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REMARKS ON GENERAL INFINITE DIMENSIONAL DUALITY WITH CONE AND EQUALITY CONSTRAINTS

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Dedicated to Professor Espedito De Pascale
on the occasion of his retirement.

ABSTRACT. We prove a strong duality result between a convex optimization problem with both cone and equality constraints and its Lagrange dual formulation, provided that a constraint qualification condition related to the notion of quasi-relative interior holds true. In such a way we overcome the difficulty that the interior of the set involved in the regularity condition is empty.

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1. INTRODUCTION

In the papers [7], [8] and [19] the authors present an infinite dimensional duality theory which guarantees the existence of strong duality between a convex optimization problem with cone constraints and its Lagrange dual formulation, provided that a generalized constraint qualification assumption called *Assumption S* (see [7], [8]) or a normal cone condition called *Assumption N* (see [19]) holds true. Both *Assumption S* and *Assumption N* are related to the notion of quasi-relative interior and the use of such notion allows to overcome the difficulty that the interior of the set involved in the regularity condition is empty (see [16] and [19] for more details).

The aim of this paper is to study the existence of strong duality between a convex optimization problem with both cone and equality constraints and its Lagrange dual. In particular, given $f : S \to \mathbb{R}$, $g : S \to Y$, $h : S \to Z$, where S is a convex subset

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of a real linear topological space X, Y is a real normed space ordered by a convex cone C, Z is a real normed space and h is an affine-linear mapping, we consider the optimization problem:

$$f(x_0) = \min_{x \in \mathbb{K}} f(x), \qquad \text{Problem 1}$$

with

$$\mathbb{K} = \{x \in S : g(x) \in -C,\ h(x) = \theta_Z\}$$

and the dual problem:

$$\max_{\substack{u \in C^* \\ v \in Z^*}} \inf_{x \in S} [f(x) + \langle u, g(x) \rangle + \langle v, h(x) \rangle], \qquad \text{Problem 2}$$

where

$$C^* = \{u \in Y^* : \langle u, y \rangle \geq 0,\ \forall y \in C\}.$$

Then, using a nice suggestion of C. Zalinescu, in Section 3, we show the following result which is an improvement of the analogous Theorem 3.1 in [7]. We refer to Section 2 for the meaning of the concepts used here.

Theorem 1.1. *Assume that $f : S \to \mathbb{R}$, $g : S \to Y$ are convex functions and that $h : S \to Z$ is an affine-linear mapping. Assume that the following Assumption S is fulfilled at the minimal solution $x_0 \in \mathbb{K}$ to Problem 1, namely*

$$T_{\widetilde{M}}(0, \theta_Y, \theta_Z) \cap]-\infty, 0[\times \{\theta_Y\} \times \{\theta_Z\} = \emptyset, \qquad \textit{Assumption S}$$

where

$$\widetilde{M} = \{(f(x) - f(x_0) + \alpha, g(x) + y, h(x)) :\ x \in S \setminus \mathbb{K},\ \alpha \geq 0,\ y \in C\}$$

and $T_{\widetilde{M}}(0, \theta_Y, \theta_Z)$ is the tangent cone to $\widetilde{M}$ at $(0, \theta_Y, \theta_Z)$. Then also Problem 2 is solvable and if $\bar{u} \in C^*$, $\bar{v} \in Z^*$ are the extremal points to Problem 2, we have:

$$\langle \bar{u}, g(x_0) \rangle = 0$$

and the extrema of the two problems are equal.

This result improves Theorem 3.1 in [7] because the additional assumptions qri $C \neq \emptyset$, $\mathrm{cl}\,(C - C) = Y$, $\mathrm{cl}\, h(S - S) = Z$ and there exists $\hat{x} \in S$ with $g(\hat{x}) \in$ - qri C and $h(\hat{x}) = \theta_Z$ are removed.

Next, in Section 4, we provide a sufficient condition for the existence of the strong duality based on the notion of normal cone. Precisely, we prove the following result.

Theorem 1.2. *Let $f : S \to \mathbb{R}$, $g : S \to Y$, $h : S \to Z$ be three functions such that the following assumption holds:*

$$\exists\, \bar{x} \in \mathbb{K},\ \exists\, (\hat{\xi}, \hat{y}^*, \hat{z}^*) \in N_M(0, \theta_Y, \theta_Z) \text{ such that}$$

$$\hat{\xi}(f(\bar{x}) - f(x_0)) + \langle \hat{y}^*, g(\bar{x}) \rangle + \langle \hat{z}^*, h(\bar{x}) \rangle < 0 \qquad \textit{Assumption N}$$

where

$$M = \{(f(x) - f(x_0) + \alpha, g(x) + y, h(x)),\ x \in S,\ \alpha \geq 0,\ y \in C\},$$

$N_M(0, \theta_Y, \theta_Z)$ is the normal cone to M at the point $(0, \theta_Y, \theta_Z)$ and $x_0 \in \mathbb{K}$ is the minimal solution to Problem 1. Then Problem 2 is solvable and the extremal values of both problems are equal.

Assumption N is strictly connected with the qri M notion. In fact, if *Assumption N* holds, then, if M is convex, namely for instance if f and g are convex functions and h is an affine-linear function, $N_M(0, \theta_Y, \theta_Z)$ cannot be a linear subspace and so (see Proposition 2.2) also the tangent cone $T_M(0, \theta_Y, \theta_Z)$ is not a linear subspace, hence $(0, \theta_Y, \theta_Z) \notin$ qri M. Vice versa, if $(0, \theta_Y, \theta_Z) \notin$ qri M, $N_M(0, \theta_Y, \theta_Z)$ cannot be a linear subspace and there exists $(\hat{\xi}, \hat{y}^*, \hat{z}^*) \neq (0, \theta_{Y^*}, \theta_{Z^*})$ such that $(\hat{\xi}, \hat{y}^*, \hat{z}^*) \in N_M(0, \theta_Y, \theta_Z)$.

The paper is organized as follows. In Section 2 we recall some useful definitions and theorems. In Section 3 we prove Theorem 1.1 and in Section 4 Theorem 1.2. Finally, Section 5 is devoted to applications of the strong duality results.

2. PRELIMINARY CONCEPTS AND RESULTS

Let X denote, if not otherwise specified, a real normed space and X^* the topological dual of all continuous linear functionals on X, and let C be a subset of X. Given a point $x \in X$, the set

$$T_C(x) = \Big\{h \in X : h = \lim_{n \to \infty} \lambda_n(x_n - x),\ \lambda_n \in \mathbb{R} \text{ and } \lambda_n > 0\ \forall n \in \mathbb{N},$$
$$x_n \in C,\ \forall n \in \mathbb{N} \text{ and } \lim_n x_n = x\Big\}$$

is called the *tangent cone* to C at x. Of course, if $T_C(x) \neq \emptyset$, then $x \in$ cl C. If $x \in$ cl C and C is convex, then we have:

$$T_C(x) = \text{ cl cone } (C - \{x\}),$$

where

$$\text{cone } (C) = \{\lambda x : x \in C,\ \lambda \in \mathbb{R},\ \lambda \geq 0\}.$$

Following Borwein and Lewis (see [2]), we give the following definition of quasi-relative interior for a convex set.

Definition 2.1. Let C be a convex subset of X. The quasi-relative interior of C, denoted by qri C, is the set of those $x \in C$ for which $T_C(x)$ is a linear subspace of X.

If we define the normal cone to C at x by

$$N_C(x) = \{\xi \in X^* : \langle \xi, y - x \rangle \leq 0,\ \forall y \in C\},$$

the following result holds true.

Proposition 2.2. *Let C be a convex subset of X and $x \in C$. Then $x \in \text{qri } C$ if and only if $N_C(x)$ is a linear subspace of X^*.*

By the way, we report the following separation theorem based on the notion of qri C.

Theorem 2.3 (Separation Theorem)**.** *Let C be a convex subset of X and $x_0 \in C\setminus$ qri C. Then there exists $\xi \neq \theta_{X^*}$ such that*

$$\langle \xi, x \rangle \leq \langle \xi, x_0 \rangle \quad \forall x \in C.$$

Vice versa, let us suppose that there exists $\xi \neq \theta_{X^*}$ such that

$$\langle \xi, x \rangle \leq \langle \xi, x_0 \rangle \quad \forall x \in C$$

and that

$$cl\, (T_C(x_0) - T_C(x_0)) = X.$$

Then $x_0 \notin \text{qri } C$.

3. PROOF OF THEOREM 1.1

For the reader's convenience, we report the first part of the proof of Theorem 3.1 in [7], which remains unchanged.

Let us recall that

$$\begin{aligned} M &= \{(f(x) - f(x_0) + \alpha, g(x) + y, h(x)) : x \in S,\ \alpha \geq 0,\ y \in C\} \\ &= (f, g, h)(S) + \mathbb{R}_+ \times C \times \theta_Z. \end{aligned}$$

The set M is convex in virtue of convexity assumptions on the functions f and g and of the affine–linearity of h.

Since Problem 1 is solvable, there exists $x_0 \in \mathbb{K}$ such that

$$f(x_0) \leq f(x) \quad \forall x \in \mathbb{K},$$

namely

$$f(x_0) \leq f(x) \quad \forall x \in S:$$
$$-g(x) \in C,\ h(x) = \theta_Z, \text{ and } -g(x_0) \in C,\ h(x_0) = \theta_Z.$$

We wish to prove that $T_M(0, \theta_Y, \theta_Z)$ is not a linear subspace of $\mathbb{R} \times Y \times Z$. In fact, let $y \in T_M(0, \theta_Y, \theta_Z)$, namely

$$y = \lim_{n\to\infty} \lambda_n \left[(f(x_n) - f(x_0) + \alpha_n, g(x_n) + y_n, h(x_n)) - (0, \theta_Y, \theta_Z)\right]$$

with $\lambda_n > 0$, $\lim\limits_{n\to\infty} (f(x_n) - f(x_0) + \alpha_n) = 0$, $\lim\limits_{n\to\infty} (g(x_n) + y_n) = \theta_Y$, $\lim\limits_{n\to\infty} h(x_n) = \theta_Z$, $\alpha_n \geq 0$, $y_n \in C$, $x_n \in S$.

Four different situations can occur:

1. If $x_n \in \mathbb{K}$, $\forall n \in \mathbb{N}$, then $g(x_n) \in -C$ and $h(x_n) = \theta_Z$; hence, $f(x_n) \geq f(x_0)$ and $f(x_n) + \alpha_n \geq f(x_0)$ (since $\alpha_n \geq 0$); moreover, $\lambda_n > 0$ and so $\lambda_n[(f(x_n) + \alpha_n) - f(x_0)] \geq 0$ which implies $\lim\limits_{n\to\infty} [\lambda_n (f(x_n) + \alpha_n - f(x_0))] \geq 0$. As a consequence, the first component of y is greater than or equal to zero.
2. If $x_n \in S \setminus \mathbb{K}$, $\forall n \in \mathbb{N}$, then $y \in T_{\widetilde{M}}(0, \theta_Y, \theta_Z)$ where

$$\begin{aligned} &T_{\widetilde{M}}(0, \theta_Y, \theta_Z) \\ &= \big\{ y = \lim_{n\to\infty} \lambda_n \left[(f(x_n) - f(x_0) + \alpha_n, g(x_n) + y_n, h(x_n)) - (0, \theta_Y, \theta_Z) \right] \\ &\quad = \lim_{n\to\infty} \lambda_n \left(f(x_n) + \alpha_n - f(x_0), g(x_n) + y_n, h(x_n) \right) : \\ &\quad x_n \in S \setminus \mathbb{K}, \lim_{n\to\infty} \left(f(x_n) + \alpha_n - f(x_0) \right) = 0, \\ &\quad \lim_{n\to\infty} (g(x_n) + y_n) = \theta_Y, \ \lim_{n\to\infty} h(x_n) = \theta_Z \big\}, \end{aligned}$$

and from *Assumption S* it results to be $T_{\widetilde{M}}(0, \theta_Y, \theta_Z) \cap \{\mathbb{R}^-, \theta_Y, \theta_Z\} = \emptyset$. Hence, y cannot be a point of the type (l, θ_Y, θ_Z) with $l < 0$.
3. If $x_n \in S \setminus \mathbb{K}$ for a finite number of indexes n, then the sequence $\{x_n\}$ definitely belongs to $\mathbb{K}$ and the conclusion of point 1. holds.
4. If $x_n \in S \setminus \mathbb{K}$ for an infinite number of indexes n, then we can consider a subsequence $x_{z_n} \in S \setminus \mathbb{K}$ and we come back to point 2.

Therefore, we obtain that if $(l, \theta_Y, \theta_Z) \in T_M(0, \theta_Y, \theta_Z)$ then l must be nonnegative. In particular,

$$(-1, \theta_Y, \theta_Z) \notin T_M(0, \theta_Y, \theta_Z) = \text{ cl cone } (M - (0, \theta_Y, \theta_Z)).$$

Since this set is a closed convex cone (because M is convex), in virtue of a well known separation theorem, there exists $(\mu, y^*, z^*) \in \mathbb{R} \times Y^* \times Z^*$ such that

$$\begin{aligned} -\mu < 0 \leq \mu(f(x) + \alpha - f(x_0)) + \langle y^*, g(x) + y \rangle + \langle z^*, h(x) \rangle \\ \forall x \in S, \ \forall \alpha \geq 0, \ \forall y \in C. \end{aligned} \tag{3.1}$$

Setting $\dfrac{y^*}{\mu} = \bar{u}$, $\dfrac{z^*}{\mu} = \bar{v}$ and assuming $\alpha = 0$, $x = x_0$, we get

$$\langle \bar{u}, g(x_0) + y \rangle \geq 0 \quad \forall y \in C,$$

and hence, assuming $y = z - g(x_0) \in C$ $\forall z \in C$, since C is a convex cone, we have:

$$\langle \bar{u}, z \rangle \geq 0, \quad \forall z \in C,$$

namely $\bar{u} \in C^*$. Moreover, from (3.1), assuming $\alpha = 0$ and $y = 0$, we get

$$f(x_0) \leq f(x) + \langle \bar{u}, g(x) \rangle + \langle \bar{v}, h(x) \rangle \quad \forall x \in S. \tag{3.2}$$

Choosing in (3.2) $x = x_0$, we obtain $\langle \bar{u}, g(x_0) \rangle \geq 0$ and, since $-g(x_0) \in C$, $\langle \bar{u}, g(x_0) \rangle \leq 0$ and so $\langle \bar{u}, g(x_0) \rangle = 0$.

Then we get:

$$f(x_0) + \langle \bar{u}, g(x_0)\rangle + \langle \bar{v}, h(x_0)\rangle \leq \inf_{x\in S}[f(x) + \langle \bar{u}, g(x)\rangle + \langle \bar{v}, h(x)\rangle] \tag{3.3}$$

and, taking into account that $\langle u, g(x_0)\rangle \leq 0$, $\forall u \in C^*$ and $\langle v, h(x_0)\rangle = 0$, $\forall v \in Z^*$, we have:

$$\inf_{x\in S}[f(x) + \langle u, g(x)\rangle + \langle v, h(x)\rangle] \leq f(x_0) + \langle u, g(x_0)\rangle + \langle v, h(x_0)\rangle \leq f(x_0)$$
$$\forall u \in C^*,\ \forall v \in Z^*. \tag{3.4}$$

Then, taking into account (3.4) and using also (3.3), we get

$$\sup_{\substack{u\in C^*\\ v\in Z^*}} \inf_{x\in S}[f(x) + \langle u, g(x)\rangle + \langle v, h(x)\rangle] \leq f(x_0) + \langle \bar{u}, g(x_0)\rangle + \langle \bar{v}, h(x_0)\rangle$$
$$\leq \inf_{x\in S}[f(x) + \langle \bar{u}, g(x)\rangle + \langle \bar{v}, h(x)\rangle]$$

and, finally, we have

$$f(x_0) + \langle \bar{u}, g(x_0)\rangle + \langle \bar{v}, h(x_0)\rangle \leq \inf_{x\in S}[f(x) + \langle \bar{u}, g(x)\rangle] + \langle \bar{v}, h(x)\rangle$$
$$\leq \sup_{\substack{u\in C^*\\ v\in Z^*}} \inf_{x\in S}[f(x) + \langle u, g(x)\rangle + \langle v, h(x)\rangle] \leq f(x_0) + \langle \bar{u}, g(x_0)\rangle + \langle \bar{v}, h(x_0)\rangle,$$

namely

$$f(x_0) = f(x_0) + \langle \bar{u}, g(x_0)\rangle + \langle \bar{v}, h(x_0)\rangle$$
$$= \max_{\substack{u\in C^*\\ v\in Z^*}} \inf_{x\in S}[f(x) + \langle u, g(x)\rangle + \langle v, h(x)\rangle] = \inf_{x\in S}[f(x) + \langle \bar{u}, g(x)\rangle + \langle \bar{v}, h(x)\rangle]. \qquad \square$$

4. PROOF OF THEOREM 1.2

The normal cone $N_M(0, \theta_Y, \theta_Z)$ to M at $(0, \theta_Y, \theta_Z) \in M$ is given by the points $(\xi, y^*, z^*) \in \mathbb{R} \times Y^* \times Z^*$ such that:

$$\xi(f(x) - f(x_0) + \alpha) + \langle y^*, g(x) + y\rangle + \langle z^*, h(x)\rangle \leq 0$$
$$\forall x \in S,\ \forall \alpha \geq 0,\ \forall y \in C, \tag{4.1}$$

which is equivalent to $\xi \leq 0$, $y^* \in C^-$ and $\xi(f(x) - f(x_0)) + \langle y^*, g(x)\rangle + \langle z^*, h(x)\rangle \leq 0$, $\forall x \in S$.

In virtue of *Assumption N*, i.e. there exist $\bar{x} \in \mathbb{K}$ and $(\hat{\xi}, \hat{y}^*, \hat{z}^*) \in N_M(0, \theta_Y, \theta_Z)$ such that (note that $\langle z^*, h(\bar{x})\rangle = 0$):

$$\hat{\xi}(f(\bar{x}) - f(x_0)) + \langle \hat{y}^*, g(\bar{x})\rangle + \langle z^*, h(\bar{x})\rangle < 0, \tag{4.2}$$

we get that $(\hat{\xi}, \hat{y}^*, \hat{z}^*)$ is different than $(0, \theta_Y, \theta_Z)$. Let us prove that $\hat{\xi} \neq 0$. In fact, if $\hat{\xi} = 0$, from (4.2) we get:

$$\langle \hat{y}^*, g(\bar{x})\rangle < 0,$$

whereas, being $-g(\bar{x}) \in C$, we have $\langle \hat{y}^*, g(\bar{x})\rangle \geq 0$. Then $\hat{\xi} < 0$ and from (4.1) rewritten with $\hat{\xi}$, setting $\dfrac{y^*}{\hat{\xi}} = \bar{u} \in C^*$, $\dfrac{z^*}{\hat{\xi}} = \bar{v} \in Z^*$, and $\alpha = 0$, we get:

$$f(x) + \langle \bar{u}, g(x)\rangle + \langle \bar{v}, h(x)\rangle \geq f(x_0) \quad \forall x \in S, \tag{4.3}$$

namely the estimate (3.2) in Section 3. Then, going on as in Theorem 1.1, we get the proof of Theorem 1.2. □

Using Theorem 1.1 or Theorem 1.2, we are able to prove the usual relationship between a saddle point of the so-called Lagrange functional

$$\mathcal{L}(x, u, v) = f(x) + \langle u, g(x)\rangle + \langle v, h(x)\rangle \quad \forall x \in S,\ \forall u \in C^*,\ \forall v \in Z^* \tag{4.4}$$

and the solution to Problem 1.

Theorem 4.1. *Let the assumptions of Theorems* 1.1 *or* 1.2 *be satisfied. Then* $x_0 \in \mathbb{K}$ *is a minimal solution to Problem* 1 *if and only if there exist* $\bar{u} \in C^*$ *and* $\bar{v} \in Z^*$ *such that* $(x_0, \bar{u}, \bar{v})$ *is a saddle point of the Lagrange function* (4.4) *and*

$$\langle \bar{u}, g(x_0)\rangle = 0.$$

5. APPLICATIONS

Assumption N or similar ones of the type $(0, \theta_Y, \theta_Z) \notin \operatorname{qri} M$ have only a theoretical value, whereas *Assumption S* is very effective in the applications. In fact, it consists in the calculation of some limits and, even if it is expressed by means of the minimal point $x_0 \in \mathbb{K}$ of Problem 1, this fact is not influential, because it is based on the behavior of the difference $f(x) - f(x_0)$. In order to be clear, let us illustrate the effectiveness of *Assumption S* through some examples.

First, let us consider the archetype problem which models all the equilibrium problems (see [5], [6], [9], [11] [12], [13], [14], [15], [17], [18], [20]), that is the variational inequality

$$\int_0^T \langle C(x_0(t)), x(t) - x_0(t)\rangle\, dt \geq 0 \quad \forall x \in \mathbb{K}, \tag{5.1}$$

where

$$\mathbb{K} = \left\{x \in L^2([0,T], \mathbb{R}^m) : x(t) \geq 0,\ \Phi x(t) = \rho(t) \text{ a.e. in } [0,T]\right\},$$

with $\rho(t) \in L^2([0,T], \mathbb{R}^l)$, $\rho(t) > 0$ a.e. in $[0,T]$, $\Phi = \{\Phi_{ij}\}_{\substack{i=1,\dots,l \\ j=1,\dots,m}}$, $\Phi_{ij} \in \{0,1\}$, and in each column there is only one entry different from zero, and $C : \mathbb{K} \to L^2([0,T], \mathbb{R}^m)$ is the cost trajectory. For this variational inequality the following theorem is proved.

Theorem 5.1. $x_0 \in \mathbb{K}$ *verifies variational inequality* (5.1) *if and only if there exist* $\tilde{C} \in L^2([0,T], \mathbb{R}^l)$ *and* $\mu \in L^2([0,T], \mathbb{R}^m)$ *such that*

$$C(x_0) - \Phi^T \tilde{C} = \mu, \quad \langle \mu,\, x_0\rangle = 0, \quad \mu \geq 0$$

(Wardrop's principle).

Proof. For the reader's convenience, we recall the proof of Theorem 5.1. Following Theorem 3.1 in [10] in which we assume $\mu = +\infty$ and $\lambda = 0$, we have that $x_0 \in \mathbb{K}$ verifies variational inequality (5.1) if and only if for all $i = 1, \dots, l$, all q, s such that $\Phi_{iq} = \Phi_{is} = 1$ and a.e. in $[0, T]$

$$C_q(x_0)(t) > C_s(x_0)(t) \Longrightarrow x_q^0(t) = 0. \tag{5.2}$$

Setting $\tilde{C}_i(t) = \min\{C_j(x_0)(t) : \Phi_{ij} = 1\} \in L^2(0, T)$, $i = 1, \dots, l$, we can rewrite (5.2) in an equivalent form a.e. in $[0, T]$ as:

$$\left(C_q(x_0)(t) - \tilde{C}_i(t)\right) x_q^0(t) = 0 \quad \forall q \text{ such that } \Phi_{iq} = 1,\ i = 1, \dots, l. \tag{5.3}$$

In fact, if (5.2) holds true and $C_q(x_0)(t) - \tilde{C}_i(t) > 0$, then $x_q^0(t) = 0$, since $\tilde{C}_i(t)$ is equal to some $C_s(x_0)(t)$. Vice versa, if $x_q^0(t) > 0$, then $C_q(x_0)(t) - \tilde{C}_i(t)$ must be zero, because if $C_q(x_0)(t) - \tilde{C}_i(t) > 0$, then $x_q^0(t)$ should be zero. Analogously, if we assume that (5.3) holds true and consider q, s such that $\Phi_{iq} = 1$, $\Phi_{is} = 1$ and $C_q(x_0)(t) > C_s(x_0)(t)$, since it is $C_s(x_0)(t) \geq \tilde{C}_i(t)$, it follows $x_q^0(t) = 0$. Denoting by $\tilde{C}(t)$ the vector $\left[\tilde{C}_1(t), \dots, \tilde{C}_l(t)\right]^T$, and taking into account that in each column of Φ there is only one entry different from zero, we can rewrite condition (5.3) in the form

$$C(x_0) - \Phi^T \tilde{C} = \mu, \quad \langle \mu, x_0 \rangle = 0,$$

with $\mu \geq 0$, $\mu \in L^2([0, T], \mathbb{R}^m)$. □

Now, assuming that $x_0 \in \mathbb{K}$ is a solution to (5.1), we can rewrite problem (5.1) in the form:

$$\min_{\mathbb{K}} f(x) = f(x_0) = 0$$

with

$$f(x) = \int_0^T \langle C(x_0(t)), x(t) - x_0(t) \rangle \, dt$$

and we can prove that *Assumption S* is fulfilled. In fact, we have to prove that if $(l, \theta_Y, \theta_Z) \in T_{\widetilde{M}}(f(x_0), \theta_Y, \theta_Z)$, where $Y = L^2([0, T], \mathbb{R}^m)$ and $Z = L^2([0, T], \mathbb{R}^l)$, namely if

$$\begin{gathered} l = \lim_n \lambda_n (f(x_n) + \alpha_n - f(x_0)),\ \theta_Y = \lim_n \lambda_n (-x_n + y_n), \\ \theta_z = \lim_n \lambda_n (\Phi x_n(t) - \rho(t)), \end{gathered} \tag{5.4}$$

with $\lambda_n > 0$, $\lim\limits_n (f(x_n) + \alpha_n - f(x_0)) = 0$, $\lim_n(-x_n + y_n) = \theta_Y$, $\lim\limits_n (\Phi x_n(t) - \rho(t)) = \theta_Z$, l must be nonnegative. In virtue of Theorem 5.1 we have

$$f(x_n) - f(x_0) = \int_0^T \langle C(x_0), x_n(t) - x_0(t) \rangle \, dt = \int_0^T \langle \Phi^T \tilde{C} + \mu, x_n(t) - x_0(t) \rangle \, dt$$

and, taking into account that $\Phi x_0(t) = \rho(t)$ and $\mu x_0 = 0$, we get:

$$\lambda_n(f(x_n) + \alpha_n - f(x_0))$$
$$= \lambda_n \int_0^T \langle \Phi^T \tilde{C}, x_n(t) - x_0(t)\rangle\, dt + \lambda_n \int_0^T \langle \mu, x_n(t) - x_0(t)\rangle\, dt + \lambda_n\, \alpha_n$$
$$= \int_0^T \langle \tilde{C}(t), \lambda_n(\Phi x_n(t) - \rho(t))\rangle\, dt + \int_0^T \langle \mu(t), \lambda_n(x_n(t) - y_n(t))\rangle\, dt$$
$$+ \int_0^T \langle \mu(t), \lambda_n y_n(t)\rangle\, dt + \lambda_n \alpha_n.$$

By means of conditions (5.4), we obtain:

$$\lim_n \int_0^T \langle \tilde{C}(t), \lambda_n(\Phi x_n(t) - \rho(t))\rangle\, dt = 0,\ \lim_n \int_0^T \langle \mu(t), \lambda_n(x_n(t) - y_n(t))\rangle\, dt = 0,$$

and, being $\mu \geq 0$, $\lambda_n > 0$, $y_n(t) \geq 0$, $\alpha_n \geq 0$, we get:

$$\lim_n \lambda_n(f(x_n) + \alpha_n - f(x_0)) \geq 0,$$

namely our assertion.

Next, let us consider the variational inequality which expresses the dynamic Cournot-Nash equilibrium, namely the dynamic oligopolistic market equilibrium problem (see [1]):

$$\text{Find } x^* \in \mathbb{K} : \ll -\nabla v(t, x^*(t)), x - x^* \gg \geq 0 \quad \forall x \in \mathbb{K} \tag{5.5}$$

where

$$\mathbb{K} = \{x \in L^2([0,T], \mathbb{R}^{mn}) : 0 \leq \lambda(t) \leq x(t) \leq \mu(t) \text{ a.e. in } [0,T]\}$$

and $v_i(t, x(t))$, $i = 1, \ldots, m$ is the profit of the firm P_i at time $t \in [0,T]$. In the paper [1] the following Lemma is proved.

Lemma 5.2. *Let $x^* \in \mathbb{K}$ be a solution to the variational inequality* (5.5)*. Then, setting:*

$$E^i_- = \{t \in [0,T] : x^*_i(t) = \lambda_i(t) \textit{ a.e. in } [0,T]\},$$
$$E^i_0 = \{t \in [0,T] : \lambda_i(t) < x^*_i(t) < \mu_i(t) \textit{ a.e. in } [0,T]\},$$
$$E^i_+ = \{t \in [0,T] : x^*_i(t) = \mu_i(t) \textit{ a.e. in } [0,T]\},$$

we have:

$$-\frac{\partial v(t, x^*(t))}{\partial x_i} \geq 0 \textit{ a.e. in } E^i_-,$$
$$\frac{\partial v(t, x^*(t))}{\partial x_i} = 0 \textit{ a.e. in } E^i_0,$$
$$-\frac{\partial v(t, x^*(t))}{\partial x_i} \leq 0 \textit{ a.e. in } E^i_+.$$

Taking into account this result and following the same technique used in the previous example, we can easily show that also in this case *Assumption S* is verified. Moreover, the procedure used for the oligopolistic market equilibrium problem can be easily adapted to show that *Assumption S* is also verified in the case of environmental pollution dynamic control problem (see [20]).

Finally we recall that *Assumption S* guarantees the existence of the Lagrange multiplier associated to the elastic-plastic torsion problem (see see [3], [4], [8], [21]).

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