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COVERING THE SPHERE AND THE BALL IN BANACH SPACES

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Dedicated to my friend Espedito, with best wishes

ABSTRACT. We collect and discuss some results concerning different, economical coverings for the unit ball or the unit sphere of Banach spaces.

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1. INTRODUCTION AND DEFINITIONS

Let X be a Banach space over the real field R; by E^n and M^n, respectively, we denote n-dimensional Euclidean and Minkowskian spaces.

We shall denote by $B(x, r)$ $(x \in X; r \geq 0)$ the closed ball centered at x, with radius r; by B_X and S_X (or simply by B and S if no confusion can arise), respectively, the unit ball $B(\theta, 1)$, and its boundary, the unit sphere of X.

We denote by $\delta(A)$ the diameter of a bounded set A.

A *covering* of a set $A \subset X$ is a family of sets $\{A_i\}_{i \in I}$ (I a family) such that $A \subset \cup_{i \in I} A_i$; a covering will be called a *ball covering* if the elements of $\{A_i\}$ are balls. Given a set C, a *ball covering from* C is a covering by balls centered at points of C. A covering (or a ball covering, or a ball covering from C) will be said *finite* if I is finite.

Results on coverings of the unit ball or the unit sphere are scattered in the literature. Some facts can be considered trivial: for example, B cannot be covered by a countable family of closed sets with nonempty interior (due to Baire theorem).

We discuss here several results mainly concerning coverings of the unit sphere or the unit ball by means of balls, eventually with radii not too large and/or by asking that the centers of balls belong to the unit sphere.

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2. SOME SIMPLE RESULTS AND GENERAL FACTS

We consider now coverings of S.

We can ask the following:

- In which cases must a covering of S contain the origin?

- Under which assumptions a covering of S is also a covering of B?

We recall an old result (see Molnár [19]).

Proposition 1. If three balls B_1, B_2, B_3 of radii ≤ 1 cover the unit sphere in the Euclidean plane, then they also cover the unit ball.

Clearly, in the above proposition 1 is a critical value.

In the paper by Asplund and Grünbaum [1], after quoting the above result, a prove of the same for any strictly convex and smooth Minkowski plane was given (see Theorem 8 there); also, it was said that the proof could be extended to any Minkowski plane.

Moreover, the following conjectures were done: maybe the same result is true for all Minkowskian (thus for Euclidean) spaces of dimension n, any $n \in N$, when coverings by $(n+1)$ balls are considered.

Another simple fact was discussed in [6]:

Proposition 1bis. The Euclidean spaces cannot be covered with circular disks having mutually disjoint interiors.

We recall the following important result.

Proposition 2. If S^{n-1}, the unit sphere of E^n, is covered by n closed sets $A_1, ..., A_n$, then there is an index $i \in \{1, ..., n\}$ such that A_i contains an antipodal pair $x, -x$ of S^{n-1}; thus $\delta(A_i) \geq 2$ for some index i.

This result is usually known as the Lyusternik - Snirel'man lemma (see Lyusternik - Snirel'man [16], p.182): it was proved in 1930 and then rediscovered by Borsuk in 1933.

Proposition 2bis. The above result is also true if we consider M^n instead of E^n (see Furi and Vignoli [11]).

As a consequence, we have the following.

Corollary 1. Let $\dim(X) = \infty$. If S_X is covered by finitely many closed sets $\{A_i\}_{i \in \{1,...,n\}}$, then $\delta(A_j) \geq 2$ for some index j.

In particular: if $\{B_i = B(x_i, r_i);\ i = 1, ..., n\}$, is a finite covering of S_X by balls centered at points of S_X, then $\theta \in \cup_{i \in \{1,...,n\}} B_i$.

Remark. It is clear that the above corollary (X any Banach space) is a consequence of Prop. 2bis. It can also be deduced by the version for X Hilbert (see Prop. 2), by means of Dvoretzki's Theorem.

In the last part of Corollary 1, the restriction on the centers of the balls is not necessary: in fact, the following result, proved by different authors, is indicated in Bagchi et al [2].

Proposition 3. Let $\dim(X) = \infty$; then any finite covering of S_X by balls contains θ.

More generally, along the lines of the proof given for the above proposition, the following more general result can be proved.

Proposition 3bis. Let C be a closed convex body; if the boundary of C is covered by n closed convex sets, $n \leq \dim(X)$, then the whole set C is covered.

We can also prove, in a different way, the following fact.

Proposition 3ter. Let X be an infinite-dimensional space. If finitely many balls cover S, then they also cover B.

Proof. Let

$$S_X \subset \bigcup_{i=1}^{n} B(x_i, r_i).$$

Now consider the bidual X^{**} (where balls are w^* compact); we have:

$w^* - clS_X \subset \bigcup_{i=1}^{n} w^* - clB(x_i, r_i)$. However $w^* - clS_X = B_{X^{**}}$, and so by taking the intersection with X we obtain the thesis. $\diamondsuit$

Another simple fact is the following.

Proposition 4. A covering of S_X by balls with centers in S_X, of radius at least 1, is also a covering of B_X.

Proof. Clearly in this case all balls of the covering contain the origin. Let $x \in B_X$; set $x' = x/||x||$. Since $x' \in S_X$, one of the balls of the covering must contain x' (together with θ), so by convexity it contains also x. $\diamondsuit$

For some other general results in this spirit, see Carl and Edmunds [5] and the references there.

3. COVERING S MISSING THE ORIGIN

Some related results received some attention recently; namely, the following problem has been considered (see Cheng [7]).

Given a space X, find a ball covering $\{A_i\}_{i\in I}$ of S_X such that $\theta \notin \cup_{i\in I} A_i$.

Call such a covering a *ball (-) covering*.

Note that a ball(-)covering of S_X does not cover B_X. The results in previous section give restrictions to the possibility to have such coverings, mainly if we wish to have centers on S_X.

We can ask what is the minimal number of balls necessary to obtain such a covering, and what is the minimal r such that a ball(-)covering by balls of radius not larger than r is possible.

The situation in this context has been clarified by some recent papers.

The following results have been proved; two balls like $B(x, r)$ and $B(-x, r)$ are said to be *symmetric*.

- If $X = M^n$, then S_X admits a ball(-)covering consisting of no more than n pairs of symmetric balls, but $n-1$ pairs are not enough; also, at least $n+1$ balls are always necessary, while $n+1$ are sufficient whenever X is smooth, but not in general (see Cheng [7], Theorems 2.2, 2.3, and Example 2.5).

Concerning the smallest radius r ($r \in [1, 2]$) of balls necessary to have a finite ball(-)covering in this case, see Shi and Zhang [22], Fu and Cheng, [10], Zhang [24]; see also in Böröczky [4], mainly for large dimensions.

- If S_X has a countable ball(-)covering (or simply, a countable ball covering) by balls of radius $\leq r$ for some $r < 1$, then X is separable (see Cheng [7], Theorem 3.1); in fact, if X is separable, there exists such covering for every positive r.

- If S_X has a countable ball(-)covering with balls of radii $\leq r$, then for every $\epsilon > 0$, B_X can be covered by a countable family of balls of radii $\leq r + \epsilon$ (see Cheng [7], Fact 3.3). A slightly more general result will be proved at the end of this section.

Also:

- ℓ_∞ (which is not separable) has a countable ball(-)covering by balls of radii equal to 1 (see Cheng [7], Example 3.4). But there is a renorming of the space lacking a countable ball(-)covering (see Cheng et al [8], Theorem 2.1).

- If S_X has a countable ball(-)covering, then X^* is w^*-separable (see Cheng [7], Proposition 4.1). The converse is not true in general (see Cheng et al [8], Corollary 2.4). Maybe it is true under some assumptions on the norm: see Cheng [7], Theorem 4.3, Corollary 4.4, Theorem 4.5.

Other results of this type with relation to quotient mappings have been considered in Cheng et al [8].

Concerning the results in Cheng [7] and in Cheng et al [8], see the reviews of those papers appearing in Zbl. Math.: #1139-46016, and to appear.

More precise results concerning coverings and w^*-separability of X have been given in Fonf and Zanco [9].

We end this section by proving a result which generalize one of the above results. Note that in next statement, we cannot substitute countable by finite (think at finite-dimensional spaces).

Proposition 5. Assume that S can be covered by a countable set of balls or radius $\leq \rho < 1$ ($\rho > 0$); then the same is true for B.

Proof. Let

$$S \subset \bigcup_{i\in N} B(x_i, r_i),\ r_i \le \rho\ \forall i\,,$$

so

$$S \subset \bigcup_{i\in N} B(x_i, \rho).$$

Set

$$S_t = \{x \in X;\ ||x|| = t\};\quad C_{\alpha,\beta} = \{x \in X;\ \alpha \le ||x|| \le \beta\};\quad (t, \alpha, \beta \in R^+).$$

We have (for every $t \ge 0$):

$$S_t \subset \bigcup_{i\in N} B(tx_i, t\rho);$$

also

$$C_{t-\epsilon,\,t+\epsilon} \subset \bigcup_{i\in N} B(tx_i, t\rho + \epsilon)\ :$$

so, if $t < 1$ ($\epsilon = \rho - t\rho$):

$$C_{t(1+\rho)-\rho, t(1-\rho)+\rho} \subset \bigcup_{i\in N} B(tx_i, \rho).$$

Now consider the following sequence: $t_0 = 0$; $t_1 = \frac{2\rho}{1+\rho}$, and for $n \ge 1$: $t_{n+1} = \frac{2\rho + t_n(1-\rho)}{1+\rho}$. Clearly $t_{n+1} > t_n$ for every $n > 1$ ($t_n < 1$ for all n; $lim_{n\to\infty} t_n = 1$).

Also, $t_n(1-\rho) + \rho = t_{n+1}(1+\rho) - \rho$; thus B is contained in

$$\bigcup_{i\in N} \big(\bigcup_{n\in N} B(t_n x_i, t_n\rho)\big) = B(\theta, \rho) \cup \big(\bigcup_{n\in N} C_{t_n(1+\rho)-\rho,\, t_n(1-\rho)+\rho}\big) = B(\theta, 1).$$

This concludes the proof. $\diamondsuit$

4. TWO RELATED CONSTANTS

The following constant was defined in Whitley [23].

$T(X) = \inf\{\epsilon > 0;\ S_X$ has a finite ball covering from S_X by balls of radii $\le \epsilon$ $\}$.

Not too many papers deal with this constant: see Papini [20], Maluta and Papini [17] (where some generalizations of this constant were considered) and Maluta and Papini [18]. But the following facts are known:

$T(X) = 0 \Leftrightarrow \dim(X) < \infty$;

For infinite-dimensional spaces the range of $T(X)$ is $[1, 2]$.

Also, it is not difficult to see that, when $\dim(X) = \infty$, we obtain an equivalent definition if we define $T(X)$ by considering coverings of B_X (see Corollary 1 and Proposition 4).

We can also consider the following constant (see for example Papini [20]).

$g'(X) = \inf\{\epsilon > 0;\ S_X$ has covering from S_X by two symmetric balls of radius $\leq \epsilon\ \}$.

Clearly, $T(X) \leq g'(X)$ always.

According to Proposition 2bis, $g'(X) \geq 1$ always: so $T(X) \neq g'(X)$ when $\dim(X) < \infty$.

But also in infinite dimensional spaces we can have $T(X) \neq g'(X)$, as the following example shows. Let c_0 denote the space of real sequences converging to 0, and ℓ_∞ the space of bounded real sequences, both endowed with the max norm.

Example 1. Consider $X = \mathbb{R} \oplus_1 \ell_\infty$, or $X = \mathbb{R} \oplus_1 c_0$, i.e. the space of bounded (respectively vanishing) sequences equipped with the norm

$$||(x_1, x_2, x_3, ...)|| = |x_1| + ||(x_2, x_3, ...)||_\infty.$$

We want to show that

a) $T(X) = 1$;

b) $g'(X) = 4/3$.

To prove a), consider $n \in N$; we want to show that there exists a $(1 + 1/n)$-net for S_X from S_X.

Given n, consider the $2n + 1$ numbers $\pm k/n$, $k = 0, 1, ..., n$.

Set $x_{\pm k} = (\frac{\pm k}{n}, \frac{n-k}{n}, 0, 0, ...)$ and $y_{\pm k} = (\frac{\pm k}{n}, \frac{k-n}{n}, 0, 0, ...)$. We say that the $4n+2$ points in S_X: $\{x_{\pm k};\ y_{\pm k},\ k = 0, 1, ..., n\}$ form a $(1 + 2/n)$-net for S_X.

Given $x = (x_1, x_2, ..., x_n, ...) \in S_X$, choose k such that, for some choice of the sign, $|\frac{\pm k}{n} - x_1| \leq \frac{1}{2n}$. Then we have, for the same choice of the sign:

$||x - x_{\pm k}|| \leq \frac{1}{2n} + 1$ if $x_2 \geq 0$; $||x - y_{\pm k}|| \leq \frac{1}{2n} + 1$ if $x_2 \leq 0$;

this proves a).

To prove b), we prove first that for $y = (\frac{1}{3}, \frac{2}{3}, 0, 0, ...)$ $(y \in S_X)$ we have $min\{||x - y||, ||x + y||\} \leq \frac{4}{3}$ for every $x \in S_X$.

Let $x = (x_1, x_2, x_3, ...)$. Then $||x \pm y|| = |x_1 \pm \frac{1}{3}| + max\{|x_2 \pm \frac{2}{3}|,\ max_{n\geq 3}|x_n|\}$.

Suppose that $min\{||x - y||, ||x + y||\} > \frac{4}{3}$. Since $|x_1 \pm \frac{1}{3}| + max_{n\geq 3}|x_n| \leq |x_1 \pm \frac{1}{3}| + 1 - |x_1| \leq \frac{4}{3}$, this implies $|x_1 \pm \frac{1}{3}| + |x_2 \pm \frac{2}{3}| > \frac{4}{3}$. But this is impossible since it is simple to see that $\{(\frac{1}{3}, \frac{2}{3}), (\frac{-1}{3}, \frac{-2}{3})\}$ is a $\frac{4}{3}$-net for the plane with the sum norm.

Now we prove that for every $y \in S_X$, $sup_{x\in S_X} min\{||x - y||, ||x + y||\} \geq \frac{4}{3}$.

Fixed $y \in S_X$, if $|y_1| \leq 1/3$ then $||x \pm y|| \geq |x_1 - y_1| + \sum_{n\geq 2} |y_n| \geq \frac{4}{3}$ for $x = (1, 0, 0, ...)$.

Now let $|y_1| \geq 1/3$: then if y_2 and y_3 have the same sign, we take $x = (0, 1, -1, 0, 0, ...)$; otherwise, take $x = (0, 1, 1, 0, 0, ...)$; in any case we thus obtain $||x \pm y|| \geq \frac{4}{3}$. This concludes the proof of b).

5. SOME OTHER RELATED PROBLEMS

We indicate some other problems which seem to be somehow connected with the above ones.

- Completely saturated coverings were discussed in Hinrichs and Richter [13].

- Borsuk and Hadwiger problems; they deal with covering problems for bounded sets in E^n. For example, Borsuk's problem asks whether every set of diameter 1 in E^n can be covered by $n+1$ sets of diameter < 1. For $n = 2, 3$ the solution is positive, while there are negative solutions for big dimensions (see Hinrichs and Richter [12]). The most famous unsolved case of Borsuk's problem is $n = 4$. Many other problems of combinatorial character concerning coverings and tilings can be found for example in Schmitt [21].

- Coverings of B_X for X a Hilbert space by some kind of unbounded sets (planks, cylinders, or ball-cut-like sets) received and are still receiving much attention: see for example Kadets [14].

Another problem, more related to ours, was formulated and studied in Kadets [15], and reconsidered also in Bezdek [3]. Given a convex body A, let $r'(A) = sup\{r;\ A$ contains a ball of radius $r\}$.

Let a convex closed body A in a Banach space X be covered by a sequence of convex closed bodies $A_n, n \in N$. Must we have $r'(A) \leq \sum_{n\in N} r'(A_n)$?

Equivalently, we can ask if $1 \leq \sum_{n\in N} r'(A_n)$ for every such covering of B_X.

The answer is yes if X is Hilbertian, but seems to be open in general.

The property $r'(A) \leq \sum_n r'(A_n)$ of coverings by convex bodies reminds a basic property of Lebesgue measure. Remark that another property of Lebesgue measure -continuity- fails in general for r'. In fact, the following example (that can be adapted to some function spaces) shows that a sequence of convex closed sets $A_1 \subset A_2 \subset A_3$ ... with $r'(A_n) = \epsilon < 1$ for every n, can cover the unit ball of c_0.

Example 2. Let $X = c_0$. Then B_X can be covered, for $\epsilon > 0$ fixed, by the union of the sequence of sets

$A_n = \{x_i \in B_X;\ |x_i| \leq \epsilon \text{ for all } i > n\}$.

Note that $r'(A_n) = \epsilon$ for every $n \in N$.

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