L.

Now for $m=1,2,\ldots$ we consider the SDE

$$X_m(t)=x+B(t)-\frac{m}{2}\int_0^t\nabla U(X_m(s))\,ds, \quad 0\le t\le T \tag{2.3}$$

and Skorohod equation $X(t)=x+B(t)+\Phi(t)$, $X(0)=x\in\overline{D}$. Then it is easy to see that the SDE (2.3) has a unique solution. We also have the following results.

Theorem 2. ([17], [15]) Suppose D satisfies Conditions (A) and (B). Then we have

$$\lim_{m\to\infty} \|X_m - X\|_T = 0 \quad \text{a.e.}$$

We assume that a domain D satisfies Conditions (A) and (B). Let $X(t)$ be a *multi-dimensional reflecting Brownian motion* the process determined by Skorohod equation $X(t) = x+B(t)+\Phi(t)$, $X(0) = x \in \overline{D}$. We also assume $T = 1$ for simplicity. Then we have the following proposition from [15].

Proposition 1. For any $\varepsilon \in (0, r_0)$ there exists a positive integer M satisfying

$$X_m(t) \in D_\varepsilon \equiv \{x \mid \operatorname{dist}(x, D) < \varepsilon\}, \quad 0 \le t \le 1,\ m \ge M.$$

Now put $w(t) = x + B(t)$ and

$$\Delta_{0,T,h}(w) = \sup_{0\le s,t\le T,\ |t-s|<h} |w(t) - w(s)|, \quad T > 0,\ h > 0$$

Then by Lemma 4.2 in [17] and Proposition 1 we know that if we can take M such as

$$\Delta_{0,1,1/m}(w) < \frac{\varepsilon}{12}e^{-L}, \quad \forall m \ge M, \tag{2.4}$$

then we have $X_m(t) \in D_\varepsilon$, $0 \le t \le 1$. On the other hand, from the continuity of the Brownian path we have

$$C_1\sqrt{\frac{1}{m}\log m} \le \Delta_{0,1,1/m}(w) \le C_2\sqrt{\frac{1}{m}\log m}, \quad m \to \infty, \tag{2.5}$$

where C_1 and C_2 are positive constants with $C_1 < C_2$. From (2.4) and (2.5), if we take m such as for each $\varepsilon \in (0, r_0)$

$$\sqrt{\frac{1}{m}\log m} < Const.\varepsilon,$$

then we have $X_m(t) \in D_\varepsilon$, $0 \le t \le 1$, which yields the following proposition.

Proposition 2. As $m \to \infty$, we have $X_m(t) \in D_\varepsilon$, $0 \le t \le 1$ for some $\varepsilon = \varepsilon(m) > 0$. Moreover, we have $\varepsilon = O\left(m^{-1/2}(\log m)^{1/2}\right), \quad m \to \infty$.

Following (2.2) define an Euler-Maruyama approximation $X_m^n(t)$ for $X_m(t)$ by

$$\begin{cases} X_m^n(t) = x, \quad 0 \le t < t_1, \\ X_m^n(t) = X_m^n(t_{k-1}) + \triangle B(t_k) - \frac{m}{2}\nabla U\left(X_m^n(t_{k-1})\right)\triangle t_k, \quad t_k \le t < t_{k+1}. \end{cases} \tag{2.6}$$

Then we have the following theorem.

Theorem 3. [1] For the Euler-Maruyama scheme for SDE we have

$$E\{\|X_m^n - X_m\|_T^p\} = o\left(|\triangle_n|^{p/2}(-\log|\triangle_n|)^\varepsilon\right), \ n \to \infty, \quad {}^\forall\varepsilon > p/2,\ 2 \le p < \infty,$$

where $\|w\|_T \equiv \sup_{0\le t\le T}|w(t)|$.

Proposition 2 and Theorem 3 imply the following result.

Theorem 4. ([15]) For any $\delta,\ \delta' > 0$ we have

$$n^{1/2-\delta}\,\|X_m^n - X_m\|_1 \to 0 \quad \text{a.e.}, \quad \text{n} \to \infty,$$
$$m^{1/4-\delta'}\,\|X_m - X\|_1 \to 0 \quad \text{a.e.}, \quad \text{m} \to \infty.$$

Remark 2.3. When we simulate a reflecting Brownin motion using $X_m^n(t)$, m should be selected appropriately with respect to n which is sufficiently large. If m is not appropriate the path of $X_m^n(t)$ goes outside from the domain D as the following example. Fig. 2 shows distributions of 100 points of $\tilde{X}_m^n\,(0.02)$ starting at $(0,2) \in \mathbb{R}^2$ when $t = 0.02$, $n = 2500$ for $m = 10$, $m = 100$, $m = 400$ and $m = 1000$, respectively. $m = 400$ is the best in these cases. In the case when $m = 1000$ the penalty is too strong to find a suitable path of the reflecting Brownian motion. On the other hand as in the case when $m = 400$ only few points located outside the domain since m and n are selected appropriately enough.

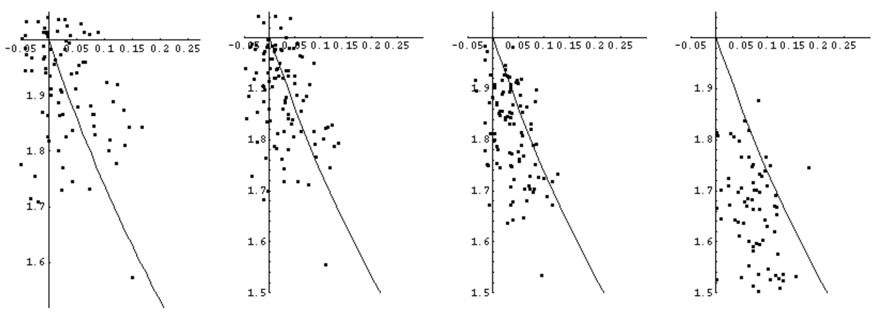

FIGURE 2. $m = 10$, $m = 100$, $m = 400$, $m = 1000$

3. NUMERICAL SIMULATION OF REFLECTING BROWNIAN MOTION WITH SEMI-REFLECTING BOUNDARIES

3.1. **What is a semi-reflecting boundary?** In this section we investigate sample paths of Brownian motion with an semi-reflecting boundary, i.e. any path of the Brownian motion does not reflect at the boundary immediately but is absorbed for a short period according to the speed of the path getting out of the boundary (Fig. 3). Such situation is observed when the electromagnetic wave reaches the ionosphere of the earth which has a kind of flexibility, not perfect elasticity. We approximate the Brownian motion with a semi-reflecting boundary by a random walk via a penalty method. We define an approximate solution of SDE with a reflecting barrier via Skorohod's equation and show some examples of their simulations.

3.2. **Some examples.** Now we first treat a reflecting Brownian motion defined in a concentric circle type domain with a semi-reflecting boundary. Since each m in (2.4) controls the power to pull the Brownian motion path into the domain D from the outside of it, the length of the period in which the path stays in the outside of D

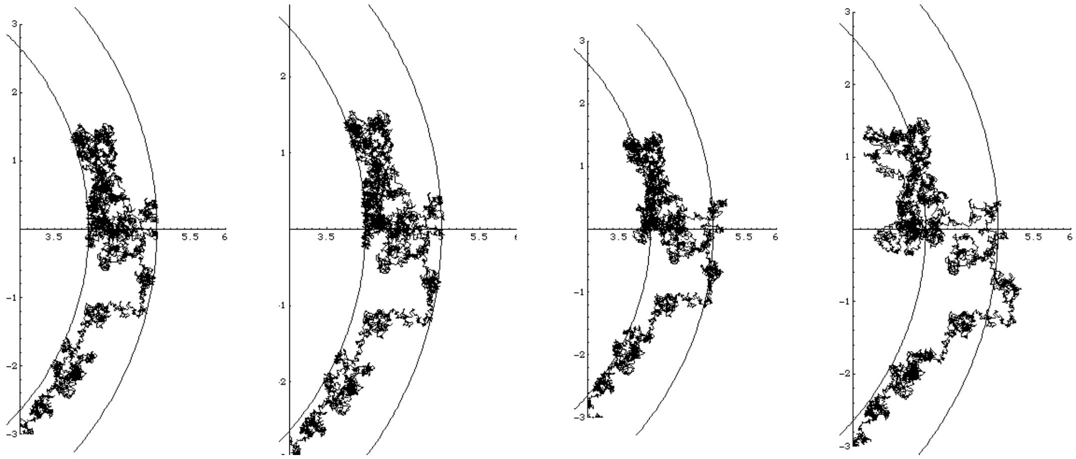

FIGURE 3. Semi-reflecting boundaries

depends on m. We next observe the relation between m and the proportion of the number of steps for $X_m^n(t_k)$ located outside of D.

Example 1. $D = \{\boldsymbol{x} = (x, y) : 16 < x^2 + y^2 < 25\} \subset \mathbb{R}^2$. The Brownian motions starts from $(0, 4.5) \in \mathbb{R}^2$ with $m = 0.01,\ 0.1,\ 0.5,\ 1, 2$,respectively. Put $n = 10000$ and observe $\{X_m^{10000}(t),\ 0 \le t \le 200\}$ which approximates $\{X_m(t),\ 0 \le t \le 200\}$. Fig. 3 shows a path of X_m^n starting at $(0, 4.5) \in \mathbb{R}^2$ with $m = 0.1, n = 10,000$. For example we observe from Fig. 3 that

$$\#\left\{1 \le k \le 10000 : X_{0.1}^{10000}(t_k) \notin D\right\} = 511.$$

The next table is a relation between $m = 0.01,\ 0.1,\ 0.5,\ 1, 2$ and the average of

$$\frac{\#\{1 \le k \le 10000 : X_m^{10000}(t_k) \notin D\}}{10000}$$

for 100 sample paths, respectively.

m	*average of proportion*
0.01	0.2
0.1	0.022
0.5	0.029
1	0.017
2	0.009

Example 2. The domain $D = \{\boldsymbol{x} = (x, y) : x^2 + y^2 < 25\}$ is the inside of a circle. Fig. 4 shows a sample path of a reflecting Brownian motion which starts from $(0, 5) \in \mathbb{R}^2$ with $m = 0.5, n = 10,000$ (Fig. 6). The next table is a relation between $m = 0.01,\ 0.1,\ 1,\ 2$ and the averages of proportions of $X_m^{10,000}(t_k)$ located outside of

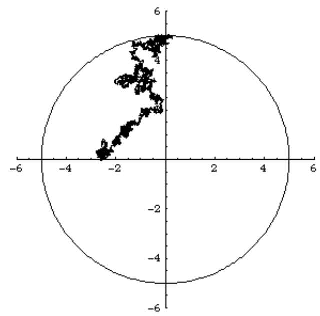

FIGURE 4. Circle

the domain D to $10,000$ steps for 100 sample paths, respectively.

m	*average of proportion*
0.01	0.096
0.1	0.032
1	0.0081
2	0.0037

Example 3. The domain is $D = \left\{ (x, y) \ ; \ 0 < y < 3 - \sqrt{10 - (x-3)^2}, \ 0 < x < 2 \right\}$. Fig. 5 shows a sample path of a reflecting Brownian motion which starts from $(0.2, 1.5) \in D$ with $m = 0.5$, $n = 1,000$. The next table is a relation between $m = 0.1, \ 0.5, \ 1$

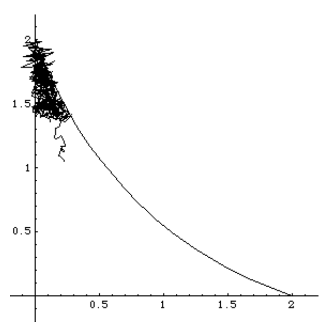

FIGURE 5. Non-convex domain

and the averages of proportions of $X_m^{1,000}(t_k)$ located outside of the domain D to $1,000$ steps for 100 sample paths, respectively.

m	*average of proportion*
0.1	0.24
0.5	0.074
1	0.0039

4. NUMERICAL SIMULATION OF OCCUPATION TIME FOR REFLECTING BROWNIAN MOTION

As for the occupation time of a standard Brownian motion it is well known that the arcsine law holds. On the other hand it is clear that the law does not hold for a reflecting Brownian motion. Therefore we investigate the occupation time of the reflecting Brownian motion using a numerical simulation and show that it is in proportion to the ratio of the area of the subset to the domain.

Remark 4.1. Suppose D satisfies Conditions (A) and (B). Then the Lebesgue measure on $\overline{D}$ is an invariant measure of the reflecting Brownian motion X ([17]). In particular, if D is bounded and the starting point of the reflecting Brownian motion is non-random, then the distribution of $X(t)$ tends to the uniform distribution on $\overline{D}$ as $t \to \infty$.

Fig. 6 shows a sample path of $X_m^n(t)$ starting from $(0,4) \in \mathbb{R}^2$ with $m = 640,000, n = 4,000,000$ for $t \in [0, 0.25]$ (Fig. 6) and the distributions of $X_m^n(t)$ with $X_m^n(0) = (0,2), m = 640,000, n = 4,000,000$ and the sample size number $= 60$ at $t = 0.05$ (Fig. 7), $t = 0.25$ (Fig. 8) and $t = 0.625$ (Fig. 9), respectively. Fig. 9 shows that the asymptotic distribution of $X_m^n(t)$ as $t \to \infty$ is the uniform distribution over the domain D. From Fig. 9 we can expect that the occupation time sojourning in

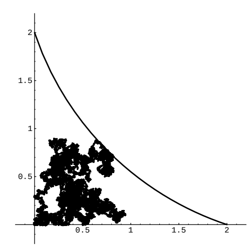

FIGURE 6. Sample path of $\{X_m^n(t), 0 \le t \le 0.25\}$

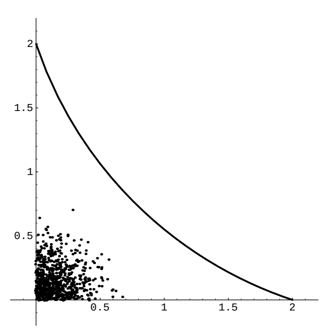

FIGURE 7. 500 sample points of $X_m^n(0.05)$

a subset in the domain is in direct proportion to the area of the subset. Therefore we verify the expectation by the following numerical simulations for occupation times.

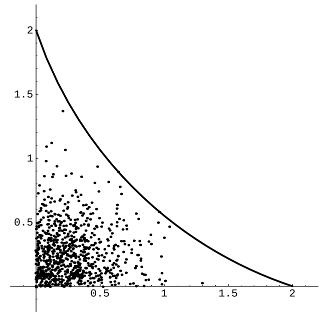

FIGURE 8. 500 sample points of $X_m^n(0.25)$

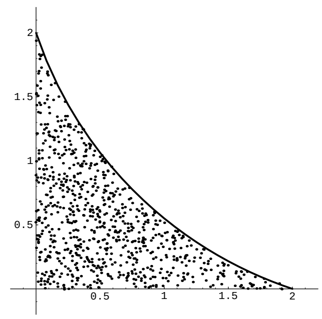

FIGURE 9. 500 sample points of $X_m^n(0.65)$

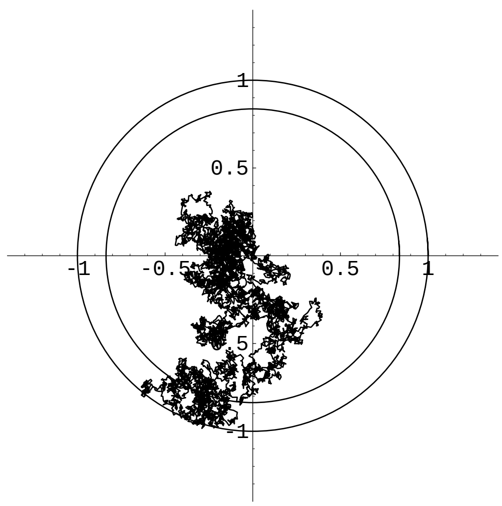

FIGURE 10. A sample path of $\{Y_m^n(t), 0 \le t \le 10\}$

Example 4. Let $D = \{\boldsymbol{x} = (x, y) : 0 < x^2 + y^2 < 1\} \subset \mathbb{R}^2$. The Brownian motion starts from $(0, 0) \in \mathbb{R}^2$. Fig. 10 shows a sample path of $\{X_m^n(t), 0 \le t \le 10\}$ with $m = 2$.

The next graphs (Figures 11-16) are samples of the occupation time $\tau_a(k)$ of $\{Y_m^n(t), 0 \le t \le k/n\}$ sojourning in $T_a = \{(x, y) |\, a \le x^2 + y^2 \le 1\}$, i.e.

$$\tau_a(k) = \left| \left\{ 1 \le i \le k |\, Y_n^m\left(\frac{i}{n}\right) \in T_a \right\} \right| \Big/ k$$

The horizontal axis and vertical axis mean "k" and $\tau_a(k)$, respectively.

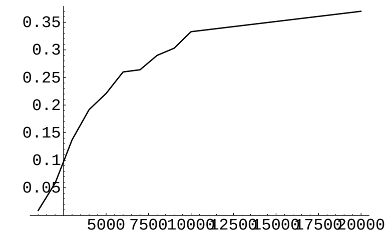

FIGURE 11. $a = 0.6$, $n = 10000$, starting from $(0,0)$

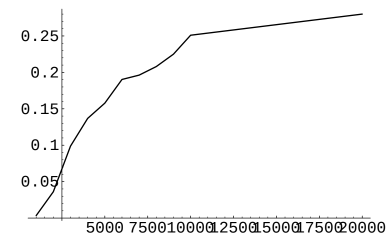

FIGURE 12. $a = 0.7$, $n = 10000$, starting from $(0,0)$

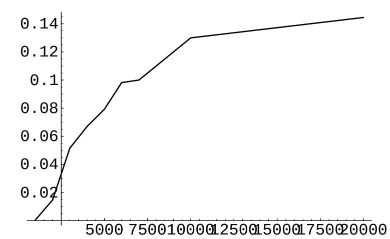

FIGURE 13. $a = 0.85$, $n = 10000$, starting from $(0,0)$

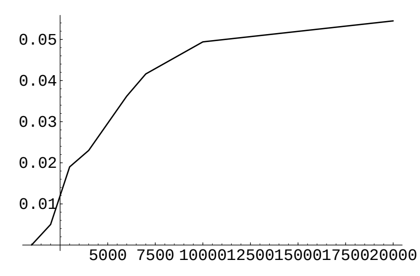

FIGURE 14. $a = 0.95$, $n = 10000$, starting from $(0,0)$

5. NEW MODEL OF SEMI-REFLECTING BROWNIAN MOTIN WITH SEMI-REFLECTING BOUNDARIES

For example, when the reflecting Brownian motion is applied to a financial modeling, the reflecting domain often changes to another domain. Therefore we consider

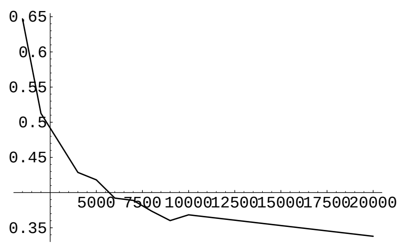

FIGURE 15. $a = 0.7$, $n = 10000$, starting from $(0, 1)$

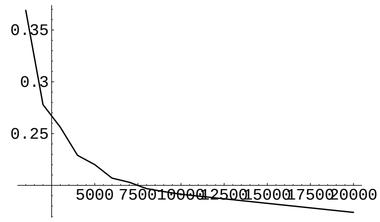

FIGURE 16. $a = 0.85$, $n = 10000$, starting point $(0, 1)$

a new model of a semi-reflecting Brownian motion with some reflecting domains in which the Brownian motion moves from a domain to another domain, one after another. Such semi-reflecting Brownian motion off course does not satisfy the Skorohod equation, however it will be applied to lots of fields. Now, as a typical example, we consider the following semi-reflecting boundaries. The reflecting domain is $W_1 \cup W_3$, where

$$W_1 = \left\{ (x,y) \mid 0 < x^2 + y^2 < 4 \right\}, \quad W_3 = \left\{ (x,y) \mid 16 < x^2 + y^2 < 64 \right\}.$$

Through the set $W_2 = \{ (x,y) \mid 4 \leq x^2 + y^2 \leq 16 \}$ between W_1 and W_3, the semi-reflecting Brownian motion moves from W_1 to W_3 or W_3 to W_1 alternately. To realize a semi-reflecting Brownian motion we consider the SDE

$$X_m(t) = x + B(t) - \frac{m}{2} \int_0^t \nabla U(X_m(s))\, ds, \quad 0 \leq t \leq T.$$

Define a penalty function $U(x,y)$ by

$$U(x,y) = \begin{cases} 0, \quad x^2 + y^2 < 4 \\ \left(\sqrt{x^2+y^2} - 2\right)^{1.1} \left(3 - \sqrt{x^2+y^2}\right) \left(4 - \sqrt{x^2+y^2}\right)^{1.1}, \\ \qquad\qquad\qquad\qquad 4 \leq x^2 + y^2 \leq 16 \\ 0, \quad 16 < x^2 + y^2 < 64 \\ \left(\sqrt{x^2+y^2} - 8\right)^{1.1} \left(11 - \sqrt{x^2+y^2}\right)^{1.1}, \quad 64 \leq x^2 + y^2 \leq 121 \\ 0, \quad 121 < x^2 + y^2 \end{cases}$$

$U(x,y)$ on $\{ (x,y) \mid 4 \leq x^2 + y^2 \leq 9 \}$ draws back the Brownian motion into the domain W_1. When the Brownian motion reaches in $\{ (x,y) \mid 9 < x^2 + y^2 \leq 16 \}$, $U(x,y)$ draws up into the domain W_3. On the other hand $U(x,y)$ on $\{ (x,y) \mid 64 \leq x^2 + y^2 \leq 121 \}$

draws back into the domain W_3. Therefore $U(x, y)$ keeps the Brownian motion inside of $\{(x, y) | 0 \le x^2 + y^2 \le 121\}$. When the penalty factor m is large enough the Brownian motion cannot move so often from a domain to another. It would remain in the domain for a long time. Then the solution of the SDE is a semi-reflecting Brownian

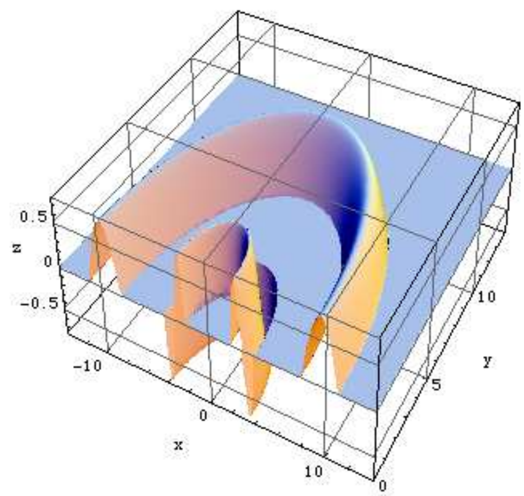

FIGURE 17. $U(x, y)$

motion with reflecting domain $W_1 \cup W_3$ and the Brownian motion moves in $W_1 \cup W_3$ through W_2. Figures 18-20 are sample paths with several time intervals, respectively. They show that the semi-reflecting Brownian motion moves from W_1 to W_3 or W_3 to W_1 alternately.

Remark 5.1. In Fig. 20 two figures of sample paths for $0 \le t \le 800$ and $0 \le t \le 1300$ are similar, respectively. However off course they are not same. Since the sample path returned from W_3 to W_1 at $t = 750$ and it stays in W_1 after $t = 750$, we cannot easily distinguish these two figures.

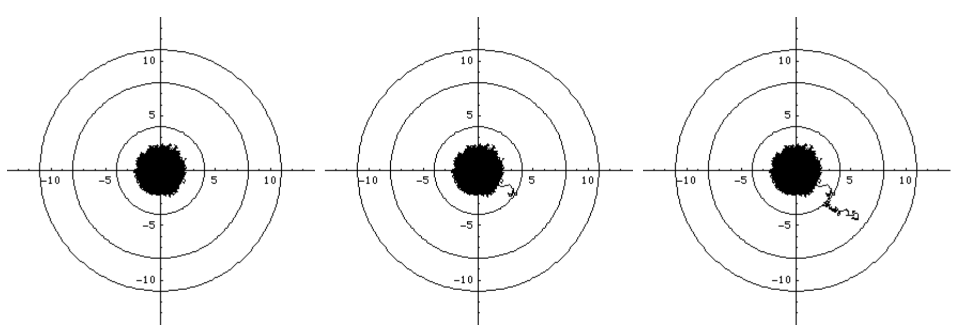

FIGURE 18. Semi-reflecting Brownian motion for $0 \le t \le 300$, $0 \le t \le 310$, $0 \le t \le 320$.

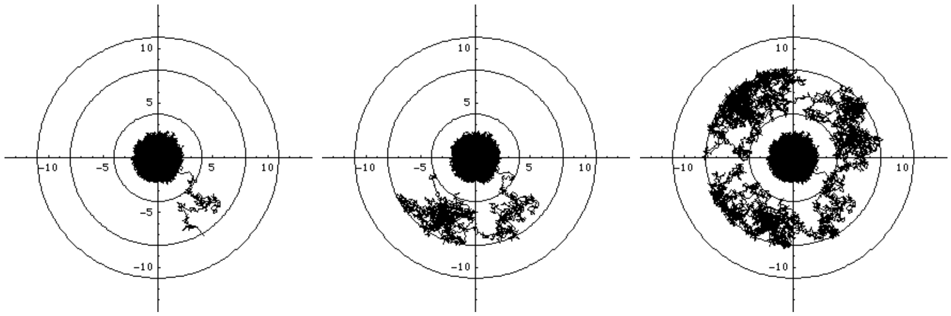

FIGURE 19. Semi-reflecting Brownian motion for $0 \le t \le 330$, $0 \le t \le 400$, $0 \le t \le 500$.

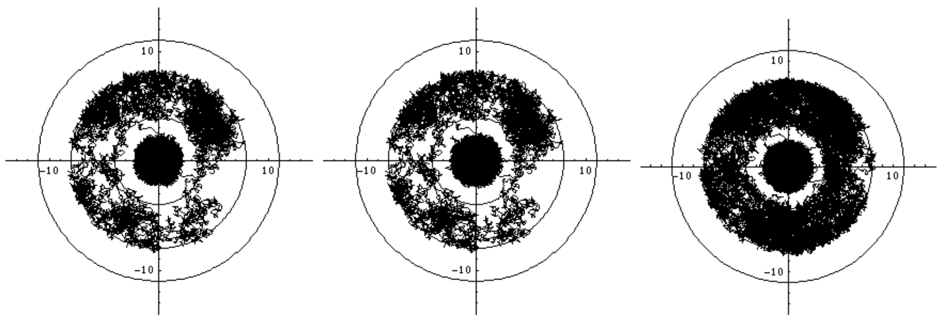

FIGURE 20. Semi-reflecting Brownian motion for $0 \le t \le 800$, $0 \le t \le 1300$, $0 \le t \le 2500$.

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REFERENCES

[1] O. Faure, *Numerical pathwise approximation of stochastic differential* equations, *Prépublication of LAMM* , Ecole Nationale des Ponts et Chaussees, Paris, 1990.

[2] S. Kanagawa, *On the rate of convergence for Maruyama's approximate solutions of stochastic differential equations*, Yokohama Mathematical Journal, **36**: 79–86, 1988.

[3] S. Kanagawa, *The rate of convergence for approximate solutions of stochastic differential equations*, *Tokyo J. Math.*, **12**: 33–48, 1989.

[4] S. Kanagawa, *Confidence intervals of discretized Euler-Maruyama approximate solutions of SDE's*, Nonlinear Analysis, Theory, Methods and Applications, **30**: 4101–4104, 1997.

[5] S. Kanagawa, *Some Remarks on Strong Approximation of Reflecting Brownian Motion Using Penalty Method*, Proceedings of Neural, Parallel and Scientific Computations, **2**: 63–70, 2002, Atlanta, U.S.A.

[6] S. Kanagawa, A. Arimoto and Y. Saisho, *Numerical simulation of reflecting geometrical Brownian motion and its application to mathematical finance*, Nonlinear Analysis, **63**: e2209–e2222, 2005.

[7] S. Kanagawa, K. Inoue, A. Arimoto and Y. Saisho, *Mean square approximation of multi dimensional reflecting fractional Brownian motion via penalty method*, Discrete and Continuous Dynamical Systems, Supplement Volume: 463–475, 2005, American Institute of Mathematical Sciences.

[8] S. Kanagawa and S. Ogawa, *Numerical Solutions of Stochastic Differential Equations and Their Applications*, Sugaku Expositions, Amer. Math. Soc., **18** : 75–99, 2005.

[9] S. Kanagawa and Y. Saisho, *Strong Approximation of Reflecting Brownian Motion Using Penalty Method and its Application to Computer Simulation*, Monte Carlo Methods Appl., **6**: 105-114, 2000.

[10] S. Kanagawa, Y. Saisho and H. Uesu, *Error Estimation for the Euler-Maruyama Approximate Solution of Reflecting Brownian Motion Using Penalty Method and Numerical Applications*, Dynamic Systems and Applications, **15**: 287–300, 2006.

[11] P.L. Lions and A.S. Sznitman, *Stochastic differential equations with reflecting boundary conditions*, Comm. Pure Appl. Math., **37**: 511–537, 1981.

[12] D. Lépingle, *Euler scheme for reflected stochastic differential equations, Proceedings of Probabilités numeriqués*, Paris 1992, in Math. Comput. Simulation, **38**: 119–126, 1995.

[13] G. Maruyama, *Continuous Markov processes and stochastic equations*, Rend. Circ. Mat. Palermo, **4**: 48–90, 1955.

[14] R. Pettersson, *Approximation for stochastic differential equations with reflecting convex boundaries*, Stoch. Processes Appl., **59**: 295–308, 1995.

[15] Y. Saisho, *Stochastic differential equations for multi-dimensional domain with reflecting boundary*, Probab. Theory Rel. Fields, **74**: 455–477, 1987.

[16] Y. Saisho and H. Tanaka, *Stochastic differential equations for mutually reflecting Brownian balls*, *Osaka J. Math.*, **23**: 725–740, 1986.

[17] Y. Saisho and H. Tanaka, *On the symmetry of a reflecting Brownian motion defined by Skorohod's equation for a multi-dimensional domain*, Tokyo J.Math., **10**: 419–435, 1987.

[18] H. Schurz, *Numerical Analysis of SDEs without tears*, in V. Lakshmikantham and D. Kannan (eds.) *Handbook of Stochastic Analysis and Applications*, Mecel Dekker, Basel, 237–359, 2002.

[19] H. Tanaka, *Stochastic differential equations with reflecting boundary condition in convex regions*, Hiroshima Math. J., **9**: 163-177, 1979.

[20] K. Yoshihara, *Weakly Dependent Stochastic Sequences and Their Applications*, **4**, Curve Estimation based on Weakly Dependent Data, Sanseido Co. Ltd., Tokyo, 1994.