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SOME NEW RESULTS IN NON-ADDITIVE MEASURE THEORY

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Dedicated with great pleasure to Espedito De Pascale

ABSTRACT. We are concerned with a wide class of non-additive functions, namely quasi - triangular functions, defined on a Boolean ring and taking values into a topological space, where no algebraic structure is required. The aim of the paper is twofold. First we prove that in some sense this class is equivalent to that one of finitely additive functions valued into a topological Abelian group. Secondly we show that a Vitali-Hahn-Saks theorem holds for exhaustive elements of it.

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1. INTRODUCTION

Given a non-negative extended real-valued function φ acting on a Boolean ring $\mathcal{R}$, for any pair of disjoint elements a, b of $\mathcal{R}$ one can consider the values $\varphi(a)$, $\varphi(b)$ and $\varphi(a \vee b)$ and various possible relations between them.

Among all these ones, broadly speaking, here our interest relies on the following two different conditions:

i) if any two of the values $\varphi(a)$, $\varphi(b)$ and $\varphi(a \vee b)$ are 'small', then the remaining one has to be 'small';

ii) if $\varphi(b)$ is 'small', then $\varphi(a)$ and $\varphi(a \vee b)$ have to be 'near'.

Both *i)* as *ii)* are clearly verified whenever φ is a finitely additive function. In this paper, we point out that simply requiring φ to satisfy either *i)* or *ii)* leads to a natural generalization of finitely additive assumption, with the further advantage that no algebraic operation is needed. Our target space will be in fact an arbitrary

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Hausdorff topological space where no algebraic structure is required too. The class of functions we are concerned with, namely quasi-triangular functions, constitutes a generalization of classical finitely additive functions in which additivity assumption is removed and no monotonicity is imposed as well. Moreover, as it will be clarified in Sect. 2 through examples, such a class properly contains various families of non-additive functions deeply studied in literature and this motivates our investigation. As it is well-known the interest for a non-additive context goes back to G. Choquet [3] as well as to L. Shapley [18]. We draw the reader attention to [6], [16] for a comprehensive exposition of results and applications in measure theory and functional analysis, whereas to [7], [14] and the references therein for the economic applications. See also [15].

In a recent paper [1] we investigated quasi-triangular and exhaustive functions on a Boolean ring satisfying the Subsequential Completeness Property and we established a Cafiero theorem as well as a Brooks-Jewett theorem. In this paper we keep working on the same class and there are two main contributions. One is that every quasi-triangular function generates a Fréchet-Nikodým topology on the underlying ring (in Sect. 3, Theorem 3.2). This result allows us to state that, in same sense (specified by Definition 3.4 below), any quasi-triangular function is equivalent to a finitely additive one, which is defined on the same ring and attains values in some topological Abelian group. The second is a Vitali-Hahn-Saks theorem for quasi-triangular and exhaustive functions on a Boolean ring satisfying the Subsequential Completeness Property in Sect. 5. Our approach to its proof relies on an improvement of a result concerning finitely additive functions due to T. Traynor in [19], that is formulated below as Lemma 4.3. We begin by defining the basic concepts more precisely and fixing the main notations.

2. BASIC DEFINITIONS AND EXAMPLES

Throughout this note we always consider a Boolean ring $\mathcal{R}$, whose least point is denoted by $0_{\mathcal{R}}$ or 0, and a nonempty set $\mathcal{S}$, where a topology τ or a uniformity $\mathcal{U}$ is given. About $\mathcal{S}$, we assume that a point $e \in \mathcal{S}$ is arbitrary fixed and use the notation $\tau[e]$ to denote one of its fundamental system of τ-neighbourhoods. When the topology τ on $\mathcal{S}$ is generated by a uniformity $\mathcal{U}$ (briefly $\tau \equiv \tau_{\mathcal{U}}$), we shall write $\mathcal{U}[e]$ in place of $\tau[e]$.

Hereafter we deal with the family of all functions $\varphi : \mathcal{R} \to \mathcal{S}$ such that $\varphi(0) = e$, that we denote as $\mathcal{M}[e]$. Formalizing conditions $i)$ and $ii)$ above, when $\varphi \in \mathcal{M}[e]$, then *i)*: φ is said to be *quasi-triangular* if for every $U \in \tau[e]$ there exists $V = V(U) \in \tau[e]$ such that for any disjoint pair (a, b) of elements of $\mathcal{R}$ it holds

$$\varphi(a) \in V,\ \varphi(b) \in V \quad \Longrightarrow \quad \varphi(a \vee b) \in U;$$

$$\varphi(a) \in V,\ \varphi(a \vee b) \in V \quad \Longrightarrow \quad \varphi(b) \in U.$$

and, whenever $\tau \equiv \tau_{\mathcal{U}}$, following [12], we say that

ii): φ is said to be *s-outer* if for every $U \in \mathcal{U}$ there exists $V = V(U) \in \mathcal{U}$ such that for any disjoint pair (a, b) of elements of $\mathcal{R}$ it holds

$$\varphi(b) \in V[e] \quad \Longrightarrow \quad \Big(\varphi(a), \varphi(a \vee b)\Big) \in U.$$

Since for any given subcollection $\Phi := (\varphi_i)_{i\in I}$ of $\mathcal{M}[e]$, the uniform version of the above definitions can be easily written down, we leave it to the reader. In the following, we adopt the convention that $U^{(0)}$ denotes the intersection of the sets $U \cap V(U)$ appearing in the previous definitions and then $U^{(n)} := U^{(n-1)} \cap V(U^{(n-1)})$, $n \in \mathbb{N}$. Besides for any function $\varphi \in \mathcal{M}[e]$, the notation $\varphi([0, a])$, or simply $\tilde{\varphi}(a)$, represents the set $\{\varphi(b) : b \in \mathcal{R},\ b \le a\}$, for each $a \in \mathcal{R}$.

It is worth noting that condition $i)$ is sharper than $ii)$. In fact, for any disjoint elements $a, b \in \mathcal{R}$, condition $i)$ requires a control on $\varphi(a)$, $\varphi(b)$ and $\varphi(a \vee b)$ for 'small' values of φ, where 'small' means 'near enough' to $e = \varphi(0)$; whereas in $ii)$, apart from the uniformity structure needed on $\mathcal{S}$, one asks that $\varphi(a)$ and $\varphi(a \vee b)$ have to be 'near' whenever $\varphi(b)$ is 'small'. We show indeed

Proposition 2.1. *If $\varphi \in \mathcal{M}[e]$ is s-outer, then it is quasi-triangular.*

Proof. Given $U \in \mathcal{U}$, pick a symmetric $U_1 \in \mathcal{U}$ such that $U_1 \circ U_1 \subseteq U$ and put $W := U_1^{(0)}$. For every disjoint pair (a, b) of elements of $\mathcal{R}$, since φ is s-outer, one notes that if $\varphi(a) \in W[e]$ and $\varphi(b) \in W[e]$, then $\big(\varphi(a \vee b), e\big) \in U_1 \circ U_1 \subseteq U$, whereas if $\varphi(a) \in W[e]$ and $\varphi(a \vee b) \in W[e]$, then $\big(\varphi(b), e\big) \in U_1 \circ U_1 \subseteq U$; therefore φ is quasi-triangular. □

The converse fails, as shown by the following example.

Example 2.2. Let $\mathcal{L}$ denote the σ-algebra of the Lebesgue measurable subsets of $\mathbb{R}$ and λ is the Lebesgue measure on $\mathcal{L}$. The function φ defined by

$$\varphi(A) := \lambda(A) \quad \text{if} \quad \lambda(A) \le 1, \qquad\qquad \varphi(A) := 2 \quad \text{if} \quad \lambda(A) > 1,$$

is a non-negative increasing set function such that $\varphi(\emptyset) = 0$. Such φ is clearly quasi-triangular, but it is not s-outer. Indeed, for any $\delta \in]0, 1[$ and any disjoint sets $A, B \in \mathcal{L}$ such that $\lambda(A) = 1$ and $\lambda(B) < \delta$, one gets that

$$\varphi(B) < \delta \qquad\text{and}\qquad \varphi(A \cup B) - \varphi(A) = 1.$$

Now we emphasize through several examples that the family of quasi-triangular functions (as well as that of s-outer ones) under investigation includes some classes of non-additive functions widely studied in literature.

Example 2.3. A non-negative extended real-valued function φ defined on a Boolean ring $\mathcal{R}$ is said to be a *measuroid* (see, e.g., [17], [21]) whenever $\varphi(0_{\mathcal{R}}) = 0$ and for any disjoint pair (a, b) of elements of $\mathcal{R}$ it satisfies the following subadditive and quasi-triangular conditions

$$\varphi(a \vee b) \leq \varphi(a) + \varphi(b), \qquad \varphi(a) \leq \varphi(a \vee b) + \varphi(b).$$

Hence every measuroid is a quasi-triangular function. As shown in [21], under the convention that $\infty - \infty = 0$, the following assertions are equivalent:

- φ is a measuroid;
- $\forall a, b \in \mathcal{R},\ a \wedge b = 0_{\mathcal{R}},$ it holds: $|\varphi(a) - \varphi(b)| \leq \varphi(a \vee b) \leq \varphi(a) + \varphi(b)$;
- $\forall a, b \in \mathcal{R},\ a \wedge b = 0_{\mathcal{R}},$ it holds: $|\varphi(a \vee b) - \varphi(a)| \leq \varphi(b)$.

From the last condition one clearly infers that every measuroid is a s-outer function with respect to the usual uniformity in $[0, +\infty]$.

Example 2.4. A non-negative real-valued function φ defined on a Boolean ring $\mathcal{R}$ is said to be k-*triangular*, for $k \in [1, +\infty[$ (see, e.g., [9], [8], [16], [20]), whenever $\varphi(0_{\mathcal{R}}) = 0$ and

- ◦ $\forall a, b \in \mathcal{R},\ a \wedge b = 0_{\mathcal{R}},$ it holds: $\varphi(a) - k\varphi(b) \leq \varphi(a \vee b) \leq \varphi(a) + k\varphi(b)$.

One can easily check that every k-triangular function is s-outer.

Example 2.5. Let $\mathcal{P} : \mathcal{A} \to [0, 1]$ be a finitely additive probability defined on a Boolean algebra $\mathcal{A}$ and let $\gamma : [0, 1] \to [0, 1]$ be an increasing function such that $\gamma(0_{\mathcal{A}}) = 0$ and $\gamma(1_{\mathcal{A}}) = 1$. The composite function $\varphi := \gamma \circ \mathcal{P}$ is said to be a *distorted probability* and the function γ is called its *distortion* (see, e.g., [6], [16]). It is easy to verify that every distorted probability determined by a concave distortion is a measuroid, thus it is a s-outer function.

Example 2.6. Given an internal composition law $\oplus$ on $[0, 1]$, $\oplus$ is said to be a t-*conorm* (see, for instance, [23], [16]) whenever $\oplus$ is commutative, associative and satisfies the following condition

- if $x, y \in [0, 1],\ x \leq y,$ then $x \oplus z \leq y \oplus z$ for all $z \in [0, 1]$,
- $x \oplus 0 = x$ for all $x \in [0, 1]$,

i.e. $\oplus$ is monotone with neutral element 0.

The mappings $\oplus_1(x, y) := \max\{x, y\}$, $\oplus_2(x, y) := x + y - xy$, and

$$\oplus_3(x, y) := \begin{cases} \max\{x, y\} & \text{if } \min\{x, y\} = 0, \\ 1 & \text{otherwise} \end{cases}$$

are standard examples of t-conorms.

A function $\varphi : \mathcal{R} \to ([0,1], \oplus)$ is said to be $\oplus$-*additive* whenever $\varphi(0_\mathcal{R}) = 0$ and $\varphi(a \vee b) = \varphi(a) \oplus \varphi(b)$ for all $a, b \in \mathcal{R}$, $a \wedge b = 0_\mathcal{R}$. If the t-conorm $\oplus$ is continuous at the point $(0,0)$, then every $\oplus$-additive function φ is quasi-triangular. It is worth noting that the foregoing t-conorms $\oplus_1$ and $\oplus_2$ are both continuous at $(0,0)$, differently from $\oplus_3$.

We conclude with two explicit examples of s-outer functions which fail to be k-triangular. The first one is shown in [2].

Example 2.7. Let $\mathcal{S} := C^o([0,1])$ be the space of real-valued continuous functions on $[0,1]$ with the sup norm $||f|| := \sup|f(x)|$ and $e \equiv 0$. In $\mathcal{S}$ consider the binary operation $\oslash$ defined by $f \oslash g := f + g - f \cdot g$, for every $f, g \in \mathcal{S}$.

A function φ, defined on a Boolean ring $\mathcal{R}$ and valued into $\{f \in C^o([0,1]) : f([0,1]) \subseteq [0,1]\}$, is called $\oslash$-*additive* whenever $\varphi(a \vee b) = \varphi(a) \oslash \varphi(b)$ for any disjoint pair (a,b) of elements of $\mathcal{R}$. Because of the following estimate

$$||\varphi(a \vee b) - \varphi(a)|| \leq ||\varphi(b)|| \quad \text{for every}\, a, b \in \mathcal{R},\ a \wedge b = 0_\mathcal{R},$$

every $\oslash$-additive function is s-outer.

Example 2.8. Let $\mathcal{L}_o$ denote the σ-algebra of the Lebesgue measurable subsets of $[0,1]$ and λ is the Lebesgue measure on $\mathcal{L}_o$. The set function $\varphi : \mathcal{L}_o \to [0,1]$ defined by $\varphi(A) := \lambda^2(A)$, $A \in \mathcal{L}_o$, is a s-outer function which is not k-triangular.

Indeed the uniform continuity of $f(x) = x^2$ in $[0,1]$ easily implies that φ is s-outer. Assume by way of contradiction that φ is k-triangular, $k \geq 1$. Thus, for any disjoint $A, B \in \mathcal{L}_o$ with positive Lebesgue measure, the condition $\varphi(A \cup B) \leq \varphi(A) + k\varphi(B)$ implies that $\lambda(A) \leq \frac{k-1}{2}\lambda(B)$ as well as $k > 1$. Hence it must hold that $\lambda(B) \geq \frac{2}{k-1}\lambda(A)$, which clearly leads to a contradiction once fixed B of small measure. Therefore φ is not a k-triangular function.

3. QUASI-TRIANGULARITY AND FINITE ADDITIVITY

First we show that every quasi-triangular function φ induces a topology Γ_φ on the Boolean ring on which is defined, which is described in terms of basic neighbourhoods. This Γ_φ will be referred to as the φ-topology on $\mathcal{R}$.

Proposition 3.1. *If $\varphi \in \mathcal{M}[e]$ is quasi-triangular and if $\mathcal{B}[e]$ is a neighbourhood base at the point $e \in \mathcal{S}$, then the family*

$$\Gamma_\varphi(a) := \Big(\{x \in \mathcal{R} :\ \tilde{\varphi}(x \triangle a)\ \subseteq\ U\}\Big)_{U \in \mathcal{B}[e]}$$

is a neighbourhood base at $a \in \mathcal{R}$ for the φ-topology.

Proof. Certainly each $\mathcal{V} \in \Gamma_{\varphi}(a)$ contains a since each $U \in \mathcal{B}[e]$ contains $\varphi(0) = e$. If $\mathcal{V}_1$ and $\mathcal{V}_2$ belong to $\Gamma_{\varphi}(a)$, determined by some U_1 and $U_2 \in \mathcal{B}[e]$, so easily does $\mathcal{V}_3 := \{x \in \mathcal{R} : \tilde{\varphi}(x \triangle a) \subseteq U_3\}$, where $U_3 \in \mathcal{B}[e]$ and $U_3 \subseteq U_1 \cap U_2$, and moreover $\mathcal{V}_3 \subseteq \mathcal{V}_1 \cap \mathcal{V}_2$. Next let $U \in \mathcal{B}[e]$ and $\mathcal{V}$ be the element of $\Gamma_{\varphi}(a)$ determined by U. Taking into account the quasi-triangularity of φ, pick $U_1 \in \mathcal{B}[e]$ such that $U_1 \subseteq V(U)$. We claim that for every $b \in \mathcal{V}_1 := \{x \in \mathcal{R} : \tilde{\varphi}(x \triangle a) \subseteq U_1\}$ then there exists $\mathcal{W} \in \Gamma_{\varphi}(b)$ such that $\mathcal{W} \subseteq \mathcal{V}$. Indeed, put $\mathcal{W} := \{x \in \mathcal{R} : \tilde{\varphi}(x \triangle b) \subseteq U_1\} \subseteq \mathcal{V}_1 \cap \mathcal{V}_2$, then for every $y \in \mathcal{W}$, one observes that

$$\tilde{\varphi}(y\triangle a) = \tilde{\varphi}\big((y\triangle b)\triangle(b\triangle a)\big) = \big\{\varphi\big(\big(z\backslash(y\triangle b)\wedge(b\triangle a)\big)\vee\big(z\backslash(b\triangle a)\wedge(y\triangle b)\big)\big) : z \in \mathcal{R}\big\};$$

since for each $z \in \mathcal{R}$ one gets

$$\varphi\big(z \setminus (y \triangle b) \wedge (b \triangle a)\big) \in U_1 \quad \text{and} \quad \varphi\big(z \setminus (b \triangle a) \wedge (y \triangle b)\big) \in U_1,$$

the quasi-triangularity of φ assures that $\tilde{\varphi}(y \triangle a) \subseteq U$. Thus $\mathcal{W} \subseteq \mathcal{V}$. $\square$

Now we prove

Theorem 3.2. *If $\varphi \in \mathcal{M}[e]$ is quasi-triangular, then the φ-topology is a Fréchet-Nikodým topology on the Abelian group $(\mathcal{R}, \triangle)$.*

Proof. Since $a \triangle a = 0$, for every $a \in \mathcal{R}$, in order to check that $(\mathcal{R}, \triangle, \Gamma_{\varphi})$ is a topological group it suffices to prove that the symmetric difference operation $\triangle$ is continuous. Let $a_o, b_o \in \mathcal{R}$ and $\mathcal{V} \in \Gamma_{\varphi}(a_o\triangle b_o)$. Since $\mathcal{V} := \{x \in \mathcal{R} : \tilde{\varphi}(x\triangle(a_o\triangle b_o) \subseteq U\}$, for some $U \in \mathcal{B}[e]$, the quasi-triangularity of φ yields that there exists some $U_1 \in \mathcal{B}[e]$ such that $U_1 \subseteq V(U)$. Clearly

$$\mathcal{V}_{a_o} := \{x \in \mathcal{R} : \tilde{\varphi}(x\triangle a_o) \subseteq U_1\} \in \Gamma_{\varphi}(a_o), \;\; \mathcal{V}_{b_o} := \{x \in \mathcal{R} : \tilde{\varphi}(x\triangle b_o) \subseteq U_1\} \in \Gamma_{\varphi}(b_o).$$

Moreover, by the quasi-triangularity of φ, using the same trick as in the proof of the previous proposition, one easily gets that $a \triangle b \in \mathcal{V}$, for every $a \in \mathcal{V}_{a_o}$ and every $b \in \mathcal{V}_{b_o}$, that is the desired continuity of the symmetric difference operation $\triangle$.

To end the proof, it remains to verify that the function $\pi_b : a \in \mathcal{R} \mapsto a \wedge b \in \mathcal{R}$ is Γ_{φ}- continuous, uniformly respect to $b \in \mathcal{R}$. To this end, let $a_o \in \mathcal{R}$ and $\mathcal{V} \in \Gamma_{\varphi}(a_o \wedge b)$. Since $\mathcal{V} := \{x \in \mathcal{R} : \tilde{\varphi}(x \triangle (a_o \wedge b)) \subseteq U\}$, for some $U \in \mathcal{B}[e]$, and $(a \wedge b) \triangle (a_o \wedge b) = (a \triangle a_o) \wedge b$, one clearly has

$$\tilde{\varphi}((a \wedge b) \triangle (a_o \wedge b)) = \tilde{\varphi}\big((a \triangle a_o) \wedge b\big) \subseteq \tilde{\varphi}(a \triangle a_o) \subseteq U \qquad \forall a \in \mathcal{V},$$

uniformly respect to $b \in \mathcal{R}$. $\square$

Remark 3.3. It is well-known (see, for instance, [5], [22]) that any Fréchet-Nikodým topology Γ_{FN} on a Boolean ring is the μ-topology on $(\mathcal{R}, \triangle)$ for some group-valued finitely additive function μ acting on $\mathcal{R}$, *i.e.* $\Gamma_{FN} \equiv \Gamma_{\mu}$. In fact, once put $\mathcal{G} := (\mathcal{R}, \triangle, \Gamma_{FN})$ and denoted as μ the identity map on $\mathcal{R}$, then Γ_{FN} coincides with Γ_{μ}.

Mymic [11], [22], we can formulate the following

Definition 3.4. *Let $(\mathcal{S}_o, \tau_o)$ be a topological space and $e_o \in \mathcal{S}_o$ be arbitrary fixed. If $\psi \in \mathcal{M}[e]$ and $\nu \in \mathcal{M}[e_o]$, then ψ is said to be ν-*continuous *($\psi << \nu$, for short) if, and only if, for every $U \in \tau[e]$ there exists some $V \in \tau_o[e_o]$ such that $\nu([0,a]) \subseteq V$ implies that $\psi([0,a]) \subseteq U$. Moreover, ψ is said to be* equivalent to ν, *in symbols $\psi \asymp \nu$, whenever both $\psi << \nu$ and $\nu << \psi$.*

The results of this section can be therefore summed up as

Corollary 3.5. *If $\varphi \in \mathcal{M}[e]$ is quasi-triangular, then there exists a finitely additive function μ, acting on the same Boolean ring $\mathcal{R}$ and valued in some topological Abelian group $(\mathcal{G}_o, \tau_o)$, with $\mu(0) = e_o$, which is equivalent to φ, i.e. $\mu \asymp \varphi$.*

4. AUXILIARY RESULTS

In this section we show some properties fulfilled by families of uniformly quasi-triangular and uniformly exhaustive functions. We recall that a function $\varphi \in \mathcal{M}[e]$ is said to be *e-exhaustive* (exhaustive, for short) whenever $\lim_k \varphi(a_k) = e$ for any disjoint sequence $(a_k)_{k\in\mathbb{N}}$ in $\mathcal{R}$, then for every subfamily Φ of $\mathcal{M}[e]$ the definition of *uniform exhaustivity* can be immediately expressed.

The first lemma exhibits a good behaviour on finite disjoint union of quasi-triangular functions. The proof can be easily determined by an inductive argument.

Lemma 4.1. *Let $a \in \mathcal{R}$ be the supremum of a finite disjoint subset $\{a_1, \ldots, a_k\}$ of $\mathcal{R}$ and $U \in \tau[e]$. If φ is quasi-triangular, then*

$$\varphi(a_i) \in U^{(i)} \qquad \forall i \in \{1, \ldots, k\} \qquad \Longrightarrow \qquad \varphi(a) \in U. \tag{1}$$

Moreover, if $\Phi \subseteq \mathcal{M}[e]$ is uniformly quasi-triangular, then (1) holds uniformly respect to $\varphi \in \Phi$.

The following result describes the action of uniformly exhaustive functions on increasing sequences in $\mathcal{R}$.

Lemma 4.2. *Let $(b_k)_{k\in\mathbb{N}}$ be an increasing sequence in $\mathcal{R}$. If $\Phi \subseteq \mathcal{M}[e]$ is uniformly exhaustive, then for every $U \in \tau[e]$ there exists some $l \in \mathbb{N}$ such that*

$$\varphi([0, b_k \setminus b_l]) \subseteq U \qquad \forall k \geq l, \ \forall \varphi \in \Phi. \tag{2}$$

Proof. Assume the contrary. Then there exist some $U_o \in \tau[e]$, a strictly increasing sequence of index $(k_l)_{l\in\mathbb{N}}$, two sequences $(\varphi_l)_{l\in\mathbb{N}}$ and $(y_l)_{l\in\mathbb{N}}$, in Φ and $\mathcal{R}$ respectively, such that

$$\varphi_l\big(y_l \wedge (b_{k_l} \setminus b_{k_{l-1}})\big) \notin U_o \qquad \forall l \in \mathbb{N}, \tag{3}$$

where $b_{k_0} := b_1$. Since the sequence $\big(y_l \wedge (b_{k_l} \setminus b_{k_{l-1}})\big)_{l\in\mathbb{N}}$ is disjoint in $\mathcal{R}$, the uniform exhaustivity of Φ contradicts (3) and this concludes the proof. $\square$

We are now in position to state the following extension to quasi-triangular functions of a result concerning finitely additive functions due to T. Traynor in [19]. We refer the reader also to (2.2) in Chap. IV of [5]. This result is one of the main ingredient in the proof of the Vitali-Hahn-Saks theorem in next section.

Lemma 4.3. *If $\Phi \subseteq \mathcal{M}[e]$ is uniformly exhaustive and uniformly quasi-triangular, then for every $U \in \tau[e]$ there exist some $W \in \tau[e]$ and a finite subset $\Phi_f \subseteq \Phi$ such that for any $a \in \mathcal{R}$*

$$\psi([0,a]) \;\subseteq\; W \qquad \forall \psi \in \Phi_f \qquad \Longrightarrow \qquad \varphi(a) \;\in\; U \qquad \forall \varphi \in \Phi. \tag{4}$$

Proof. Arguing by contradiction, then there exists some $U_o \in \tau[e]$ such that for every $W \in \tau[e]$ and every finite $\Phi_f \subseteq \Phi$ there are an element $a_o \in \mathcal{R}$ and a function $\varphi \in \Phi$ verifying

$$\psi([0,a_o]) \;\subseteq\; W \quad \forall \psi \in \Phi_f, \qquad \varphi(a_o) \;\notin\; U_o. \tag{5}$$

Because Φ is uniformly quasi-triangular, starting from $U_o \in \tau[e]$, one can consider the decreasing sequence $(U_o^{(k)})_{k\in\mathbb{N}_o}$ in $\tau[e]$ and then, by induction in (5), determine two sequences $(a_k)_{k\in\mathbb{N}}$ in $\mathcal{R}$ and $(\psi_k)_{k\in\mathbb{N}}$ in Φ such that for every $k \in \mathbb{N}$ it holds

$$\psi_i([0,a_k]) \;\subseteq\; U_o^{(k+1)} \qquad \forall i \in \{1,\ldots,k\}, \qquad \psi_{k+1}(a_k) \;\notin\; U_o. \tag{6}$$

Now fix $h \in \mathbb{N}$, and define $b_k := \vee_{j=h+1}^{k} a_j$, $k > h$. Since $(b_k)_{k\in\mathbb{N}}$ is increasing and $a_k \le b_k$, Lemma 4.2 guarantees the existence of an index $l \in \mathbb{N}$ with $l \ge h+1$, such that

$$\varphi([0,a_k \setminus b_l]) \;=\; \tilde{\varphi}(a_k \setminus \vee_{j=h+1}^{l} a_j) \;\subseteq\; U_o^{(h+1)} \qquad \forall k \ge l,\ \forall \varphi \in \Phi. \tag{7}$$

Therefore, going on by induction, one can determine a strictly increasing sequence of index $(l_n)_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$ one has that

$$\tilde{\varphi}(a_k \setminus \vee_{j=l_n+1}^{l_{n+1}} a_j) \;\subseteq\; U_o^{(l_n+1)} \qquad \forall k \ge l_{n+1},\ \forall \varphi \in \Phi. \tag{8}$$

We claim that there exists an index $n_o \in \mathbb{N}$ such that $\psi_{l_{n+1}}(a_{l_n}) \;\in\; U_o$, $\forall n \ge n_o$.

To prove this, for every $n \in \mathbb{N}$, write a_{l_n} as disjoint union of $c_n := a_{l_n} \wedge \big(\vee_{j=l_n+1}^{l_{n+1}} a_j\big)$ and $d_n := a_{l_n} \setminus \vee_{j=l_n+1}^{l_{n+1}} a_j$; moreover define $d_n' := d_n \setminus \vee_{m=1}^{n-1} d_m$ and $d_n'' := d_n \wedge (\vee_{m=1}^{n-1} d_m)$. Therefore each a_{l_n} is the pairwise disjoint union of c_n, d_n' and d_n'', that is

$$a_{l_n} = c_n \vee d_n' \vee d_n'' \qquad \forall n \in \mathbb{N}. \tag{9}$$

Since

$$c_n \;:=\; a_{l_n} \wedge \big(\vee_{j=l_n+1}^{l_{n+1}}\ a_j\big) \;=\; \vee_{j=l_n+1}^{l_{n+1}} \big(y_j \wedge a_j\big) \qquad \forall n \in \mathbb{N}, \tag{10}$$

where y_j's are suitable elements in $\mathcal{R}$ such that the $y_j \wedge a_j$'s are mutually disjoint, from (6) one deduces that

$$\psi_{l_n+1}([0,y_j \wedge a_j]) \;\subseteq\; U_o^{(j+1)} \qquad \forall j \in \{l_n+1,\ldots,l_{n+1}\},$$

thus, Lemma 4.1 and (10) imply that

$$\psi_{l_n+1}([0,c_n]) \subseteq U_o^{(l_n+1)} \subseteq U_o^{(2)} \qquad \forall n \in \mathbb{N}. \tag{11}$$

Corresponding to the neighbourhood $U_o^{(2)}$, since $(d'_n)_{n\in\mathbb{N}}$ is disjoint as well as Φ is uniformly exhaustive, then there exists an index $n_o \in \mathbb{N}$ such that

$$\varphi(d'_n) \in U_o^{(2)} \qquad \forall n \geq n_o,\ \forall \varphi \in \Phi. \tag{12}$$

Accordingly to (11) and (12), as $\{\psi_{l_n} : n \in \mathbb{N}\} \subseteq \Phi$ is uniformly quasi-triangular and $c_n \wedge d'_n = 0$, one obtains that

$$\psi_{l_n+1}(c_n \vee d'_n) \in U_o^{(1)} \qquad \forall n \geq n_o. \tag{13}$$

Let us now observe that

$$d''_n := d_n \wedge (\vee_{m=1}^{n-1} d_m) = \vee_{m=1}^{n-1}\Big(\big(d_n \setminus \vee_{s=1}^{m-1} d_s\big) \wedge d_m\Big) \qquad \forall n \in \mathbb{N}; \tag{14}$$

moreover for each $m \in \{1,\dots,n-1\}$ one also has

$$\big(d_n \setminus \vee_{s=1}^{m-1} d_s\big) \wedge d_m = \big(a_{l_n} \setminus \vee_{j=l_m+1}^{l_{m+1}} a_j\big) \wedge \big(a_{l_m} \setminus \vee_{s=1}^{m-1} d_s\big) \wedge \big(\setminus \vee_{j=l_n+1}^{l_{n+1}} a_j\big) \tag{15}$$

as well as $l_m \leq l_n$. But, from (8),

$$\tilde{\varphi}(a_{l_n} \setminus \vee_{j=l_m+1}^{l_{m+1}} a_j) \subseteq U_o^{(l_m+1)} \subseteq U_o^{(m+1)} \qquad \forall m \in \{1,\dots,n-1\},\ \forall \varphi \in \Phi,$$

hence (15) yields in particular that

$$\varphi\Big(\big(d_n \setminus \vee_{s=1}^{m-1} d_s\big) \wedge d_m\Big) \in U_o^{(m+1)} \qquad \forall m \in \{1,\dots,n-1\},\ \forall \varphi \in \Phi.$$

Then, from Lemma 4.1 and (14) one infers that

$$\varphi(d''_n) \in U_o^{(1)} \qquad \forall n \in \mathbb{N},\ \forall \varphi \in \Phi. \tag{16}$$

Thus, by (13),(16) and (9), the uniform quasi-triangularity of Φ assures that

$$\psi_{l_n+1}(c_n \vee d'_n \vee d''_n) = \psi_{l_n+1}(a_n) \in U_o \qquad \forall n \geq n_o.$$

This proves the claim and obviously contradicts (6). This contradiction completes the proof. □

5. CONVERGENCE THEOREMS

In this section we always assume that $(\mathcal{S},\tau)$ is a Hausdorff topological space and the ring $\mathcal{R}$ verifies the *Subsequential Completeness Property* (briefly *SCP*), *i.e.* for every disjoint sequence $(a_k)_{k\in\mathbb{N}}$ in $\mathcal{R}$ then there exists an infinite subset M of $\mathbb{N}$ such that the supremum of the set $\{a_k : k \in M\}$ exists. We refer the reader to [4], [10], [21] for more details.

In a previous paper [1] we have proved the following two results

Theorem 5.1 (Cafiero). *If $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of exhaustive and uniformly quasi-triangular elements of $\mathcal{M}[e]$, then $(\varphi_n)_{n\in\mathbb{N}}$ is uniformly exhaustive if, and only if, the following condition holds*

($\star$): *for every $U \in \tau[e]$ and every disjoint sequence $(a_k)_{k\in\mathbb{N}}$ in $\mathcal{R}$ there exist some $k_o, n_o \in \mathbb{N}$ such that $\varphi_n(a_{k_o}) \in U$ for every $n \geq n_o$.*

Theorem 5.2 (Brooks-Jewett). *Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of exhaustive and uniformly quasi-triangular elements of $\mathcal{M}[e]$. If $(\varphi_n)_{n\in\mathbb{N}}$ converges pointwise in $\mathcal{R}$ to a exhaustive element φ of $\mathcal{M}[e]$, i.e.*

$$\lim_{n\to+\infty} \varphi_n(a) = \varphi(a), \qquad a \in \mathcal{R}, \tag{17}$$

then $(\varphi_n)_{n\in\mathbb{N}}$ is uniformly exhaustive.

Here, combining the previous theorem with Lemma 4.3, we are able to state the following

Theorem 5.3 (Vitali-Hahn-Saks). *Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of exhaustive and uniformly quasi-triangular elements of $\mathcal{M}[e]$ converging pointwise in $\mathcal{R}$ to a exhaustive element φ of $\mathcal{M}[e]$. If every φ_n is ν-continuous, where ν belongs to $\mathcal{M}[e_o]$ and e_o is an arbitrary fixed point in some topological space $(\mathcal{S}_o, \tau_o)$, then the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is ν-equicontinuous.*

Proof. First, by Theorem 5.2, $\Phi := \{\varphi_n : n \in \mathbb{N}\}$ is uniformly exhaustive and uniformly quasi-triangular. Next, let $U \in \tau[e]$ be given. Then Lemma 4.3 assures that there are some $W \in \tau[e]$ and a finite subset $\Phi_f \subseteq \Phi$ such that for any $a \in \mathcal{R}$ condition (4) holds, *i.e.*

$$\psi([0,\, a]) \;\subseteq\; W \qquad \forall\psi \in \Phi_f \qquad \Longrightarrow \qquad \varphi(a) \;\in\; U \qquad \forall\varphi \in \Phi.$$

Since every $\psi \in \Phi_f$ is ν-continuous, then corresponding to $W \in \tau[e]$ there exists some $V \in \tau_o[e_o]$ such that

$$\nu([0,\, a]) \;\subseteq\; V \qquad \Longrightarrow \qquad \psi([0,\, a]) \;\subseteq\; W \qquad \forall\psi \in \Phi_f.$$

Hence the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is ν-equicontinuous and the proof is complete. □

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