

MICROPOLAR LINEARLY ELASTIC RODS

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ABSTRACT. We use Γ -convergence to recover the behaviour of solutions of the equilibrium problem for a linearly elastic micropolar rod.

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1. Introduction

A micropolar body is a continuous collection of *material particles* whose configuration is given by a placement and an orientation [5] in the three-dimensional Euclidean space. This formal structure is shared by theories of thin bodies which take into account both *bending* and *shearing*, such as the *Timoshenko beam* and the *Reissner-Mindlin plate*. Timoshenko’s theory pictures an elastic beam as a one-dimensional collection of two-dimensional *sections*, whereas Reissner-Mindlin’s theory pictures an elastic plate as a two-dimensional collection of one-dimensional *fibers*; in both cases, the configuration of the body is assigned by prescribing the displacement and the rotation of, respectively, the sections and the fibers. This similarity is at the basis of the conjecture [6] that three-dimensional micropolar elasticity might be effectively used as starting point for a “rational” deduction of theories of shearable beams and plates by some dimension reduction process.

No matter whether the “target” theory describes a beam or a plate, the dimension reduction may be achieved starting from a three-dimensional body having the shape of a right-cylinder and then appropriately scale geometry, loads, and possibly material moduli by multiplying each datum by some power of a small parameter [7]. The choice of the scaling laws, if done appropriately, leads in the limit to the energy functional of the target structural theory. For elastic plates, this program has been carried out [1, 8, 9]. For rods, it seems to us that much has to be done.

This paper constitutes a first step towards building the Timoshenko's beam theory from micropolar elasticity. We take a linearly elastic, homogeneous micropolar body having the shape of a right cylinder of fixed height and cross section a disk of radius h . Equilibrium configurations for such a body are pairs of *displacement* \mathbf{u} and *microrotation* $\boldsymbol{\omega}$ which minimize a suitable quadratic energy functional. Following the same line of reasoning as [9], we are able to identify the Γ limit, as h tends to zero, of the energy functionals, and we show that this functional is “one dimensional”, in the sense that its effective domain consists on pairs $(\mathbf{u}, \boldsymbol{\omega})$ that depend on the axial coordinate only. The interpretation of this result and its connection with existing structural theories is going to be scrutinized in a forthcoming paper.

2. Notation

We denote by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{z}$ the standard orthonormal basis of \mathbb{R}^3 . For $h > 0$, we set $B^{(h)} := \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < h\}$ and, for $L > 0$, we consider domains $\Omega^{(h)} \subset \mathbb{R}^3$ having the following form:

$$\Omega^{(h)} = B^{(h)} \times (0, L). \quad (2.1)$$

The *lateral mantle* of $\Omega^{(h)}$ is:

$$\Gamma^{(h)} = \partial B^{(h)} \times (0, L).$$

Moreover we set:

$$B := B^{(1)}, \quad \Omega := \Omega^{(1)}, \quad \Gamma := \Gamma^{(1)}.$$

If $x \equiv (x_1, x_2)$ belongs to $B^{(h)}$, we denote by (\tilde{x}, ζ) and (x, ζ) the elements of $\Omega^{(h)}$ and Ω respectively; in particular we regard \tilde{x} as the result of the scaling:

$$x_1 = \tilde{x}_1/h, \quad x_2 = \tilde{x}_2/h, \quad (2.2)$$

which maps $\Omega^{(h)}$ one-to-one onto Ω .

We denote by $|\cdot|$ the Euclidean norm in \mathbb{R}^n . For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, $\mathbf{a} \cdot \mathbf{b}$ is the Euclidean scalar product of \mathbf{a} and \mathbf{b} and $\mathbf{a} \otimes \mathbf{b} \in \mathbb{R}^{3 \times 3}$ is the matrix whose entries are $(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j$. For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, $\mathbf{a} \times \mathbf{b}$ denotes the vector product of \mathbf{a} and \mathbf{b} . The space of symmetric 3×3 matrices will be denoted by “*Sym*”.

Given a matrix $\mathbf{T} \in \mathbb{R}^{3 \times 3}$, we denote by \mathbf{T}_\perp the $\mathbb{R}^{3 \times 2}$ matrix obtained from \mathbf{T} by removing the third column:

$$\mathbf{T}_\perp = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \\ T_{31} & T_{32} \end{pmatrix}.$$

With a slight abuse of notation, we also set

$$\mathbf{T}_\perp + \mathbf{a} \otimes \mathbf{z} = \begin{pmatrix} T_{11} & T_{12} & a_1 \\ T_{21} & T_{22} & a_2 \\ T_{31} & T_{32} & a_3 \end{pmatrix}. \quad (2.3)$$

Likewise, if \mathbf{v} is a vector field defined on Ω_\perp or $\Omega^{(h)}$, we define the *surface gradient* of \mathbf{v} as:

$$\nabla_\perp \mathbf{v} = \begin{pmatrix} \partial_{x_1} v_1 & \partial_{x_2} v_1 \\ \partial_{x_1} v_2 & \partial_{x_2} v_2 \\ \partial_{x_1} v_3 & \partial_{x_2} v_3 \end{pmatrix}.$$

If \mathbf{v} is a vector field defined on $\Omega^{(h)}$, we define the *scaled gradient* of \mathbf{v} as:

$$\nabla^{(h)} \mathbf{v} = \frac{1}{h} \nabla_\perp \mathbf{v} + \partial_\zeta \mathbf{v} \otimes \mathbf{z}. \quad (2.4)$$

Given $\boldsymbol{\omega} \in \mathbb{R}^3$, we denote by $\mathbf{A}_\boldsymbol{\omega}$ the unique skew-symmetric matrix such that $\mathbf{A}_\boldsymbol{\omega} \mathbf{v} = \boldsymbol{\omega} \times \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^3$. Note that

$$|\mathbf{A}_\boldsymbol{\omega}| = \sqrt{2} |\boldsymbol{\omega}|. \quad (2.5)$$

If \mathcal{U} is an open subset of \mathbb{R}^2 , we denote by $L^2(\mathcal{U}; \mathbb{R}^3)$ and $H^1(\mathcal{U}; \mathbb{R}^3)$ the usual Hilbert (respectively Lebesgue and Sobolev) spaces of functions defined on \mathcal{U} and taking values on \mathbb{R}^3 , and we set:

$$\mathbf{L}^2(\mathcal{U}) := L^2(\mathcal{U}; \mathbb{R}^3) \times L^2(\mathcal{U}; \mathbb{R}^3) \quad \text{and} \quad \mathbf{H}^1(\mathcal{U}) := H^1(\mathcal{U}; \mathbb{R}^3) \times H^1(\mathcal{U}; \mathbb{R}^3).$$

We moreover define the spaces of *kinematically admissible fields*:

$$\mathcal{K}(\Omega^{(h)}) := \{(\mathbf{u}, \boldsymbol{\omega}) \in \mathbf{H}^1(\Omega^{(h)}) : \mathbf{u} = \boldsymbol{\omega} = \mathbf{0} \text{ on } B^{(h)} \times \{0, L\}\},$$

$$\mathcal{K}_\perp(\Omega) = \{(\mathbf{u}, \boldsymbol{\omega}) \in \mathcal{K}(\Omega) : \nabla_\perp \mathbf{u} = \nabla_\perp \boldsymbol{\omega} = \mathbf{0} \text{ in } \Omega\}.$$

Given a differentiable function $\sigma : \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$, we shall denote by $\partial \sigma : \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3}$ the differential of σ , and by $\partial_{\mathbf{E}} \sigma$ and $\partial_{\mathbf{G}} \sigma$ the matrices of partial derivatives of $\sigma(\mathbf{E}, \mathbf{G})$ with respect to \mathbf{E} and \mathbf{G} respectively.

3. A family of equilibrium problems for micropolar rod-like bodies

In linear micropolar elasticity the kinematics of a body, occupying in its *reference shape* a region $\mathcal{U} \subset \mathbb{R}^3$, is described by a *displacement field* $\mathbf{u} : \mathcal{U} \rightarrow \mathbb{R}^3$ and a *microrotation field* $\boldsymbol{\omega} : \mathcal{U} \rightarrow \mathbb{R}^3$ [5]. The *ordinary stress tensor*, denoted by \mathbf{S} , is accompanied by a *couple stress tensor*, denoted by \mathbf{C} . Moreover, the loads applied to the body consist not only in a distance force and a contact force (as in standard elasticity), but also in a distance couple and a contact couple.

In the sequel in order to simplify notation, we set $\mathbf{W} := \mathbf{A}_\omega$.

The *strains* relevant to the theory are the tensor fields defined by:

$$\begin{aligned}\mathbf{E} &= \mathbf{E}(\mathbf{u}, \omega) = \nabla \mathbf{u} - \mathbf{W}, \\ \mathbf{G} &= \mathbf{G}(\omega) = \nabla \omega.\end{aligned}\tag{3.1}$$

The stress descriptors \mathbf{S} and \mathbf{C} depend on the strains by means of *constitutive equations* of the form:

$$\mathbf{S} = \mathbb{S}[\mathbf{E}], \quad \mathbf{C} = \mathbb{C}[\mathbf{G}],$$

where $\mathbb{S} : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3}$ and $\mathbb{C} : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3}$ are linear functions (called *constitutive mappings*) that carry the relevant information on the *composition* of the body. We assume that the constitutive mappings do not depend on the position $x \in \mathcal{U}$. In this case the body is said to be *homogeneous*.

Let $\sigma : \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ be the function (called *stored-energy density*) defined by

$$\sigma(\mathbf{E}, \mathbf{G}) = \frac{1}{2} \mathbb{S}[\mathbf{E}] \cdot \mathbf{E} + \frac{1}{2} \mathbb{C}[\mathbf{G}] \cdot \mathbf{G}.\tag{3.2}$$

The linear mappings \mathbb{S} and \mathbb{C} satisfy:

$$\mathbb{S}[\mathbf{A}] \cdot \mathbf{B} = \mathbb{S}[\mathbf{B}] \cdot \mathbf{A}, \quad \mathbb{C}[\mathbf{A}] \cdot \mathbf{B} = \mathbb{C}[\mathbf{B}] \cdot \mathbf{A} \quad \text{for all } \mathbf{A}, \mathbf{B} \in \mathbb{R}^{3 \times 3},\tag{3.3}$$

and are such that σ is positive definite, i.e.:

$$\sigma(\mathbf{E}, \mathbf{G}) \geq C(|\mathbf{E}|^2 + |\mathbf{G}|^2) \quad \text{for all } \mathbf{E}, \mathbf{G} \in \mathbb{R}^{3 \times 3}.\tag{3.4}$$

3.1. Micropolar rod-like bodies. We restrict our attention to *homogeneous rod-like bodies*, i.e. homogeneous bodies whose undeformed shape is a region $\Omega^{(h)}$ having the form specified in (2.1). Let σ be a function fulfilling (3.4) and, for each $h > 0$, let $(\tilde{\mathbf{f}}^{(h)}, \tilde{\mathbf{c}}^{(h)}) \in \mathbf{L}^2(\Omega^{(h)})$ be a pair of distance loads. We define the *total energy* as:

$$\tilde{\Pi}^{(h)}(\tilde{\mathbf{u}}, \tilde{\omega}) = \int_{\Omega^{(h)}} \sigma(\mathbf{E}(\tilde{\mathbf{u}}, \tilde{\omega}), \mathbf{G}(\tilde{\omega})) - \int_{\Omega^{(h)}} \{\tilde{\mathbf{f}}^{(h)} \cdot \tilde{\mathbf{u}} + \tilde{\mathbf{c}}^{(h)} \cdot \tilde{\omega}\}\tag{3.5}$$

and, for each $h > 0$, we look for a solution $(\tilde{\mathbf{u}}^{(h)}, \tilde{\omega}^{(h)})$ of the following *equilibrium problem*:

$$\min_{(\mathbf{u}, \omega) \in \mathcal{K}(\Omega^{(h)})} \tilde{\Pi}^{(h)}(\mathbf{u}, \omega).\tag{3.6}$$

Remark 3.1. *Applying the Direct Methods of Calculus of Variation (see for instance [3]), we get existence and uniqueness. In fact the convexity of σ implies existence and uniqueness of the solution of the equilibrium problem (3.6). In fact, convexity ensures the lower semicontinuity of the functional (with respect to the strong topology of $\mathbf{L}^2(\Omega^{(h)})$) and inequality (3.4) allows us to get compactness (with respect to the strong topology of $\mathbf{L}^2(\Omega^{(h)})$) for sequences having bounded energy.*

We study the asymptotic behavior of the minimizers $(\tilde{\mathbf{u}}^{(h)}, \tilde{\boldsymbol{\omega}}^{(h)})$ as h tends to 0. To work with functions defined on the same space for all h , we blow up the domain $\Omega^{(h)}$ using the change of variable (2.2). Thus, to each pair $(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\omega}}) \in \mathcal{K}(\Omega^{(h)})$ we associate the pair $(\mathbf{u}, \boldsymbol{\omega}) \in \mathcal{K}(\Omega)$ defined by:

$$\mathbf{u}(x, \zeta) := \tilde{\mathbf{u}}(hx, \zeta), \quad \boldsymbol{\omega}(x, \zeta) := \tilde{\boldsymbol{\omega}}(hx, \zeta), \quad \forall (x, \zeta) \in \Omega. \quad (3.7)$$

We shall study the scaled functional $\Pi^{(h)} : \mathcal{K}(\Omega) \rightarrow \mathbb{R}$ defined by the requirement that

$$\Pi^{(h)}(\boldsymbol{\omega}, \mathbf{u}) = \frac{1}{h^2} \tilde{\Pi}^{(h)}(\tilde{\boldsymbol{\omega}}, \tilde{\mathbf{u}})$$

whenever $(\boldsymbol{\omega}, \mathbf{u})$ is related to $(\tilde{\boldsymbol{\omega}}, \tilde{\mathbf{u}})$ by (3.7). The introduction of the scaling factor $\frac{1}{h^2}$ is essential. Without this factor, the Γ -limit in Theorem 4.7 would be the null functional.

It is immediate to see that the pair $(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)})$, corresponding to the solution $(\tilde{\mathbf{u}}^{(h)}, \tilde{\boldsymbol{\omega}}^{(h)})$ of the equilibrium problem (3.6), is the unique minimizer of $\Pi^{(h)}$. We now provide the explicit representation of $\Pi^{(h)}$. For every $(\mathbf{u}, \boldsymbol{\omega}) \in \mathcal{K}(\Omega)$ we set

$$\mathbf{E}^{(h)}(\mathbf{u}, \boldsymbol{\omega}) = \nabla^{(h)} \mathbf{u} - \mathbf{W}, \quad \mathbf{G}^{(h)}(\boldsymbol{\omega}) = \nabla^{(h)} \boldsymbol{\omega}. \quad (3.8)$$

Then (3.7) implies:

$$\mathbf{E}^{(h)}(\mathbf{u}, \boldsymbol{\omega}) = \mathbf{E}(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\omega}}), \quad \mathbf{G}^{(h)}(\boldsymbol{\omega}) = \mathbf{G}(\tilde{\boldsymbol{\omega}}).$$

Therefore, by defining the *scaled stored-energy density* as:

$$\sigma^{(h)}(\mathbf{E}, \mathbf{G}) = \sigma\left(\frac{1}{h} \mathbf{E}_\perp + \mathbf{E} \mathbf{z} \otimes \mathbf{z}, \frac{1}{h} \mathbf{G}_\perp + \mathbf{G} \mathbf{z} \otimes \mathbf{z}\right),$$

we have:

$$\sigma(\mathbf{E}^{(h)}(\mathbf{u}, \boldsymbol{\omega}), \mathbf{G}^{(h)}(\boldsymbol{\omega})) = \sigma^{(h)}(\mathbf{E}(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\omega}}), \mathbf{G}(\tilde{\boldsymbol{\omega}})).$$

It is easy to check that

$$\Pi^{(h)}(\mathbf{u}, \boldsymbol{\omega}) = \Sigma^{(h)}(\mathbf{u}, \boldsymbol{\omega}) - \Lambda^{(h)}(\mathbf{u}, \boldsymbol{\omega}), \quad (3.9)$$

where

$$\Sigma^{(h)}(\mathbf{u}, \boldsymbol{\omega}) := \int_{\Omega} \sigma^{(h)}(\mathbf{E}(\mathbf{u}, \boldsymbol{\omega}), \mathbf{G}(\boldsymbol{\omega})), \quad \Lambda^{(h)}(\mathbf{u}, \boldsymbol{\omega}) := \int_{\Omega} \{\mathbf{f}^{(h)} \cdot \mathbf{u} + \mathbf{c}^{(h)} \cdot \boldsymbol{\omega}\}, \quad (3.10)$$

with

$$\mathbf{f}^{(h)}(x_1, x_2, \zeta) := \tilde{\mathbf{f}}^{(h)}(hx_1, hx_2, \zeta), \quad \mathbf{c}^{(h)}(x_1, x_2, \zeta) := \tilde{\mathbf{c}}^{(h)}(hx_1, hx_2, \zeta), \quad \forall (x, \zeta) \in \Omega, \quad (3.11)$$

the *scaled loads*.

4. Asymptotic behavior of minimizers

We assume that the family $\{(\mathbf{f}^{(h)}, \mathbf{c}^{(h)})\}$ satisfies:

$$(\mathbf{f}^{(h)}, \mathbf{c}^{(h)}) \rightarrow (\mathbf{f}, \mathbf{c}) \text{ in } \mathbf{L}^2(\Omega). \tag{4.1}$$

Let us introduce the *limit stored-energy density* $\sigma_{\perp} : \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ defined by:

$$\sigma_{\perp}(\mathbf{E}, \mathbf{G}) := \min_{\mathbf{A}_{\perp}, \mathbf{B}_{\perp} \in \mathbb{R}^{3 \times 2}} \sigma(\mathbf{A}_{\perp} + \mathbf{Ez} \otimes \mathbf{z}, \mathbf{B}_{\perp} + \mathbf{Gz} \otimes \mathbf{z}), \tag{4.2}$$

and the *limit loads*:

$$\mathbf{f}_{\perp}(\zeta) = \int_B \mathbf{f}(x, \zeta) dx_1 dx_2, \quad \mathbf{c}_{\perp}(\zeta) = \int_B \mathbf{c}(x, \zeta) dx_1 dx_2. \tag{4.3}$$

Remark 4.1. *It follows from the definition of σ_{\perp} that:*

$$\sigma_{\perp}(\mathbf{E}, \mathbf{G}) = \sigma_{\perp}(\mathbf{Ez} \otimes \mathbf{z}, \mathbf{Gz} \otimes \mathbf{z}).$$

Furthermore σ_{\perp} is a positive-definite quadratic function if it is restricted to the elements of \mathbf{Ez} and \mathbf{Gz} (see also [9]) and this ensures existence and uniqueness for the solution of the limit equilibrium problem (4.7).

We define the *limit functional* $\Pi_{\perp} : \mathcal{K}_{\perp}(\Omega) \rightarrow \mathbb{R}$ by setting:

$$\Pi_{\perp}(\mathbf{u}, \boldsymbol{\omega}) := \Sigma_{\perp}(\mathbf{u}, \boldsymbol{\omega}) - \Lambda_{\perp}(\mathbf{u}, \boldsymbol{\omega}) \quad \forall (\mathbf{u}, \boldsymbol{\omega}) \in \mathcal{K}(\Omega_{\perp}), \tag{4.4}$$

where

$$\Sigma_{\perp}(\mathbf{u}, \boldsymbol{\omega}) := \int_0^L \sigma_{\perp}(\mathbf{E}(\mathbf{u}, \boldsymbol{\omega}), \mathbf{G}(\boldsymbol{\omega})), \quad \Lambda_{\perp}(\mathbf{u}, \boldsymbol{\omega}) := \int_0^L \{\mathbf{f}_{\perp} \cdot \mathbf{u} + \mathbf{c}_{\perp} \cdot \boldsymbol{\omega}\}. \tag{4.5}$$

4.1. Γ -convergence. In order to describe the convergence of minimum problems, we will use the notion of Γ -convergence. We recall its definition and the main properties we will use. For more details see for instance [2, 4]).

Definition 4.2. *Let X be a metric space ¹. For $n \in \mathbb{N}$ let F_n and F be functionals defined on X and taking values on $\mathbb{R} \cup \{+\infty\}$.*

We say that F_n Γ -converges to F if the two following conditions are satisfied for each $x \in X$:

(i) *for every sequence $\{x_n\}$ converging to x , there holds:*

$$\liminf_{n \rightarrow \infty} F_n(x_n) \geq F(x);$$

(ii) *for every $\eta > 0$ there exists $\{x_n\}$ converging to x such that:*

$$\limsup_{n \rightarrow \infty} F_n(x_n) \leq F(x) + \eta.$$

¹Actually, this requirement can be weakened taking on X a topology fulfilling the first axiom of countability

Usually conditions (i) and (ii) are respectively referred to as *Liminf Inequality* and *Approximate Limsup Inequality*. The latter is indeed equivalent to the standard *Limsup Inequality* [2, §1.2].

Definition 4.3. A sequence of functionals $F_n : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be sequentially equi-coercive if, for any sequence $\{x_n\}$ such that $\sup_n F_n(x_n) < +\infty$, there exists a convergent subsequence.

Theorem 4.4. Let $\{F_n\}$ be a sequence of sequentially equicoercive functionals defined on X and Γ -converging to F . Then there exists $\min_X F$ and $\min_X F = \liminf_{n \rightarrow \infty} \min_X F_n$. Moreover, if x_n is a minimizer of F_n , then every limit of a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is a minimizer of F .

Theorem 4.5. Let $\{F_n\}$ be a sequence of functionals Γ -converging to F and $\{G_n\}$ be a sequence of continuous (with respect to the same topology for which the Γ -limit is computed) functionals uniformly converging to G . Then the family $\{F_n + G_n\}$ Γ -converges to $F + G$.

Definition 4.6. Let $\{F_h\}_{h>0}$ be a family of functionals labeled by a continuous parameter h . We say that $\{F_h\}_{h>0}$ Γ -converges to F as h goes to 0, if $\{F_{h_n}\}_{n \in \mathbb{N}}$ Γ -converges to F for every subsequence $\{h_n\}_{n \in \mathbb{N}}$ converging to zero as n goes to $+\infty$. In simple way the statements of Theorem 4.4, and Theorem 4.5 can be rephrased in this framework.

We now state our main results.

Theorem 4.7 (Γ -convergence for the total energies). Let us consider the extended functionals $\overline{\Pi}^{(h)}, \overline{\Pi}_\perp : \mathbf{L}^2(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ defined as:

$$\overline{\Pi}^{(h)}(\mathbf{u}, \boldsymbol{\omega}) := \begin{cases} \Pi^{(h)}(\mathbf{u}, \boldsymbol{\omega}) & \text{if } (\mathbf{u}, \boldsymbol{\omega}) \in \mathcal{K}(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

$$\overline{\Pi}_\perp(\mathbf{u}, \boldsymbol{\omega}) := \begin{cases} \Pi_\perp(\mathbf{u}, \boldsymbol{\omega}) & \text{if } (\mathbf{u}, \boldsymbol{\omega}) \in \mathcal{K}_\perp(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Then $\overline{\Pi}^{(h)}$ Γ -converges to $\overline{\Pi}_\perp$ with respect to the strong topology of $\mathbf{L}^2(\Omega)$.

Theorem 4.8 (Compactness for the total energies). Let $\{(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)})\} \subset \mathcal{K}(\Omega)$ be a sequence such that

$$\sup_{h>0} \Pi^{(h)}(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)}) < +\infty.$$

Then there exist a subsequence of $\{(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)})\}$ (not relabeled) and $(\mathbf{u}, \boldsymbol{\omega}) \in \mathbf{H}^1(\Omega)$ such that $(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)})$ converges to $(\mathbf{u}, \boldsymbol{\omega})$ with respect to the strong topology of $\mathbf{L}^2(\Omega)$.

Corollary 4.9. *Let assumptions (3.4) and (4.1) hold. Then, as h tends to 0, we have:*

$$\min_{(\mathbf{u}, \boldsymbol{\omega}) \in \mathcal{K}(\Omega)} \Pi^{(h)}(\mathbf{u}, \boldsymbol{\omega}) \rightarrow \min_{(\mathbf{u}, \boldsymbol{\omega}) \in \mathcal{K}_\perp(\Omega)} \Pi_\perp(\mathbf{u}, \boldsymbol{\omega}), \tag{4.6}$$

and the solutions $(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)})$ of problems $\min_{(\mathbf{u}, \boldsymbol{\omega}) \in \mathcal{K}(\Omega)} \Pi^{(h)}(\mathbf{u}, \boldsymbol{\omega})$ converge, strongly in $\mathbf{H}^1(\Omega)$, to the unique solution $(\mathbf{u}, \boldsymbol{\omega})$ of the limit equilibrium problem:

$$\min_{(\mathbf{u}, \boldsymbol{\omega}) \in \mathcal{K}_\perp(\Omega)} \Pi_\perp(\mathbf{u}, \boldsymbol{\omega}). \tag{4.7}$$

Proof. By Theorems 4.4, 4.7, 4.8 it immediately follows that (4.6) holds. Moreover, arguing as in [9, §6.3] (with some minor modification), the convergence of the solutions of minimum problems with respect to the strong topology of $\mathbf{H}^1(\Omega)$ is achieved. \square

4.2. Proof of Theorem 4.7. Since the sequence $\Lambda^{(h)}$ converges to Λ_\perp uniformly with respect to the strong topology of $\mathbf{L}^2(\Omega)$, recalling Theorem 4.5, it is enough to study the Γ -limit of $\Sigma^{(h)}$.

We here introduce some additional notation. To avoid nested subscripts, we replace h_n with h in the following statement and, as a rule, we do not relabel subsequences. Positive constants are denoted by C or by C_i . For $\{\boldsymbol{\omega}^{(h)}\} \subset H^1(\Omega; \mathbb{R}^3)$ we set $\mathbf{W}^{(h)} := \mathbf{A}_{\boldsymbol{\omega}^{(h)}}$. Moreover we use the shorthand notation: $\mathbf{E}^{(h)} = \mathbf{E}^{(h)}(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)})$, $\mathbf{G}^{(h)} = \mathbf{G}^{(h)}(\boldsymbol{\omega}^{(h)})$ (cf. (3.8)). Then, recalling (2.4), we write:

$$\mathbf{E}^{(h)} = \frac{1}{h} \nabla_\perp \mathbf{u}^{(h)} + \partial_\zeta \mathbf{u}^{(h)} \otimes \mathbf{z} - \mathbf{W}^{(h)}, \quad \mathbf{G}^{(h)} = \frac{1}{h} \nabla_\perp \boldsymbol{\omega}^{(h)} + \partial_\zeta \boldsymbol{\omega}^{(h)} \otimes \mathbf{z},$$

and

$$\mathbf{E}_\perp^{(h)} = \nabla_\perp \mathbf{u}^{(h)} - \mathbf{W}_\perp^{(h)}, \quad \mathbf{G}_\perp^{(h)} = \nabla_\perp \boldsymbol{\omega}^{(h)}.$$

Let us consider the extended functionals $\overline{\Sigma}^{(h)}$, $\Sigma_\perp : \mathbf{L}^2(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ defined as:

$$\overline{\Sigma}^{(h)}(\mathbf{u}, \boldsymbol{\omega}) := \begin{cases} \Sigma^{(h)}(\mathbf{u}, \boldsymbol{\omega}) & \text{if } (\mathbf{u}, \boldsymbol{\omega}) \in \mathcal{K}(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

$$\overline{\Sigma}_\perp(\mathbf{u}, \boldsymbol{\omega}) := \begin{cases} 2 \int_{\Omega_\perp} \sigma_\perp(\mathbf{E}_\perp(\mathbf{u}, \boldsymbol{\omega}), \mathbf{G}_\perp(\boldsymbol{\omega})) dx_1 dx_2 & \text{if } (\mathbf{u}, \boldsymbol{\omega}) \in \mathcal{K}_\perp(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Then the family $\{\overline{\Sigma}^{(h)}\}$ Γ -converges to $\overline{\Sigma}_\perp$ with respect to the strong topology of $\mathbf{L}^2(\Omega)$. The proof of the *Liminf Inequality* closely follows the steps in [9, §6.2] and we omit it. We now prove the *Approximate Limsup Inequality*, which takes the form: *for every $(\mathbf{u}, \boldsymbol{\omega}) \in \mathcal{K}_\perp(\Omega)$ and for every $\eta > 0$ there exists a sequence $\{(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)})\} \subset \mathcal{K}(\Omega)$ such that*

$$(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)}) \rightarrow (\mathbf{u}, \boldsymbol{\omega}) \text{ in } \mathbf{L}^2(\Omega)$$

and

$$\limsup_{h \rightarrow 0} \Sigma^{(h)}(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)}) \leq \Sigma_{\perp}(\mathbf{u}, \boldsymbol{\omega}) + \eta.$$

Let $(\mathbf{u}, \boldsymbol{\omega}) \in \mathcal{K}_{\perp}(\Omega)$ and $\eta > 0$ be fixed, $\mathbf{E} = \nabla \mathbf{u} - \mathbf{W}$ and $\mathbf{G} = \nabla \boldsymbol{\omega}$. For every $(x, \zeta) \in \Omega$, let $(\mathbf{A}_{\perp}(x, \zeta), \mathbf{B}_{\perp}(x, \zeta))$ be the unique pair such that the minimum in (4.2) with $\mathbf{E} \equiv \mathbf{E}(x, \zeta)$ and $\mathbf{G} \equiv \mathbf{G}(x, \zeta)$ is attained. We have

$$\Sigma_{\perp}(\mathbf{u}, \boldsymbol{\omega}) = \sigma(\mathbf{A}_{\perp} + \mathbf{Ez} \otimes \mathbf{z}, \mathbf{B}_{\perp} + \mathbf{Gz} \otimes \mathbf{z}). \quad (4.8)$$

For every $\varepsilon > 0$, let $(\mathbf{A}_{\varepsilon}, \mathbf{B}_{\varepsilon}) \in H^1(\Omega; \mathbb{R}^{3 \times 2}) \times H^1(\Omega; \mathbb{R}^{3 \times 2})$ be such that

$$\|(\mathbf{A}_{\varepsilon} - \mathbf{A}_{\perp}, \mathbf{B}_{\varepsilon} - \mathbf{B}_{\perp})\|_{L^2(\Omega; \mathbb{R}^{3 \times 2}) \times L^2(\Omega; \mathbb{R}^{3 \times 2})} \leq \varepsilon. \quad (4.9)$$

Actually, \mathbf{u} does not depend on x , thus we shall henceforth write, with abuse of notation, $\mathbf{u}(\zeta)$ in place $\mathbf{u}(x, \zeta)$. We shall adopt such convention for $\boldsymbol{\omega}$, \mathbf{A}_{\perp} , \mathbf{B}_{\perp} , \mathbf{A}_{ε} , and \mathbf{B}_{ε} as well. For every (x, ζ) in Ω we define

$$\mathbf{u}_{\varepsilon}^{(h)}(x, \zeta) := \mathbf{u}(\zeta) + h(x_1 \mathbf{A}_{\varepsilon}(\zeta) \mathbf{e}_1 + x_2 \mathbf{A}_{\varepsilon}(\zeta) \mathbf{e}_2 + \boldsymbol{\omega}(\zeta) \times \mathbf{z})$$

and

$$\boldsymbol{\omega}_{\varepsilon}^{(h)}(x, \zeta) := \boldsymbol{\omega}(\zeta) + hx_1 \mathbf{B}_{\varepsilon}(\zeta) \mathbf{e}_1 + hx_2 \mathbf{B}_{\varepsilon}(\zeta) \mathbf{e}_2.$$

We claim that for ε small enough the sequence $\{\mathbf{u}_{\varepsilon}^{(h)}, \boldsymbol{\omega}_{\varepsilon}^{(h)}\}$ satisfies the Approximate Limsup Inequality. This sequence converges to $(\mathbf{u}, \boldsymbol{\omega})$, moreover, for suitable $\mathbf{E}_1, \mathbf{E}_2$ belonging to $L^2(\Omega; \mathbb{R}^{3 \times 3})$ we have

$$\begin{aligned} \mathbf{E}^{(h)}(\mathbf{u}_{\varepsilon}^{(h)}, \boldsymbol{\omega}_{\varepsilon}^{(h)}) &= \frac{1}{h} \nabla_{\perp} \mathbf{u}_{\varepsilon}^{(h)} + \partial_{\zeta} \mathbf{u}_{\varepsilon}^{(h)} \otimes \mathbf{z} - \mathbf{A}_{\boldsymbol{\omega}_{\varepsilon}^{(h)}} \\ &= \mathbf{A}_{\varepsilon} + \mathbf{Ez} \otimes \mathbf{z} + h \mathbf{E}_1 + h^2 \mathbf{E}_2, \end{aligned}$$

and

$$\mathbf{G}^{(h)}(\boldsymbol{\omega}_{\varepsilon}^{(h)}) = \mathbf{B}_{\varepsilon} + \partial_{\zeta} \boldsymbol{\omega} \otimes \mathbf{z}.$$

Since σ is a quadratic function, it immediately follows:

$$\Sigma^{(h)}(\mathbf{u}_{\varepsilon}^{(h)}, \boldsymbol{\omega}_{\varepsilon}^{(h)}) = \int_{\Omega} \sigma(\mathbf{A}_{\varepsilon} + \mathbf{Ez} \otimes \mathbf{z}, \mathbf{B}_{\varepsilon} + \mathbf{Gz} \otimes \mathbf{z}) + o_h(1), \quad (4.10)$$

where $\lim_{h \rightarrow 0} o_h(1) = 0$. Moreover, we have

$$\begin{aligned} \int_{\Omega} \sigma(\mathbf{A}_{\varepsilon} + \mathbf{Ez} \otimes \mathbf{z}, \mathbf{B}_{\varepsilon} + \mathbf{Gz} \otimes \mathbf{z}) &\leq \int_{\Omega} \sigma(\mathbf{A}_{\perp} + \mathbf{Ez} \otimes \mathbf{z}, \mathbf{B}_{\perp} + \mathbf{Gz} \otimes \mathbf{z}) + \frac{\eta}{2} + \\ &\quad + C_{\eta} \|(\mathbf{A}_{\varepsilon} - \mathbf{A}_{\perp}, \mathbf{B}_{\varepsilon} - \mathbf{B}_{\perp})\|_{L^2(\Omega; \mathbb{R}^{3 \times 2}) \times L^2(\Omega; \mathbb{R}^{3 \times 2})} \end{aligned}$$

for $C_{\eta} > 0$ large enough. Therefore, selecting $\varepsilon \leq C_{\eta}^{-1} \eta/2$ and passing to the limit in (4.10), we obtain:

$$\lim_{h \rightarrow 0} \Sigma^{(h)}(\mathbf{u}_{\varepsilon}^{(h)}, \boldsymbol{\omega}_{\varepsilon}^{(h)}) = \int_{\Omega} \sigma(\mathbf{A}_{\perp} + \mathbf{Ez} \otimes \mathbf{z}, \mathbf{B}_{\perp} + \mathbf{Gz} \otimes \mathbf{z}) + \eta,$$

and the thesis follows from (4.8).

4.3. Proof of Theorem 4.8. First we show that, if $\{(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)})\}$ is such that $\sup_{h>0} \Pi^{(h)}(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)}) < +\infty$, then it holds:

$$\sup_{h>0} \Sigma^{(h)}(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)}) < +\infty. \quad (4.11)$$

To this aim, we note that:

$$\Pi^{(h)}(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)}) \leq \Pi^{(h)}(\mathbf{0}, \mathbf{0}) = 0, \quad (4.12)$$

and therefore recalling (3.9), (3.10) and using Hölder's inequality, we get:

$$\Sigma^{(h)}(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)}) \leq \|(\mathbf{f}^{(h)}, \mathbf{c}^{(h)})\|_{\mathbf{L}^2(\Omega)} \|(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)})\|_{\mathbf{L}^2(\Omega)} \leq C \|(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)})\|_{\mathbf{L}^2(\Omega)}, \quad (4.13)$$

where the last inequality follows from assumption (4.1). Moreover, using Poincaré's inequality, equation (2.5) and the inequality $|\mathbf{A} - \mathbf{B}|^2 + |\mathbf{B}|^2 \geq C(|\mathbf{A}|^2 + |\mathbf{B}|^2)$ (holding for all 3×3 matrices \mathbf{A} and \mathbf{B}), we get:

$$\begin{aligned} \Sigma^{(h)}(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)}) &\geq C_1 \int_{\Omega} |\nabla^{(h)} \mathbf{u}^{(h)} - \mathbf{W}^{(h)}|^2 + |\nabla^{(h)} \boldsymbol{\omega}^{(h)}|^2 \\ &\geq C_2 \int_{\Omega} |\nabla^{(h)} \mathbf{u}^{(h)} - \mathbf{W}^{(h)}|^2 + |\mathbf{W}^{(h)}|^2 \\ &\geq C_3 \int_{\Omega} |\nabla^{(h)} \mathbf{u}^{(h)}|^2 + |\mathbf{W}^{(h)}|^2 \\ &\geq C_4 \|(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)})\|_{\mathbf{L}^2(\Omega)}^2. \end{aligned}$$

Hence, using the so-called Young's inequality with ε , we obtain:

$$C_5 \|(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)})\|_{\mathbf{L}^2(\Omega)} - \frac{1}{2\varepsilon} \|(\mathbf{f}^{(h)}, \mathbf{c}^{(h)})\|_{\mathbf{L}^2(\Omega)} - \frac{\varepsilon}{2} \|(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)})\|_{\mathbf{L}^2(\Omega)}^2 \leq \Pi^{(h)}(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)}).$$

Choosing $\varepsilon > 0$ small enough and using (4.12), we have:

$$\|(\mathbf{u}^{(h)}, \boldsymbol{\omega}^{(h)})\|_{\mathbf{L}^2(\Omega)} \leq C.$$

The last inequality combined with (4.13) yields (4.11). By (4.11), using the coercivity condition (3.4), we get:

$$\sup_{h>0} \left(\|\mathbf{E}^{(h)}\|_{L^2(\Omega; \mathbb{R}^3)}^2 + \|\mathbf{G}^{(h)}\|_{L^2(\Omega; \mathbb{R}^3)}^2 \right) < +\infty.$$

Hence $\frac{1}{h} \nabla_{\perp} \boldsymbol{\omega}^{(h)}$ and $\partial_{\zeta} \boldsymbol{\omega}^{(h)}$ are bounded in $L^2(\Omega; \mathbb{R}^{3 \times 2})$ and $L^2(\Omega; \mathbb{R}^3)$ respectively; thus, by extracting a subsequence (not relabeled), we get:

$$\nabla_{\perp} \boldsymbol{\omega}^{(h)} \rightharpoonup \mathbf{0}, \quad \frac{1}{h} \nabla_{\perp} \boldsymbol{\omega}^{(h)} \rightharpoonup \mathbf{B} \quad \text{in } L^2(\Omega; \mathbb{R}^{3 \times 2}).$$

By Poincaré's inequality, $\boldsymbol{\omega}^{(h)}$ is bounded in $H^1(\Omega; \mathbb{R}^3)$ and, by extracting a further subsequence, it follows:

$$\boldsymbol{\omega}^{(h)} \rightharpoonup \boldsymbol{\omega} \text{ in } H^1(\Omega; \mathbb{R}^3) \text{ with } \nabla_{\perp} \boldsymbol{\omega} = \mathbf{0}. \quad (4.14)$$

By definition:

$$\frac{1}{h} \nabla_{\perp} \mathbf{u}^{(h)} + \partial_{\zeta} \mathbf{u}^{(h)} \otimes \mathbf{z} = \mathbf{E}^{(h)} + \mathbf{W}^{(h)}. \quad (4.15)$$

The right-hand side of (4.15) is a bounded sequence in $L^2(\Omega; \mathbb{R}^{3 \times 3})$ and, arguing as in the derivation of (4.14), we have:

$$\mathbf{u}^{(h)} \rightharpoonup \mathbf{u} \text{ in } H^1(\Omega; \mathbb{R}^3) \text{ with } \nabla_{\perp} \mathbf{u} = \mathbf{0} .$$

Finally we observe that, by well known results of functional analysis, the selected subsequence converges with respect to the strong topology of $\mathbf{L}^2(\Omega)$.

Remark 4.10. *Since $\frac{1}{h}(\nabla_{\perp} \mathbf{u}^{(h)}, \nabla_{\perp} \boldsymbol{\omega}^{(h)})$ is bounded in $\mathbf{L}^2(\Omega)$, we have that*

$$(\nabla_{\perp} \mathbf{u}^{(h)}, \nabla_{\perp} \boldsymbol{\omega}^{(h)}) \rightarrow 0 \text{ in } \mathbf{L}^2(\Omega) .$$

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