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STOCHASTIC DYNAMIC MODEL FOR POROUS MEDIA EQUATION DESCRIBING UNDERGROUND RESOURCES

N. U. AHMED[1] AND S. KERBAL[2]

[1]School of Information Technology and Engineering , University of Ottawa
Ottawa, 800 King Edward Avenue, K1N6N5 Canada
E-mail: ahmed@site.uottawa.ca

[2]Department of Mathematics and Statistics, Sultan Qaboos University
Muscat,P.O. Box 36, PC 126, Al-Khod, Sultanate of Oman
E-mail: skerbal@squ.edu.om

ABSTRACT. We consider a stochastic model for porous media flow. This is a degenerate nonlinear stochastic parabolic partial differential equation which may change type from parabolic to elliptic and vice versa. We present existence and regularity of weak solutions. We formulate certain control problems for aquifers (under ground water reservoirs) and discuss some open problems.

Keywords. Porous media equations, stochastic models, nonlinear stochastic partial differential equations, aquifers, optimal feedback control laws

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1. INTRODUCTION

Dynamics of flow in porous media has direct application in recovery of underground resources like oil, gas and water. Deterministic models based on Darcy's law or more generalized versions of such laws are well known [1,2,5] and recently such models have been used with controls included for optimal extraction of underground water resources from aquifers. In this paper, we present a stochastic model which takes care of uncertainties in the recharge process that feeds the system. We prove existence and regularity properties of weak solutions and discuss some control problems. For continuity of presentation, Sections 2 and 3 are briefly devoted to model description and some recent results on existence of weak solutions. The stochastic model is introduced in Section 4. In Section 5, we present the main results of this paper.

2. BASIC DETERMINISTIC SYSTEM MODEL

Let Σ denote the porous media which we assume to be an open, bounded set with piecewise smooth boundary $\partial\Sigma$. Let $\rho(t, \cdot)$ denote the spatial distribution of fluid in

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Σ at time $t \geq 0$. The temporal and spatial evolution of the density ρ is governed by a nonlinear partial differential equation of the form

$$(\partial/\partial t)(c\rho) - \triangle\Phi(\rho) = f \text{ on } I \times \Sigma, \tag{2.1}$$

$$\Phi(\rho) = 0 \text{ on } I \times \partial\Sigma, \tag{2.2}$$

$$\rho(0) = \rho_0,\ \xi \in \Sigma, \tag{2.3}$$

where $c(\xi)$ is a scalar valued function defined on Σ and taking values $0 \leq c(\xi) \leq 1$, representing porosity of the medium. See [1, 2] for more on the model.

Flux of Vector flow Rate: In general Φ is a monotone increasing function of density ρ. The flux of vector flow rate J is given by $J \equiv \bigtriangledown\Phi$. For simplicity, we have assumed that Φ is independent of the spatial variable $\xi \in \Sigma$. For porous media, the typical form of Φ is

$$\Phi(\rho) = \beta\rho^{\gamma}, \beta > 0, \gamma = 1 + (1/\alpha), 0 < \alpha < \infty,$$

and the pressure $P \equiv F(\rho) = b\rho^{\gamma-1}, b > 0$. The exact expression for Φ is dependent on Darcy's law [4,1].

Source Term: The function $f \equiv f(t,x)$, $t \geq 0$, $x \in \Sigma$, is nonnegative and represents the natural source term giving the rate at which resources are replenished by nature. Detailed construction of the model based on physical arguments can be found in [1,2] and the references therein.

Remark 2.1. Since $\Phi^{'}(0) = 0$, the system is degenerate parabolic (not strictly parabolic). Further, if the set

$$\Sigma^o \equiv \{\xi \in \Sigma : c(\xi) = 0\}$$

is nonempty and has positive Lebesgue measure, system (2.1) may change type. It is elliptic on Σ^o and parabolic on $\Sigma \setminus \Sigma^o$. [See also Remark 3.3].

3. EXISTENCE AND REGULARITY OF SOLUTIONS (Deterministic)

Here we consider deterministic systems and present a brief review of some recent results from [1,2]. This is useful for the study of stochastic system to follow. Define the operator:

$$G \equiv (-\triangle)^{-1} \tag{3.1}$$

subject to homogeneous Dirichlet boundary condition. Using this operator, system (2.1)–(2.3) can be written as an abstract differential equation on a suitable Banach space,

$$\begin{aligned}(d/dt)(G(c\rho)) + \Phi(\rho) &= Gf,\\ \rho(0) &= \rho_0.\end{aligned} \tag{3.2}$$

For this problem, suitable Banach spaces are the Gelfand triple

$$V \hookrightarrow H \hookrightarrow V^*$$

where $H \equiv L_2(\Sigma)$ and it is identified with its dual,

$$V \equiv W_0^{1,p}(\Sigma) \text{ and } V^* \equiv W^{-1,q}(\Sigma),$$

with q being the conjugate of p, that is, $(1/p + 1/q) = 1$, and $1 < q \leq 2 \leq p < \infty$. Since G is a positive self adjoint operator in H , its square root is well defined and hence the operator $G_c^{1/2}(\varphi) \equiv G^{1/2}(c\varphi)$ is well defined. We introduce the vector (function) space W as follows

$$W \equiv \{\rho \ : \ G_c^{1/2}\rho \in L_p(I,V) \ \& \ G_c^{1/2}\dot{\rho} \in L_q(I,V^*)\}.$$

Furnished with the norm topology, given by

$$\| \rho \|_W = \| G_c^{1/2}\rho \|_{L_p(I,V)} + \| G_c^{1/2}\dot{\rho} \|_{L_q(I,V^*)},$$

it is a Banach space and the embedding $W \hookrightarrow C(I,H)$ is continuous. The proof of this embedding is similar to that given in [4, Theorem 1.2.15, p. 27].

In Showalter [5], considering $c(\xi) = 1$, a simple and elegant proof is given on page 142, Example 6.6. In [2, Theorem 3.1] we have presented a differ ent and constructive proof based on finite dimensional projection and limiting arguments without assuming $c(\xi) = 1$. This classical approach is also useful for both approximation and computation as required for real physical applications.

Theorem 3.1 (Existence & regularity)**.** *Consider system* (3.2) *with the following assumptions:*

(A1): $\exists c_1 \in (0,1]$ *such that* $\inf_{\xi\in\Sigma} c(\xi) = c_1$.
(A2): $\Phi : R \longrightarrow R$ *is continuous and m-monotone.*
(A3): $\exists\ p \geq 2$ *and constants* $c_2 > 0, c_3 > 0$*, such that*

$$(1) : \Phi(r)r \geq c_2|r|^p \quad \text{and}$$
$$(2) : |\Phi(r)| \leq c_3|r|^{p-1} \ \forall\ r \in R.$$

Then, for each $\rho_0 \in V^*$ *satisfying* $G_c^{1/2}\rho_0 \in H$ *and* $f \in L_q(I,V^*)$ *satisfying* $G^{1/2}f \in L_2(I,H)$*, system* (3.2) *has a unique weak solution* $\rho \in L_p(I, L_p(\Sigma_c))$*, in the sense that the following identity*

$$-\int_I \langle \rho, (d/dt)G_c\phi\rangle dt + \int_I < \Phi(\rho), \phi >_{V^*,V} dt$$
$$= (E_c^{1/2}\rho(0), G_c^{1/2}\phi(0))_H + \int_I \langle G^{1/2}f, \phi\rangle_{V^*,V} dt$$

holds for all $\phi \in L_p(I,V)$ *satisfying* $G_c^{1/2}\dot{\phi} \in L_2(I,H)$ *and* $\phi(T) = 0$. *Further, the solution has the following regularity properties:*

$$\left\{G_c^{1/2}\rho \in L_p(I,V), \quad G_c^{1/2}\dot{\rho} \in L_q(I,V^*) \quad \& \quad G_c^{1/2}\rho \in L_\infty(I,H) \cap C(I,H)\right\}.$$

Proof. See [Ref. 2, Theorem 3.1]. □

Remark 3.2. Recall that the porosity coefficient $0 \leq c(\xi) \leq 1$. Let

$$L_p(\Sigma_c) \equiv \left\{\varphi : \text{ (measurable) } : \int_\Sigma c(\xi)|\varphi(\xi)|^p < \infty\right\}.$$

Furnished with norm topology :

$$\| \varphi \|_{L_p(\Sigma_c)} = \left(\int_\Sigma c(\xi)|\varphi|^p d\xi\right)^{1/p}.$$

This is a Banach space and in general $L_p(\Sigma) \hookrightarrow L_p(\Sigma_c)$ is continuous. In Theorem 3.1, we assumed that the porosity coefficient $c(\xi)$, $\xi \in \Sigma$, is bounded away from zero. This is used to ensure that $L_p(\Sigma_c) \cong L_p(\Sigma)$, which, in turn, is used to prove the regularity of solutions as stated in the Theorem 3.1. It would be interesting to relax this assumption.

Remark 3.3. Recall the set Σ^o as introduced in Remark 2.1. On this set the system is elliptic. Since Φ is strictly monotone, the solution in the elliptic phase is given by

$$\rho(t,\xi) = \Phi^{-1}(Gf)(t,\xi), \quad (t,\xi) \in I \times \Sigma^o$$

provided the data satisfies the compatibility condition

$$\lim_{t\downarrow 0} \Phi^{-1}(Gf)(t,\xi) = \rho_0(\xi), \quad \xi \in \Sigma_0.$$

On the other hand, for physical reasons it is evident that the fluid content of any part of the medium that has zero porosity must be zero. Hence, the initial condition and the data f must be identically zero on Σ^o.

4. STOCHASTIC SYSTEM MODEL

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, t \geq 0, P)$ denote a complete filtered probability space with $\mathcal{F}_t, t \geq 0$, being an increasing family of right continuous (having left limits) subsigma algebras of the sigma algebra $\mathcal{F}$, P is the probability measure defined for any $\mathcal{F}$ measurable set in Ω. Without further notice, we assume that all the random processes introduced in this paper are $\mathcal{F}_t$ adapted. For any $\mathcal{F}$-measurable random variable z, we let $E(z) \equiv \int_\Omega z(\omega)P(d\omega)$ denote the mathematical expectation of the random variable z. Let X be any separable Hilbert space and, for any $1 \leq p \leq \infty$, let $L_p(\Omega, X)$ denote the space of $\mathcal{F}$-measurable X valued random variables whose X norms are p-th power P integrable. For $1 \leq p \leq \infty$, we let $L_p(I, L_p(\Omega, X))$ denote the class of stochastic processes with values in X and adapted to the filtration $\mathcal{F}_t, t \geq 0$, satisfying the integrability condition

$$E \int_I |x(t)|_X^p dt < \infty.$$

By virtue of Fubini's theorem, it is clear that $E \int_I |x(t)|_X^p \, dt = \int_I E|x(t)|_X^p \, dt$. Furnished with the natural norm topology, these are Banach spaces. One particular Banach space of significant interest is the space $L_\infty(I, L_p(\Omega, X))$ which is used in the study of the stochastic system introduced here.

Now, returning to our system model (3.2), it is reasonable to assume that the recharge process is a stochastic process. It may have a deterministic, as well as a stochastic component. The recharge process f is a deterministic process. So we must add a stochastic term. The standard practice is to use Brownian motion. Clearly the density ρ must always be a nonnegative quantity. So we can not use an additive noise term. To preserve positivity it has to be multiplicative satisfying some additional properties. We consider the following model,

$$(\partial/\partial t)(c\rho) - \triangle\Phi(\rho) = f + \sigma(\rho)\dot{W}, I \times \Sigma, \tag{4.1}$$

$$\Phi(\rho) = 0, \text{ for } (t, \xi) \in I \times \partial\Sigma, \tag{4.2}$$

$$\rho(0, \xi) = \rho_0(\xi), \text{ for } \xi \in \Sigma, \tag{4.3}$$

where $\sigma(\rho)\dot{W}$ is the multiplicative term with $\dot{W}$ denoting the space time white noise, that is, the distributional derivative of the space time Brownian motion $W(t, \xi)$, $t \geq 0$, $\xi \in \Sigma$. For Hilbert space canonical representation, we shall write $W(t), t \geq 0$, as the $H \equiv L_2(\Sigma)$ valued Brownian motion adapted to $\mathcal{F}_t$ having mean zero, $E(W(t), h) = 0, h \in H$, and covariance operator Q, given by

$$E(W(t), h)_H^2 = t(Qh, h), h \in H,$$

where Q is a positive self adjoint operator in H. Since Brownian motion is not differentiable in the classical sense, we shall soon rewrite this system in the canonical form involving stochastic Ito integral.

In order to preserve positivity of ρ, we can take any σ satisfying the following properties:

Basic properties of $\sigma(\cdot)$:

(S1): $\sigma : R \longrightarrow R$,

(S2): $\sigma(r) \geq 0$ for $r \geq 0$, $\sigma(r) = 0$ for $r \leq 0$,

(S3): $\sup\{\sigma(r), r \in R\} \leq b < \infty$ and Lipschitz with Lipschitz constant K.

Under the assumptions on Φ and σ, given that $f \geq 0$ and $\rho_0 \geq 0$, it is easy to verify that every solution of the system (4.1)–(4.3), if one exists, is nonnegative. We consider the Nemytski operator associated with the map σ and denote this by the same symbol. Under the given assumption on σ, as a function, it is easy to verify that

σ, considered as the Nemytski operator, maps H to $\mathcal{L}(H)$. Indeed, for any $\nu \in H$ and $h \in H$, it is evident that $|\sigma(\nu)h|_H \leq b|h|_H$ and $\sigma : H \longrightarrow \mathcal{L}(H)$ is Lipschitz continuous. Using this Nemytski operator σ and the Dirichlet map G we can rewrite the system (4.1)–(4.3) as an abstract stochastic differential equation on the Hilbert space H as follows:

$$d(G(c\rho)) + \Phi(\rho)dt = Gf dt + G\sigma(\rho)dW, t \in I,\ \rho(0) = \rho_0. \tag{4.4}$$

This is the canonical form for stochastic differential equations on infinite dimensional spaces obtained from stochastic PDE.

A class of stochastic porous media equations have also been considered by Barbu, Da Prato and Rockner in [6] where they assume that the porosity is homogeneous and $c \equiv 1$ and the recharge process $f \equiv 0$. These assumptions are not suitable for aquifer modeling. Thus we have eliminated them. Further, our method of proof is constructive, rather than abstract and very different from that of [6]. For existence of weak solutions, we use finite dimensional projection and limiting arguments.

5. EXISTENCE AND REGULARITY OF SOLUTIONS

For the proof of existence of solution for the stochastic system, we need the following a-prior bounds as stated in the following lemma.

Lemma 5.1. *Suppose the basic assumptions of Theorem* 3.1 *hold, and σ satisfy the assumptions (S1)–(S3) and that W is an H-valued Brownian motion with nuclear covariance Q. Then, if ρ is any weak solution of the system* (4.4)*, it must satisfy the following inequality:*

$$E|G_c^{1/2}\rho(t)|_H^2 \leq \exp(T)\left\{E|G_c^{1/2}\rho_0|_H^2 + E\left(\int_I |G^{1/2}f|_H^2 ds\right)\right\}. \tag{5.1}$$

Thus, the data to the solution map,

$$(\rho_0, f) \longrightarrow \rho,$$

is continuous from $L_2(\Omega, V^*) \times L_q(I, L_q(\Omega, V^*))$ to $L_\infty(I, L_2(\Omega, V^*))$.

Proof. Recall that c is a nonnegative bounded function and that G is a positive selfadjoint operator on the Hilbert space $H \equiv L_2(\Sigma)$ and so it admits a positive square root. Taking the scalar product with respect to $\{V^*, V\}$ pairing of equation (4.4) with $c\rho$, and integrating by parts, we obtain,

$$\begin{aligned} E|G_c^{1/2}\rho(t)|_H^2 + 2E\int_0^t \langle \Phi(\rho), c\rho\rangle ds &= E|G_c^{1/2}\rho_0|_H^2 \\ &+ 2E\int_0^t (G^{1/2}f, G_c^{1/2}\rho)ds + 2E\int_0^t (\sigma(\rho)dW, G_c\rho), \quad t \in I, \end{aligned} \tag{5.2}$$

where $\langle z, y\rangle_{V^*,V} = (z, y)_H$ whenever $z \in H \subset V^*$. To proceed further, we must verify that the expected value of the stochastic integral in the expression (5.2) vanishes. If the expected value is proved to be finite, then by use of iterated conditional expectations relative to the family of σ-algebras $\mathcal{F}_t$, one can easily verify that the expected value vanishes. This is very similar to the finite dimensional case. To prove that the expected value is finite, it suffices to verify that

$$E\biggl(\int_0^t (\sigma(\rho)dW, G_c\rho)\biggr)^2 < \infty.$$

Since σ, considered as Nemytski map from H to $\mathcal{L}(H)$, is uniformly bounded with bound $b > 0$, and G is a bounded positive selfadjoint operator on H, we have

$$\begin{aligned} E\biggl(\int_0^t (\sigma(\rho)dW, G_c\rho)\biggr)^2 &= E\int_0^t (Q(\sigma(\rho))^* G_c(\rho), (\sigma(\rho))^* G_c(\rho))ds \\ &= E\int_0^t (Q(\sigma(\rho))G_c(\rho), (\sigma(\rho))G_c(\rho))ds \\ &\le Tr(Q)E\int_0^t |(\sigma(\rho))G_c(\rho)|_H^2 ds \\ &\le b^2 \parallel G^{1/2} \parallel_{\mathcal{L}(H)}^2 Tr(Q)\int_0^t |G_c^{1/2}\rho|_H^2 ds, \end{aligned} \tag{5.3}$$

where $Tr(Q)$ stands for trace of Q.

Using this estimate in (5.2) we obtain the following inequality

$$\begin{aligned} E|G_c^{1/2}\rho(t)|_H^2 + 2E\int_0^t \langle\Phi(\rho), c\rho\rangle ds &\le 1 + E|G_c^{1/2}\rho_0|_H^2 \\ &+ 2E\int_0^t (G^{1/2}f, G_c^{1/2}\rho)ds + \tilde{b}\int_0^t |G_c^{1/2}\rho|_H^2 ds, t \in I, \end{aligned} \tag{5.4}$$

where $\tilde{b} \equiv b \parallel G^{1/2} \parallel_{\mathcal{L}(H)} \sqrt{T}r(Q)$. Since $\Phi(r)r \ge 0$ and $G^{1/2}f \in L_2(I, H)$ almost surely, it follows from the above inequality that

$$\begin{aligned} E|G_c^{1/2}\rho(t)|_H^2 \le 1 + E|G_c^{1/2}\rho_0|_H^2 &+ E\int_0^t |G^{1/2}f|_H^2 ds \\ &+ (1+\tilde{b})E\int_0^t |G_c^{1/2}\rho|_H^2 ds, t \in I. \end{aligned} \tag{5.5}$$

Using Gronwall's inequality, this leads to the following estimate

$$E|G_c^{1/2}\rho(t)|_H^2 \le \biggl(1 + E|G_c^{1/2}\rho_0|_H^2 + E\int_I |G^{1/2}f|_H^2 ds\biggr)\exp(1+\tilde{b})T \tag{5.6}$$

for all $t \in I$. From this it is evident that,

$$G_c^{1/2}\rho \in L_\infty(I, L_2(\Omega, H)) \subset L_2(I, L_2(\Omega, H)).$$

Using this fact it follows from (5.3) that the stochastic integral in the expression (5.2) is well defined and hence the expectation of the integral is zero. Also recalling that the porosity coefficient $c(\xi) \geq 0$ for $\xi \in \Sigma$ and $\Phi(r)r \geq 0$, it follows from equation (5.2) that

$$E|G_c^{1/2}\rho(t)|_H^2 \leq E|G_c^{1/2}\rho_0|_H^2 + E\int_I |G^{1/2}f|_H^2 ds + \int_0^t E|G_c^{1/2}\rho|_H^2 ds$$

for all $t \in I$. Again, by Gronwall inequality, this leads to the estimate as stated in the Lemma. □

From this result we have also other useful estimates as stated in the following corollary.

Corollary 5.2. *Under the assumptions of Lemma* 5.1*, we have*

$$\rho \in L_p(I, L_p(\Omega, V)), \Phi(\rho) \in L_q(I, L_q(\Omega, V^*)).$$

Proof. Using the identity (5.2) and the properties of the monotone map Φ, in particular the assumption (A3), and recalling that $c(\xi) \geq c_0 > 0$, it is easy to see that

$$\begin{aligned} E|G_c^{1/2}\rho(t)|_H^2 + 2c_0c_2 \int_0^t E|\rho(s)|_V^p ds &\leq E|G_c^{1/2}\rho_0|_H^2 \\ &+ E\int_0^t |G_c^{1/2}\rho(s)|_H^2 ds + \int_0^t |G^{1/2}f|_H^2 ds. \end{aligned} \tag{5.7}$$

Thus, it follows from Lemma 5.1, that there exists a positive constant $\gamma \equiv \gamma(T, c_0, c_2)$, dependent on the parameters indicated, so that

$$E\int_I |\rho(s)|_V^p ds \leq \gamma\Big\{E|G_c^{1/2}\rho_0|_H^2 + E\int_I |G^{1/2}f|_H^2 ds\Big\}. \tag{5.8}$$

This shows that $\rho \in L_p(I, L_p(\Omega, V)) \equiv L_p(I \times \Omega, V)$. Using this estimate, the assumption (A3)(2), and the embedding constant $V \hookrightarrow V^*$, denoted by $\| i \|$, it is easy to verify that

$$E\int_I \langle \Phi(\rho), \phi\rangle_{V^*,V} dt \leq c_3(\| i \|^{p/q}) \| \rho \|_{L_p(I,L_p(\Omega,V))}^{p/q} \| \phi \|_{L_p(I,L_p(\Omega,V))} . \tag{5.9}$$

This is true for arbitrary $\phi \in L_p(I, L_p(\Omega, V))$ and hence $\Phi(\rho) \in L_q(I, L_q(\Omega, V^*))$. This completes the proof. □

Now we are prepared to consider the question of existence of solutions of the stochastic differential equation (4.4). We need the following definition.

Definition 5.3. A process $\rho \in L_p(I, L_p(\Omega, V))$ with $G_c^{1/2}\rho \in L_\infty(I, L_2(\Omega, H))$, is said to be a weak solution of equation (4.4) if, for every $\varphi \in L_p(I, V)$ with $G^{1/2}\dot{\varphi} \in L_2(I, H)$, the following identity holds P-a.s. for all $t \in I$:

$$\begin{aligned} &((G_c^{1/2}\rho)(t), G^{1/2}\varphi(t)) - \int_0^t (G_c^{1/2}\rho, G^{1/2}\dot{\varphi})dt + \int_0^t \langle\Phi(\rho), \varphi\rangle_{V^*,V} \\ &\quad = (G_c^{1/2}\rho_0, G^{1/2}\varphi(0)) + \int_0^t (G^{1/2}f, G^{1/2}\varphi)_H ds + \int_0^t (G^*\varphi, \sigma(\rho)dW). \end{aligned} \tag{5.10}$$

To prove existence of solution of equation (4.4), we use finite dimensional approximation and limiting arguments. Let $\{v_i\}$ be a complete basis for the Gelfand triple $\{V, H, V^*\}$ with $\{v_i\}$ being orthonormal in H and orthogonal in both V and V^*. Since the embedding $V \hookrightarrow H$ is compact, such a basis exists. Let X_n denote the finite dimensional space given by the closure of the $span\{v_i, 1 \leq i \leq n\}$. Define $\rho^n \equiv \sum_{i=1}^n x_i^n(t) v_i$ and consider the following system of equations

$$d(G_c \rho^n, v_i) + \langle \Phi(\rho^n), v_i \rangle dt = (Gf, v_i)dt + (G^* v_i, \sigma(\rho^n) dW^n), \tag{5.11}$$

for all $1 \leq i \leq n$, where W^n is the projection of the infinite dimensional Brownian motion to X_n given by

$$W^n(t, \cdot) = \sum_{i=1}^n \sqrt{\lambda}_i v_i(\cdot) w_i(t)$$

with $\{w_i\}$ being an infinite family of independent standard real valued Brownian motions. Clearly, the trace of the covariance operator Q_n corresponding to W^n is given by

$$Tr(Q_n) = \sum_{i=1}^n \lambda_i.$$

The system (5.11) can be written as a stochastic differential equation in R^n of Ito type. This is given by

$$\Gamma dx^n(t) + \tilde{\Phi}(x^n) dt = f^n(t) dt + \Lambda(x^n) dw^n, x^n(0) = x_0^n, \tag{5.12}$$

where $x_0^n \equiv \{(\rho_0, v_i), 1 \leq i \leq n\}$. The rest of the system parameters

$$\{\Gamma, \tilde{\Phi}, f^n, \Lambda, w^n\}$$

appearing in (5.12) are given as follows:

(1): $\Gamma \in M(n \times n)$ is a positive square matrix with elements given by

$$\Gamma_{i,j} = (G_c v_j, v_i)_H, 1 \leq i, j \leq n.$$

(2): $\tilde{\Phi} : R^n \longrightarrow R^n$ is a continuous monotone map with elements given by

$$\tilde{\Phi}_i(x) \equiv \left\langle \Phi\left(\sum_{j=1}^n x_j v_j\right), v_i \right\rangle_{V^*, V}, \quad 1 \leq i \leq n.$$

(3): The function $f^n(t) = \{f_i^n(t), 1 \leq i \leq n\}$ where the components are given by

$$f_i^n(t) = ((Gf)(t), v_i)_H.$$

(4): The operator $\Lambda : R^n \longrightarrow \mathcal{L}(R^n, R^n) \equiv M(n \times n)$ with elements given by

$$\Lambda_{i,j}(x) = \sqrt{\lambda}_j (G^* v_i, \sigma(\sum_{k=1}^n x_k v_k) v_j)_H, \quad 1 \leq i, j \leq n.$$

(5) Finally the n-dimensional Brownian motion is given by

$$w^n \equiv \{w_i^n, 1 \leq i \leq n\},$$

where the components are given by

$$w_i^n(t) \equiv (1/\sqrt{\lambda_i})(W^n(t,\cdot), v_i)_H.$$

We are now prepared to prove the existence of solution of the finite dimensional SDE (5.12). For convenience of presentation, we remove the superscript n and consider the system,

$$\Gamma dx(t) + \tilde{\Phi}(x)dt = \tilde{f}(t)dt + \Lambda(x)dw, x(0) = x_0, \tag{5.13}$$

as a n-dimensional SDE with $\tilde{f} \equiv f^n$. For the proof of existence of solution of the infinite dimensional problem (4.4), first we prove the existence of solution of the finite dimensional system (5.13) and then use the a-priori bounds and limiting arguments to complete the proof. We prove the following result.

Lemma 5.4. *Consider the system* (5.13) *and suppose that x_0 is $\mathcal{F}_0(\subset \mathcal{F}_t)$ measurable having finite second moment, $\tilde{f} \in L_2(I \times \Omega, R^n)$ adapted to the sigma algebra $\mathcal{F}_t$ and the operators $\{\Gamma, \tilde{\Phi}, \Lambda\}$ are as defined above. Then system* (5.13) *has at least one solution x adapted to $\mathcal{F}_t$ having finite second moment for all $t \in I$, that is, $x \in L_\infty(I, L_2(\Omega, R^n))$.*

Proof. We use an implicit difference scheme to prove the existence. Partition the interval $I = [0, T]$ into $k(\in N)$ equal subintervals

$$0 = t_0^k \leq t_1^k \leq t_2^k \leq \cdots \leq t_{k-1}^k \leq t_k^k = T$$

and define $\Delta_k \equiv (t_{i+1}^k - t_i^k)$. Approximate the equation (5.13) by

$$\begin{aligned}(\Gamma + \Delta_k\tilde{\Phi})(x(t_{i+1}^k)) &= \Gamma x(t_i^k) + \int_{t_i^k}^{t_{i+1}^k} \tilde{f}(s)ds \\ &\quad + \Lambda(x(t_i^k))(w(t_{i+1}^k) - w(t_i^k)), \quad 0 \leq i \leq k-1.\end{aligned} \tag{5.14}$$

Since Φ is m-monotone, the reader can easily verify that $\tilde{\Phi}$ is m-monotone in R^n. Recall that $c(\xi) \geq 0, \xi \in \Sigma$, and G is a positive selfadjoint operator on H. Thus, the (finite dimensional) operator Γ, as defined above, is (symmetric) positive in R^n. Hence the sum $(\Gamma + \Delta_k\tilde{\Phi})$ is an m-monotone operator in R^n and therefore has a unique inverse. Thus, it follows from (5.14) that

$$x(t_{i+1}^k) = (\Gamma + \Delta_k\tilde{\Phi})^{-1}\left(\Gamma x(t_i^k) + \int_{t_i^k}^{t_{i+1}^k} \tilde{f}(s)ds + \Lambda(x(t_i^k))(w(t_{i+1}^k) - w(t_i^k))\right) \tag{5.15}$$

is well defined for all $0 \leq i \leq k-1$. The argument of the above map (expression within the parenthesis) is evidently measurable relative to the sigma algebra $\mathcal{F}_t$ for all $t \geq t_{i+1}^k$ and the inverse map is (Lipschitz) continuous. Hence $x(t_{i+1}^k)$ is measurable relative to $\mathcal{F}_t, t \geq t_{i+1}^k$. Thus, given the data x_0 and $\tilde{f}$, we can construct a piecewise approximate solution of equation (5.13) by the implicit difference scheme as displayed above. Since the map $(\Gamma+\Delta_k\tilde{\Phi})^{-1}$ is Lipschitz continuous and Λ is uniformly bounded on R^n because σ is uniformly bounded on R and x_0 has finite second moment and $E\int_I |\tilde{f}|^2 ds < \infty$, it follows from the above expression that $E\{|x(t_i^k)|^2_{R^n}\} < \infty$ for all $1 \le i \le k$. Now we can construct a sequence of stochastic processes $\{x_k(t), t \in I\}$ as follows

$$x_k(t) = x(t_i^k) \ t \in [t_i^k, t_{i+1}^k], 0 \le i \le k-1.$$

Clearly this process has bounded second moments and therefore it is bounded in probability. Following the same procedure as in Skorohod [3, Lemma 3], one can verify that

$$\lim_{h\to 0} \overline{\lim_{k\to\infty}} \sup_{|\tau - s| \le h} P\{|x_k(\tau) - x_k(s)| > \varepsilon\} = 0.$$

Hence, one can follow the compactness arguments of Skorohod (finite difference method) [3, p. 59–73] for the finite dimensional distributions of the processes $\{x_k\}$ to prove that the limit process $\lim_{k\to\infty} x_k(t) = x(t), t \in I$, exists and is a solution of equation (5.13). □

Now we are prepared to prove our main result giving the existence of solution of the infinite dimensional problem (4.4).

Theorem 5.5. *Consider the system* (4.4) *and suppose the assumptions (A1)–(A3) of Theorem* 3.1 *hold and* σ *is a nondecreasing function satisfying the properties (S1)–(S3). Further, suppose* $\rho_0 \in L_2(\Omega, V^*)$ *satisfying* $G_c^{1/2}\rho_0 \in L_2(\Omega, H)$, $f \in Lq(I, L_q(\Omega, V^*))$ *satisfying* $G^{1/2} f \in L_2(I, L_2(\Omega, H))$. *Then system* (4.4) *has at least one weak solution in the sense of Definition* 5.3.

Proof. By Lemma 5.4, the n dimensional system given by the SDE (5.13) has a solution x which we denote by x^n. Thus, the system (5.11) has a solution given by $\rho^n(t) \equiv \sum_{i=1}^n x_i^n(t) v_i, t \in I$. Take any function $\eta \in C^1(I)$ and multiply on either side of equation (5.11) and integrate by parts to obtain

$$\begin{aligned}(G_c^{1/2}\rho^n(t), G^{1/2}\eta(t)v_i)_H &- \int_0^t (G_c^{1/2}\rho^n(s), G^{1/2}\dot{\eta}(s)v_i)_H ds \\ &+ \int_0^t \langle \Phi(\rho^n), \eta v_i\rangle_{V^*,V} ds = (G_c^{1/2}\rho_0^n, G^{1/2}\eta(0)v_i)_H \\ &+ \int_0^t (G^{1/2} f, G^{1/2}\eta(s)v_i)_H ds + \int_0^t (G^*(\eta(s)v_i), \sigma(\rho^n(s)) dW^n(s)). \qquad (5.16)\end{aligned}$$

By virtue of Lemma 5.1, the sequence $\{G_c^{1/2}\rho^n\}$ is contained in a bounded subset of the Banach space $L_\infty(I, L_2(\Omega, H))$ and by Corollary 5.2, the sequence $\{\rho^n\}$ is contained in a bounded subset of $L_p(I, L_p(\Omega, V))$ and the sequence $\{\Phi(\rho^n)\}$ is contained in bounded set in $L_q(I, L_q(\Omega, V^*))$. Hence, there exists a subsequence of the sequence

$\{\rho^n\}$, relabeled as ρ^n, and an element ρ^o such that

$$G_c^{1/2}\rho^n \xrightarrow{w^*} G_c^{1/2}\rho^o \text{ in } L_\infty(I, L_2(\Omega, H)), \tag{5.17}$$

$$G_c^{1/2}\rho^n \xrightarrow{w} G_c^{1/2}\rho^o \text{ in } L_2(I, L_2(\Omega, H)), \tag{5.18}$$

$$\rho^n \xrightarrow{w} \rho^o \text{ in } L_p(I, L_p(\Omega, V)). \tag{5.19}$$

Since $q > 1$ and V^* is reflexive, it is clear that $Lq(I, L_q((\Omega, V^*))$ is reflexive. Hence, the boundedness of the sequence $\{\Phi(\rho^n)\}$ implies that there exists an element $\zeta \in L_q(I, L_q(\Omega, V^*))$ such that, along a subsequence if necessary,

$$\Phi(\rho^n) \xrightarrow{w} \zeta \text{ in } L_q(I, L_q(\Omega, V^*)). \tag{5.20}$$

Since $\{v_i\}$ is a basis for the triple $\{V, H, V^*\}$, and $\rho_0^n \equiv \sum_i^n x_0^n v_i$, it is clear that

$$G_c^{1/2}\rho_0^n \xrightarrow{s} G_c^{1/2}\rho_0 \text{ in } L_2(\Omega, H). \tag{5.21}$$

For the stochastic integral in (5.16), we show that the Brownian motion W^n converges strongly in $L_2(I, L_2(\Omega, H))$. Recall that

$$W^n(t,\xi) = \sum_{i=1}^n \sqrt{\lambda}_i v_i(\xi) w_i(t) \text{ and } W(t,\xi) = \sum_{i=1}^\infty \sqrt{\lambda}_i v_i(\xi) w_i(t).$$

From this representation, it follows that

$$E\int_{I\times\Sigma} |W(t,\xi) - W^n(t,\xi)|^2 d\xi dt = (T^2/2)\sum_{i\geq n+1}^\infty \lambda_i.$$

Since the covariance operator Q of the Brownian motion W is assumed to be nuclear, the above sum converges to zero as $n \to \infty$ which implies that $W^n \xrightarrow{s} W$ in $L_2(I, L_2(\Omega, H))$. Define

$$\sigma_n(t) \equiv \sigma(\rho^n(t)), t \in I.$$

By assumption, σ is uniformly bounded and hence the sequence

$$\{\sigma_n(\cdot)G^*(\eta(\cdot)v_i), n \in N\}$$

is contained in a bounded subset of $L_2(I, L_2(\Omega, H))$ and therefore has a weakly convergent subsequence, relabeled as the original sequence, giving

$$\sigma_n(\cdot)G^*(\eta(\cdot)v_i) \xrightarrow{w} \sigma_o(\cdot)G^*(\eta(\cdot)v_i), \text{ in } L_2(I, L_2(\Omega, H)) \tag{5.22}$$

for some σ_o which is uniformly bounded. Now multiplying the identity (5.16) by an arbitrary $\mathcal{F}$-measurable random variable $Z \in L_\infty(\Omega)$ and taking the (mathematical) expectation and letting $n \to \infty$, it follows from the convergence results (5.17)–(5.22) that

$$E\{Z(G_c^{1/2}\rho^o(t),G^{1/2}\eta(t)v_i)_H\} - E\{Z\int_0^t (G_c^{1/2}\rho^o(s),G^{1/2}\dot{\eta}(s)v_i)_H ds\}$$
$$+ E\{Z\int_0^t \langle\zeta,\eta v_i\rangle_{V^*,V}ds\} = E\{Z(G_c^{1/2}\rho_0,G^{1/2}\eta(0)v_i)_H\}$$
$$+ E\{Z\int_0^t (G^{1/2}f,G^{1/2}\eta(s)v_i)_H ds\} + E\{Z\int_0^t (\sigma_o(s)G^*(\eta(s)v_i),dW(s))\}. \quad (5.23)$$

To complete the proof, we must show that $\zeta = \Phi(\rho^o)$ and $\sigma_o = \sigma(\rho^o)$. Consider the first identity. Since Φ is monotone, we have

$$\int_0^t \langle\Phi(\rho) - \Phi(\rho^n), \rho - \rho^n\rangle_{V^*,V} ds \geq 0, \quad P\text{-a.s.} \quad (5.24)$$

for all $t \in I$ and for all $\rho \in L_p(I \times \Omega, V)$. Since (along a subsequence if necessary), ρ^n converges weakly in $L_p(I, L_p(\Omega, V))$, by Mazur's theorem, we can construct another sequence $\{\rho^{n_k}\}$ by a proper convex combination of the original sequence $\{\rho^n\}$ such that $\rho^{n_k} \xrightarrow{s} \rho^o$ in $L_p(I, L_p(\Omega, V))$. Letting $n \to \infty$ along the subsequence, it follows from (5.24) that

$$\int_0^t \langle\Phi(\rho) - \zeta, \rho - \rho^o\rangle_{V^*,V} ds \geq 0, \quad P\text{-a.s.}, \quad (5.25)$$

for all $\rho \in L_p(I, L_p(\Omega, V))$. Hence, taking $\rho = \rho^o + \varepsilon\varrho$ for any $\varrho \in L_p(I, L_p(\Omega, V))$ and $\varepsilon > 0$, it follows from the above inequality that

$$\int_0^t \langle\Phi(\rho^o + \varepsilon\varrho) - \zeta, \varrho\rangle_{V^*,V} ds \geq 0, \quad P\text{-a.s.}, \quad (5.26)$$

for all $\varepsilon > 0$ and $\varrho \in L_p(I, L_p(\Omega, V))$. Since Φ, as a real valued function, is continuous on the real line, it is easy to verify that Φ, considered as an operator on $L_p(I, L_p(\Omega, V))$, is semicontinuous. Thus by letting $\varepsilon \to 0$, it follows from the above inequality that for all $t \in I$,

$$\int_0^t \langle\Phi(\rho^o) - \zeta, \varrho\rangle_{V^*,V} ds \geq 0, \quad P\text{-a.s.}, \quad \forall\varrho \in L_p(I, L_p(\Omega, V)). \quad (5.27)$$

This is possible if and only if $\zeta = \Phi(\rho^o)$. This proves the first identity. For the second, we can follow similar procedure. Since σ is monotone nondecreasing, it is evident that

$$\int_{I\times\Sigma} [\sigma(\rho(t,\xi)) - \sigma(\varrho(t,\xi))](\rho(t,\xi) - \varrho(t,\xi)) d\xi dt \geq 0, \quad P\text{-a.s.}, \quad (5.28)$$

for all $\rho, \varrho \in L_p(I, L_p(\Omega, V)) \subset L_2(I, L_2(\Omega, H))$. Thus, along the same subsequence, as constructed by use of Mazur's theorem and relabeled as $\{\rho^m\} \subset L_p(I, L_p(\Omega, V)) \subset L_2(I, L_2(\Omega, H))$, we have

$$\int_{I\times\Sigma} [\sigma(\rho(t,\xi)) - \sigma(\rho^m(t,\xi))](\rho(t,\xi) - \rho^m(t,\xi)) d\xi dt \geq 0, \quad P\text{-a.s.} \quad (5.29)$$

Letting $m \to \infty$, it follows from this that

$$\int_{I\times\Sigma}[\sigma(\rho(t,\xi)) - \sigma_o](\rho(t,\xi) - \rho^o(t,\xi))d\xi dt \geq 0, \quad P\text{-a.s.} \tag{5.30}$$

This holds for all $\rho \in L_2(I, L_2(\Omega, H))$, and hence, by similar arguments as employed in the case of Φ, we arrive at the conclusion that $\sigma_o = \sigma(\rho^o)$, P-a.s. Using these results in equation (5.23) we obtain the following identity,

$$\begin{aligned} &E\{Z(G_c^{1/2}\rho^o(t), G^{1/2}\eta(t)v_i)_H\} - E\{Z\int_0^t (G_c^{1/2}\rho^o(s), G^{1/2}\dot{\eta}(s)v_i)_H ds\} \\ &\quad + E\{Z\int_0^t \langle\Phi(\rho^o), \eta v_i\rangle_{V^*,V} ds\} = E\{Z(G_c^{1/2}\rho_0, G^{1/2}\eta(0)v_i)_H\} \\ &\quad + E\{Z\int_0^t (G^{1/2}f, G^{1/2}\eta(s)v_i)_H ds\} + E\{Z\int_0^t (\sigma(\rho^o)G^*(\eta(s)v_i), dW(s))\}. \end{aligned} \tag{5.31}$$

This is true for all $Z \in L_\infty(\Omega)$, and hence, the following identity holds P-a.s.

$$\begin{aligned} &(G_c^{1/2}\rho^o(t), G^{1/2}\eta(t)v_i)_H - \int_0^t (G_c^{1/2}\rho^o(s), G^{1/2}\dot{\eta}(s)v_i)_H ds \\ &\quad + \int_0^t \langle\Phi(\rho^o), \eta v_i\rangle_{V^*,V} ds = (G_c^{1/2}\rho_0, G^{1/2}\eta(0)v_i)_H \\ &\quad + \int_0^t (G^{1/2}f, G^{1/2}\eta(s)v_i)_H ds + \int_0^t (\sigma(\rho^o)G^*(\eta(s)v_i), dW(s)) \end{aligned} \tag{5.32}$$

for all $t \in I$ and for every $\eta \in C^1(I)$ and every v_i. Since $\eta \in C^1(I)$ is arbitrary and $\{v_i\}$ is the basis of the triple, we conclude that

$$\begin{aligned} &(G_c^{1/2}\rho^o(t), G^{1/2}\varphi(t))_H - \int_0^t (G_c^{1/2}\rho^o(s), G^{1/2}\dot{\varphi}(s))_H ds \\ &\quad + \int_0^t \langle\Phi(\rho^o), \varphi\rangle_{V^*,V} ds = (G_c^{1/2}\rho_0, G^{1/2}\varphi(0))_H \\ &\quad + \int_0^t (G^{1/2}f, G^{1/2}\varphi(s))_H ds + \int_0^t (G^*(\varphi(s)), \sigma(\rho^o)dW(s)), \quad t \in I, \end{aligned} \tag{5.33}$$

for all $\varphi \in L_p(I, V)$ with $G^{1/2}\dot{\varphi} \in L_2(I, H)$. Thus, by Definition 5.3, ρ^o is a weak solution of the stochastic PDE (4.4). This completes the proof. □

Remark 5.6 (Uniqueness)**.** If both Φ and σ are strictly monotone, the weak solution of the system (4.4) is unique.

6. A CONTROL PROBLEM

In references [1,2], the problem of optimal management and control of underground resources, in particular aquifers, were considered proving existence of optimal policies and necessary conditions of optimality. Considering the model presented in

[2], and assuming that the positions of the wells are fixed, the stochastic model will take the form,

$$d(G(c\rho)) + \Phi(\rho)dt = Gfdt + GBudt + G\sigma(\rho)dW, \quad t \in I, \quad \rho(0) = \rho_0. \tag{6.1}$$

The operator $B : R^N \longrightarrow H$ and $u(t) \in U$ denotes the N-vector of extraction rates with $U \subset R^N_+$ being a compact set. The set of admissible controls is given by the set $\mathcal{U}_{ad} = L_\infty(I, U) \subset L_\infty(I, R^N)$. The cost functional is given by

$$J(u) = E \int_0^T \ell(t, c\rho(t), u(t))dt,$$

for a suitable function ℓ, [see 1,2]. For stochastic problems, such as this, it is essential to consider fully observed or partially observed feedback control. There are two possible techniques as described below.

1. Feedback Control Via HJB Equation: A standard technique for construction of optimal feedback controls is to use the HJB equation. Define the value function

$$V(t, cz) = \inf_{u \in U} E \int_t^T \ell(s, c\rho(s, z), u)ds, \quad t \in I, quadz \in H,$$

where $\rho = z$ at time t. Let DV denote the Frechet derivative of V on H. Using Bellman's optimality principle and the system (6.1), formally it is easy to derive the following HJB equation,

$$\begin{aligned} -\partial V/\partial t &= \inf_{u \in U}\{\ell(t, cz, u) + (B^* DV, u)\} \\ &\quad + (1/2)Tr(\sigma\sigma^* D^2 V) + (DV, \triangle\Phi + f), (t, z) \in I \times H \\ V(T, cz) &= 0, z \in H. \end{aligned} \tag{6.2}$$

This is a partial differential equation on an infinite dimensional Hilbert space H. The first hurdle is the question of existence of solutions (in some generalized sense such as viscosity sense). Given the existence, the second difficulty is in the determination (or computation) of V, and then constructing the control law from the expression: $u^o = arg\{\inf\{\ell(t, cz, u) + (B^* DV, u)\}\}$. All these questions are nontrivial, may be even formidable, and require extensive work.

2. Feedback Control from a Given Class: Another possibility is to select a feedback control law from a given class of feasible control laws. Clearly this will produce relatively (relative with respect to the class) optimal control laws. In other words, $u(t) = F(\rho(t))$ or $u(t) = F(K(\rho(t)))$ where F is a suitable feedback control law to be chosen and $K : H \longrightarrow H_0 \subset H$ is a given observation/measurement operator satisfying necessary regularity properties. The objective (cost) functional is given by

$$J(F) \equiv E \int_I \ell(t, \rho(t), F(\rho(t)))dt$$

in the fully observed case. In the case of partially observed problem, the feedback operator $F : H_0 \longrightarrow U$ and the cost functional is given by

$$J(F) \equiv E \int_I \ell(t, \rho(t), F(K(\rho(t))))dt.$$

Let $F_{ad} \subset C(H_0, U)$ denote the class of admissible feedback control laws furnished with the topology of point wise convergence. We assume that F_{ad} is compact with respect to this topology. Our problem is to find an element $F^o \in F_{ad}$ so that

$$J(F^o) \leq J(F), \ \forall \ F \in F_{ad}.$$

In order to prove the existence of such an optimal feedback control law, we need compactness as given in [7, Theorem 42.3] and continuity $F \longrightarrow \rho \longrightarrow J(F)$. This kind of compactness and continuity results have been used in deterministic problems [8, Theorem 4.3]. For the stochastic problem, this requires substantial work and we leave it as an open problem for future research.

Remark 6.1. The second method is relatively simpler. To construct the optimal control one must use necessary conditions of optimality and the associated numerical code. This is far less demanding than solving a nonlinear PDE (HJB equation) on infinite dimensional Hilbert space.

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