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POSITIVE PERIODIC SOLUTIONS FOR SYSTEMS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We consider classes of systems of first order functional differential equations. Criteria are established for the existence of positive T-periodic solutions of the systems under consideration. One example is also included to illustrate the applications of our results.

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1. INTRODUCTION

Let $n \geq 1$ be an integer, $T \geq 0$, and $\mathbb{R}_+ = [0, \infty)$. For $i = 1, \ldots, n$, let $a_i, \tau_i \in C(\mathbb{R}, \mathbb{R})$, $b_i \in C(\mathbb{R}, \mathbb{R}_+)$ be T-periodic functions and $f_i \in C(\mathbb{R}_+^n, \mathbb{R}_+)$. In this paper, we are concerned with the existence of positive T-periodic solutions of the system of first order functional differential equations

$$u'(t) = -A(t)u(t) + B(t)f(g(u(t))), \tag{1.1}$$

where

$$\begin{aligned} u(t) &= (u_1(t), \ldots, u_n(t))^T, \\ A(t) &= \text{diag}\,[a_1(t), \ldots, a_n(t)], \\ B(t) &= \text{diag}\,[b_1(t), \ldots, b_n(t)], \\ f(g(u(t))) &= (f_1(g(u(t))), \ldots, f_n(g(u(t))))^T, \end{aligned}$$

and

$$g(u(t)) = (u_1(t - \tau_1(t)), \ldots, u_n(t - \tau_n(t)))^T.$$

We also obtain the existence of positive T-periodic solutions of the associated eigenvalue problem

$$u'(t) = -A(t)u(t) + \lambda B(t)f(g(u(t))), \tag{1.2}$$

where λ is a positive parameter. By a positive T-periodic solution of (1.1), we mean a function $u \in C^1(\mathbb{R}, \mathbb{R}_+^n)$ such that $u(t)$ satisfies (1.1) and at least one component of $u(t)$ is positive on $\mathbb{R}$. Similar definition also applies for system (1.2).

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Functional differential equations with periodic delays appear in a number of applications, such as in the model of blood cell productions in an animal [6, 16], the control of testosterone levels in the blood stream [13], and so on. In recent years, the existence of positive periodic solutions of such equations has been investigated by many authors; see, for example, [1, 2, 5, 9, 10, 11, 12, 15, 18] and the references therein. In particular, the scalar case of system (1.2) has been studied in [1, 2, 12, 15]. In this paper, by means of fixed point index theory, we obtain several sufficient conditions for the existence of positive T-periodic solutions of systems (1.1) and (1.2). Some of our results involve the smallest positive characteristic values of some related linear operators to the systems. The technique used in this paper has been previously employed in the literature such as in the papers [3, 7, 14]. But all of these papers treated scalar differential equations. To the best of our knowledge, this paper is the first work to establish some eigenvalue criteria for systems of differential equations.

We assume throughout, and without further mention, that the following assumption holds:

(H) $\int_0^T a_i(v)dv > 0$ and $\int_0^T b_i(v)dv > 0$ for $i = 1, \ldots, n$.

The rest of this paper is organized as follows. Section 2 contains some preliminary lemmas, Sections 3 contains the main results of this paper and one simple example for demonstration, and the proofs of the main results are presented in Section 4.

2. PRELIMINARY RESULTS

For $i = 1, \ldots, n$ and $t, s \in \mathbb{R}^2$, define

$$G_i(t,s) = \frac{\exp\left(\int_t^s a_i(v)dv\right)}{\exp\left(\int_0^T a_i(v)dv\right) - 1},$$

$$c_i = \min_{0 \le s,t \le T} G_i(t,s), \text{ and } d_i = \max_{0 \le s,t \le T} G_i(t,s). \tag{2.1}$$

Then, it is easy to see that $d_i > c_i > 0$,

$$c_i \le G_i(t,s) \le d_i \quad \text{if } t \le s \le t+T. \tag{2.2}$$

For $i = 1, \ldots, n$, consider the scalar equation

$$y' = -a_i(t)y + h(t) \tag{2.3}$$

where $h \in C(\mathbb{R}, \mathbb{R})$ is a T-periodic function.

The following lemma can be directly verified.

Lemma 2.1. *For $i = 1, \ldots, n$, $y(t)$ is a T-periodic solution of* (2.3) *if and only if*

$$y(t) = \int_t^{t+T} G_i(t,s)h(s)ds.$$

Let X be a Banach space and $L : X \to X$ be a linear operator. We recall that λ is an eigenvalue of L with a corresponding eigenvector ϕ if ϕ is nontrivial and $L\phi = \lambda\phi$. The reciprocals of eigenvalues are called the characteristic values of L. The spectral radius of L, denoted by $r(L)$, is given by the well known spectral radius formula $r(L) = \lim_{k\to\infty} ||L^k||^{1/k}$. Recall also that a cone P in X is called a total cone if $X = \overline{P - P}$.

We refer the reader to [4, Theorem 19.2] or [17, Proposition 7.26] for the following well known Krein-Rutman theorem.

Lemma 2.2. *Assume that P is a total cone in a real Banach space X. Let $L : X \to X$ be a compact linear operator with $L(P) \subseteq P$ and $r(L) \in (0, \infty)$. Then $r(L)$ is an eigenvalue of L with an eigenvector in P.*

Throughout this paper, let the Banach space X be defined by

$$X = \{u \in C(\mathbb{R}, \mathbb{R}^n) \ : \ u(t + T) = u(t) \text{ for } t \in \mathbb{R}\}$$

equipped with the norm $||u|| = \sum_{i=1}^n ||u_i||_\infty$, where $u = (u_1, \dots, u_n)$ and $||u_i||_\infty = \sup_{t\in\mathbb{R}} |u_i(t)|$. Define a cone P in X by

$$P = \{u \in X \ : \ u(t) \geq 0 \text{ on } \mathbb{R}\}. \tag{2.4}$$

Let

$$\sigma_i = \frac{c_i}{d_i}, \quad i = 1, \dots, n, \quad \text{and} \quad \sigma = \min_{1\leq i\leq n} \sigma_i. \tag{2.5}$$

We also define a subcone K of P by

$$K = \{u \in P \ : \ u = (u_1, \dots, u_n), \ u_i(t) \geq \sigma ||u_i||_\infty \text{ on } \mathbb{R}\}. \tag{2.6}$$

For $u = (u_1, \dots, u_n) \in X$, let the linear operator $L : X \to X$ be defined by

$$Lu(t) = (L_1u(t), \dots, L_nu(t))^T, \tag{2.7}$$

where

$$L_iu(t) = \int_t^{t+T} G_i(t, s)b_i(s) \left(\sum_{j=1}^n u_j(s - \tau_j(s))\right) ds \tag{2.8}$$

for $i = 1, \dots, n$.

The next two lemmas provide some information about the operator L.

Lemma 2.3. *The operator L maps P into K and is compact.*

Proof. We first show $L(P) \subseteq K$. For $u = (u_1, \dots, u_n) \in P$, $t \in \mathbb{R}$, and $i = 1, \dots, n$, from (2.2), we have

$$L_iu(t) \geq c_i \int_0^T b_i(s) \left(\sum_{j=1}^n u_j(s - \tau_j(s))\right) ds$$

and

$$L_i u(t) \le d_i \int_0^T b_i(s) \left(\sum_{j=1}^n u_j(s - \tau_j(s)) \right) ds,$$

from which it follows that

$$L_i u(t) \ge (c_i/d_i)||L_i u||_\infty = \sigma_i ||L_i u||_\infty \ge \sigma ||L_i u||_\infty.$$

Hence, $L(P) \subseteq K$. A standard argument can be used to show that L is compact and we omit the details here. This completes the proof of the lemma. □

Lemma 2.4. *The spectral radius,* $r(L)$*, of* L *satisfies* $r(L) \in (0, \infty)$*. Moreover,* $r(L)$ *is an eigenvalue of* L *with an eigenvector* $\phi_L \in P$*.*

Proof. By the spectral theory in Banach spaces (see, for example, [17]), it is clear that $r(L) < \infty$. In the following, we show $r(L) > 0$. Let $u = (u_1, \dots, u_n) \in K$ and $t \in \mathbb{R}$. For $i = 1, \dots, n$, we have

$$\begin{aligned} L_i u(t) &\ge c_i \int_0^T b_i(s) \left(\sum_{j=1}^n u_j(s - \tau_j(s)) \right) ds \\ &\ge \sigma \left(\sum_{j=1}^n ||u_j||_\infty \right) c_i \int_0^T b_i(s) ds = \sigma ||u|| c_i \int_0^T b_i(s) ds \end{aligned} \tag{2.9}$$

and

$$L^2 u(t) = (L_1(Lu(t)), \dots, L_n(Lu(t)))^T.$$

For $i = 1, \dots, n$, from (2.9), we have

$$\begin{aligned} L_i(Lu(t)) &= \int_t^{t+T} G_i(t,s) b_i(s) \left(\sum_{j=1}^n L_j u(s - \tau_j(s)) \right) ds \\ &\ge c_i \int_t^{t+T} b_i(s) \left(\sigma ||u|| \sum_{j=1}^n c_j \int_0^T b_j(s) ds \right) ds \\ &= \sigma ||u|| \left(\sum_{j=1}^n c_j \int_0^T b_j(s) ds \right) c_i \int_0^T b_i(s) ds. \end{aligned}$$

Then,

$$||L_i(Lu)||_\infty \ge \sigma ||u|| \left(\sum_{j=1}^n c_j \int_0^T b_j(s) ds \right) c_i \int_0^T b_i(s) ds,$$

which in turn implies that

$$\begin{aligned} ||L^2 u|| &= \sum_{i=1}^n ||L_i(Lu)||_\infty \\ &\ge \sigma ||u|| \left(\sum_{i=1}^n c_i \int_0^T b_i(s) ds \right)^2. \end{aligned}$$

For $k \in \mathbb{N}$, note that

$$L^k u(t) = (L_1(L^{k-1}u(t)), \ldots, L_n(L^{k-1}u(t)))^T.$$

By induction, we can obtain that

$$||L^k u|| \geq \sigma ||u|| \left(\sum_{i=1}^{n} c_i \int_0^T b_i(s) ds \right)^k.$$

Hence,

$$||L^k||\, ||u|| \geq ||L^k u|| \geq \sigma ||u|| \left(\sum_{i=1}^{n} c_i \int_0^T b_i(s) ds \right)^k.$$

As a result,

$$||L^k|| \geq \sigma \left(\sum_{i=1}^{n} c_i \int_0^T b_i(s) ds \right)^k.$$

Then, from Assumption (H), we have

$$r(L) = \lim_{k \to \infty} ||L^k||^{1/k} \geq \sum_{i=1}^{n} c_i \int_0^T b_i(s) ds > 0.$$

Now, since $r(L) \in (0, \infty)$ and the cone P defined by (2.4) is a total cone, the "moreover" part readily follows from Lemmas 2.2 and 2.3. This completes the proof of the lemma. □

Let $r(L)$ and ϕ_L be given as in Lemma 2.4 and let

$$\phi_L = (\phi_{L,1}, \ldots, \phi_{L,n})$$

and

$$\mu_L = \frac{1}{r(L)}. \tag{2.10}$$

Then, it is clear that μ_L is the smallest positive characteristic value of L satisfying $\phi_L = \mu_L L \phi_L$.

Define

$$\xi = \frac{1}{\sum_{i=1}^{n} d_i \int_0^T b_i(s) ds} \quad \text{and} \quad \eta = \frac{1}{\sigma \sum_{i=1}^{n} c_i \int_0^T b_i(s) ds}. \tag{2.11}$$

The following lemma give some useful estimates for μ_L.

Lemma 2.5. *The characteristic value μ_L satisfies $\xi \leq \mu_L \leq \eta$.*

Proof. For $i = 1, \ldots, n$ and $t \in \mathbb{R}$, from $\phi_L = \mu_L L \phi_L$, it follows that $\phi_{L,i} = \mu_L L_i \phi_L$, i.e.,

$$\phi_{L,i}(t) = \mu_L \int_t^{t+T} G_i(t,s) b_i(s) \left(\sum_{j=1}^{n} \phi_{L,j}(s - \tau_j(s)) \right) ds. \tag{2.12}$$

Then,

$$\phi_{L,i}(t) \leq \mu_L \left(\sum_{j=1}^{n} ||\phi_{L,j}||_\infty \right) d_i \int_t^{t+T} b_i(s)ds = \mu_L ||\phi_L|| d_i \int_0^T b_i(s)ds,$$

and so

$$||\phi_{L,i}||_\infty \leq \mu_L ||\phi_L|| d_i \int_0^T b_i(s)ds.$$

Hence,

$$||\phi_L|| = \sum_{i=1}^{n} ||\phi_{L,i}|| \leq \mu_L ||\phi_L|| \sum_{i=1}^{n} d_i \int_0^T b_i(s)ds.$$

Thus,

$$\mu_L \geq \frac{1}{\sum_{i=1}^{n} d_i \int_0^T b_i(s)ds} = \xi.$$

On the other hand, from (2.12), we have

$$\phi_{L,i}(t) \geq \mu_L \sigma \left(\sum_{j=1}^{n} ||\phi_{L,j}||_\infty \right) c_i \int_0^T b_i(s)ds = \mu_L \sigma ||\phi_L|| c_i \int_0^T b_i(s)ds.$$

Then,

$$||\phi_{L,i}||_\infty \geq \mu_L \sigma ||\phi_L|| c_i \int_0^T b_i(s)ds.$$

Hence,

$$||\phi_L|| = \sum_{i=1}^{n} ||\phi_{L,i}|| \geq \mu_L \sigma ||\phi_L|| \sum_{i=1}^{n} c_i \int_0^T b_i(s)ds,$$

from which we have

$$\mu_L \leq \frac{1}{\sigma \sum_{i=1}^{n} c_i \int_0^T b_i(s)ds} = \eta.$$

This completes the proof of the lemma. □

We also need the following two well known lemmas. We refer the reader to [8, Corollary 2.3.1. and Lemma 2.3.1.], respectively, for their proofs.

Lemma 2.6. *Let X be a Banach space and $K \subseteq X$ be a cone. Assume that Ω is a bounded open subset of X and that $T : K \cap \overline{\Omega} \to K$ is compact. If there exists $u_0 \in K \setminus \{0\}$ such that*

$$u - Tu \neq \tau u_0 \quad \textit{for all } u \in K \cap \partial\Omega \textit{ and } \tau \geq 0.$$

Then the fixed point index

$$i(T, K \cap \Omega, K) = 0.$$

Lemma 2.7. *Let X be a Banach space and $K \subseteq X$ be a cone. Assume that Ω is a bounded open subset of X with $0 \in \Omega$ and that $T : K \cap \overline{\Omega} \to K$ is compact. If*

$$u \neq \tau T u \quad \textit{for all } u \in K \cap \partial\Omega \textit{ and } \tau \in [0, 1].$$

Then the fixed point index

$$i(T, K \cap \Omega, K) = 1.$$

3. MAIN RESULTS

In this section, we state our existence results. For convenience, we introduce the following notations. For any $x = (x_1, \dots, x_n) \in \mathbb{R}^n_+$, let $|x| = \sum_{j=1}^n x_j$, and for $i = 1, \dots, n$, define

$$f_{i,0} = \liminf_{|x|\to 0^+} \frac{f_i(x)}{|x|}, \quad f_{i,\infty} = \liminf_{|x|\to\infty} \frac{f_i(x)}{|x|},$$

$$F_{i,0} = \limsup_{|x|\to 0^+} \frac{f_i(x)}{|x|}, \quad F_{i,\infty} = \limsup_{|x|\to\infty} \frac{f_i(x)}{|x|},$$

In the sequel, we let μ_L be defined by (2.10) and ξ and η be given in (2.11). We now state a result for system (1.1).

Theorem 3.1. *Assume either*

$$F_{i,0} < \mu_L < f_{i,\infty} \quad \textit{for } i = 1, \dots, n, \tag{3.1}$$

or

$$F_{i,\infty} < \mu_L < f_{i,0} \quad \textit{for } i = 1, \dots, n. \tag{3.2}$$

Then system (1.1) *has at least one positive T-periodic solution.*

The following corollaries are immediate consequences of Theorem 3.1.

Corollary 3.2. *Assume either*

$$\frac{F_{i,0}}{\xi} < 1 < \frac{f_{i,\infty}}{\eta} \quad \textit{for } i = 1, \dots, n, \tag{3.3}$$

or

$$\frac{F_{i,\infty}}{\xi} < 1 < \frac{f_{i,0}}{\eta} \quad \textit{for } i = 1, \dots, n. \tag{3.4}$$

Then system (1.1) *has at least one positive T-periodic solution.*

Corollary 3.3. *Assume either*

$$\frac{\eta}{f_{i,\infty}} < \lambda < \frac{\xi}{F_{i,0}} \quad \textit{for } i = 1, \dots, n, \tag{3.5}$$

or

$$\frac{\eta}{f_{i,0}} < \lambda < \frac{\xi}{F_{i,\infty}} \quad \textit{for } i = 1, \dots, n. \tag{3.6}$$

Then system (1.2) *has at least one positive T-periodic solution.*

In what follows, we establish several results for the existence of multiple positive T-periodic solutions of system (1.1). To do so, we first introduce the following conditions that describe the "smallness" and "largeness" of $f(x)$ on "slabs" of $\mathbb{R}_+^n$.

(A1) For $i = 1, \dots, n$, there exists $p_1 > 0$ such that

$$f_i(x) < p_1\xi \quad \text{for any } x = (x_1, \dots, x_n) \in \mathbb{R}_+^n \text{ with } \sum_{j=1}^{n} x_j \leq p_1.$$

(A2) For $i = 1, \dots, n$, there exists $p_2 > 0$ such that

$$f_i(x) > \sigma p_2 \eta \quad \text{for any } x = (x_1, \dots, x_n) \in \mathbb{R}_+^n \text{ with } \sigma p_2 \leq \sum_{j=1}^{n} x_j \leq p_2,$$

where σ is defined in (2.5).

The following criteria provide sufficient conditions for the existence of multiple positive T-periodic solutions of system (1.1).

Theorem 3.4. *Assume one of the following conditions holds:*

(B1) $F_{i,0} < \mu_L$ *for* $i = 1, \dots, n$, *and (A1) and (A2) hold with* $p_1 > p_2$*;*
(B2) $f_{i,\infty} > \mu_L$ *for* $i = 1, \dots, n$, *and (A1) and (A2) hold with* $p_1 > p_2$*;*
(B3) $f_{i,0} > \mu_L$ *for* $i = 1, \dots, n$, *and (A1) and (A2) hold with* $p_1 < p_2$*;*
(B4) $F_{i,\infty} < \mu_L$ *for* $i = 1, \dots, n$, *and (A1) and (A2) hold with* $p_1 < p_2$*;*
(B5) $f_{i,0} > \mu_L$ *and* $f_{i,\infty} > \mu_L$ *for* $i = 1, \dots, n$, *and (A1) holds;*
(B6) $F_{i,0} < \mu_L$ *and* $F_{i,\infty} < \mu_L$ *for* $i = 1, \dots, n$, *and (A2) holds.*

Then system (1.1) *has at least two positive* T*-periodic solutions.*

Moreover, if either both (B1) *and* (B2) *hold or both* (B3) *and* (B4) *hold, then system* (1.1) *has at least three positive* T*-periodic solutions.*

Corollary 3.5. *Assume one of the following conditions holds:*

(C1) $F_{i,0} < \xi$ *for* $i = 1, \dots, n$, *and (A1) and (A2) hold with* $p_1 > p_2$*;*
(C2) $f_{i,\infty} > \eta$ *for* $i = 1, \dots, n$, *and (A1) and (A2) hold with* $p_1 > p_2$*;*
(C3) $f_{i,0} > \eta$ *for* $i = 1, \dots, n$, *and (A1) and (A2) hold with* $p_1 < p_2$*;*
(C4) $F_{i,\infty} < \xi$ *for* $i = 1, \dots, n$, *and (A1) and (A2) hold with* $p_1 < p_2$*;*
(C5) $f_{i,0} > \eta$ *and* $f_{i,\infty} > \eta$ *for* $i = 1, \dots, n$, *and (A1) holds;*
(C6) $F_{i,0} < \xi$ *and* $F_{i,\infty} < \xi$ *for* $i = 1, \dots, n$, *and (A2) holds.*

Then system (1.1) *has at least two positive* T*-periodic solutions.*

Moreover, if either both (C1) *and* (C2) *hold or both* (C3) *and* (C4) *hold, then system* (1.1) *has at least three positive* T*-periodic solutions.*

Corollary 3.6. *In system* (1.1)*, assume that, for* $i = 1, \ldots, n$*,* $a_i(t) = a > 0$*,* $b_i(t) = b > 0$*,* $\tau_i \in C(\mathbb{R}, \mathbb{R})$ *is* T*-periodic, and for* $x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$*,* f_i *satisfies either*

$$\begin{cases} f_i(x) = l_{i,1}|x|^{k_{i,1}} & \text{if } 0 \le |x| \le m \max_{1 \le i \le n}(l_{i,1}C^{-1})^{1/(1-k_{i,1})}, \\ f_i(x) \ge \nu_1 D|x| & \text{if } |x| \text{ is large enough}, \end{cases} \tag{3.7}$$

or

$$\begin{cases} f_i(x) = l_{i,2}|x|^{k_{i,2}} & \text{if } 0 \le |x| \le m e^{aT} \max_{1 \le i \le n}(l_{i,2}^{-1}D)^{1/(k_{i,2}-1)}, \\ f_i(x) \le \nu_2 C|x| & \text{if } |x| \text{ is large enough}, \end{cases} \tag{3.8}$$

where $0 < k_{i,1} < 1$*,* $k_{i,2} > 1$*,* $l_{i,1}, l_{i,2} > 0$*,* $m > 1$*,* $\nu_1 > 1$*,* $\nu_2 < 1$*, and*

$$C = \frac{e^{aT} - 1}{nbTe^{aT}}, \quad D = \frac{e^{aT}(e^{aT} - 1)}{nbT}. \tag{3.9}$$

Then system (1.1) *has at least two positive* T*-periodic solutions.*

Results similar to Theorem 3.4 and Corollaries 3.5 and 3.6 can be easily formulated for system (1.2). We leave this to the interested reader.

We conclude this section with the following example.

Example. For $i = 1, 2$, let $a_i(t)$ and $b_i(t)$ be nonnegative T-periodic continuous functions satisfying (H), and for $(y, z) \in \mathbb{R}_+^2$, define

$$f_i(y, z) = \begin{cases} (y+z)^{\alpha_i}, & y + z < 1, \\ \frac{100^{\beta_i} - 1}{99}(y + z - 1) + 1, & 1 \le y + z \le 100, \\ (y+z)^{\beta_i}, & y + z > 100, \end{cases} \tag{3.10}$$

where $\alpha_i, \beta_i \in \mathbb{R}_+$. Clearly, $f_i \in C(\mathbb{R}_+^2, \mathbb{R}_+)$.

Let ξ and η be defined by (2.11) with the above $a_i(t)$ and $b_i(t)$. Then we claim that if either

(D1) $\alpha_i < 1$, $\beta_i > 1$, $i = 1, 2$, and $\xi > 1$, or
(D2) $\alpha_i > 1$, $\beta_i < 1$, $i = 1, 2$, and $\eta < 1/100^{1-\hat{\beta}}$, where $\hat{\beta} = \min\{\beta_i, \beta_2\}$,

then the system

$$\begin{cases} u'(t) = -a_1(t)u(t) + b_1(t)f_1(u(t), v(t)), \\ v'(t) = -a_2(t)v(t) + b_1(t)f_2(u(t), v(t)), \end{cases}$$

has at least two positive T-periodic solutions.

Proof of the Claim. We first assume (D1) holds. Then, for $i = 1, 2$, from (3.10), we see that

$$f_{i,0} = f_{i,\infty} = \infty. \tag{3.11}$$

Moreover, from $\xi > 1$, we have $(1/\xi)^{1/(1-\alpha_i)} < 1$. Then, we can choose a constant p_1 such that

$$(1/\xi)^{1/(1-\alpha_i)} < p_1 < 1.$$

Thus, $p_1\xi > p_1^{\alpha_i}$. Hence, for $(y,z) \in \mathbb{R}_+^2$ with $y+z \le p_1$, from (3.10), we have

$$f_i(y,z) = (y+z)^{\alpha_i} \le p_1^{\alpha_i} < p_1\xi,$$

i.e., $(A1)$ holds. Then, in view of (3.11), (C5) of Corollary 3.5 holds. The claim now follows from Corollary 3.5.

Next, we assume (D2) holds. Then, for $i = 1, 2$, from (3.10), we see that

$$F_{i,0} = F_{i,\infty} = 0. \tag{3.12}$$

Moreover, from $\eta < 1/100^{1-\hat{\beta}}$, we have $(1/\eta)^{1/(1-\hat{\beta})} > 100$. Then, we can choose a constant p_2 such that

$$100 < \sigma p_2 < (1/\eta)^{1/(1-\hat{\beta})},$$

where σ is defined in (2.5). Thus, $(\sigma p_2)^{\hat{\beta}} > \sigma p_2 \eta$. Hence, for $(y,z) \in \mathbb{R}_+^2$ with $\sigma p_2 \le y+z \le p_2$, from (3.10), we have

$$f_i(y,z) = (y+z)^{\beta_i} \ge (y+z)^{\hat{\beta}} \ge (\sigma p_2)^{\hat{\beta}} > \sigma p_2 \eta,$$

i.e., $(A2)$ holds. Then, in view of (3.12), (C6) of Corollary 3.5 holds. The claim again follows from Corollary 3.5.

4. PROOFS OF THE MAIN RESULTS

Define an operator $\mathcal{T} : X \to X$ by $\mathcal{T}u = (\mathcal{T}_1 u, \ldots, \mathcal{T}_n u)$, where

$$\mathcal{T}_i u(t) = \int_t^{t+T} G_i(t,s) b_i(s) f_i(g(u(s)))ds. \tag{4.1}$$

By Lemma 2.1, a T-periodic solution of system (1.1) is equivalent to a fixed point of the operator $\mathcal{T}$. Let K be defined by (2.6). Using a similar argument as in the proof of Lemma 2.3, it is easy to see that $\mathcal{T}(K) \subseteq K$. Moreover, a standard argument shows that $\mathcal{T}$ is compact.

Proof of Theorem 3.1. We first assume (3.1) holds. For $i = 1, \ldots, n$, since $f_{i,\infty} > \mu_L$, there exists $R_1 > 0$ such that

$$f_i(x) \ge \mu_L |x| = \mu_L \sum_{j=1}^n x_j \tag{4.2}$$

for any $x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$ with $|x| = \sum_{j=1}^n x_j > \sigma R_1$. Let

$$\Omega_1 = \{u \in X \ : \ ||u|| < R_1\}.$$

Then, for $u = (u_1, \ldots, u_n) \in K \cap \partial\Omega_1$ and $t \in \mathbb{R}$, we have

$$\sum_{j=1}^n u_j(t - \tau_j(t)) \ge \sigma \sum_{j=1}^n ||u_j||_\infty = \sigma ||u|| = \sigma R_1.$$

Hence, from (4.2),

$$f_i(g(u(t))) \ge \mu_L \sum_{j=1}^n u_j(t - \tau_j(t)). \tag{4.3}$$

Now, in view of (2.8), (4.1), and (4.3), we reach that

$$\mathcal{T}_i u(t) \geq \mu_L \int_t^{t+T} G_i(t,s) b_i(s) \sum_{j=1}^n u_j(s-\tau_j(s))ds = \mu_L L_i u(t) > 0.$$

Thus,

$$\mathcal{T}u(t) \geq \mu_L L u(t) > 0. \tag{4.4}$$

We may suppose that $\mathcal{T}$ has no fixed point on $K \cap \partial\Omega_1$. Otherwise, we can see that system (1.1) has a positive T-periodic solution and the proof is then finished. Let ϕ_L be given as in Lemma 2.4. Then, $\phi_L(t) > 0$ on $\mathbb{R}$ and $\phi_L = \mu_L L \phi_L$. In the following, we show that

$$u - \mathcal{T}u \neq \tau\phi_L \quad \text{for all } u \in K \cap \partial\Omega_1 \text{ and } \tau \geq 0. \tag{4.5}$$

If this is not the case, then there exists $u^* \in K \cap \partial\Omega_1$ and $\tau^* \geq 0$ such that $u^* - \mathcal{T}u^* = \tau^*\phi_L$. Thus, $\tau^* > 0$ and

$$u^* = \mathcal{T}u^* + \tau^*\phi_L > \tau^*\phi_L.$$

Define

$$\tau_1 = \sup\{\tau \ : \ u^* \geq \tau\phi_L\}.$$

Then, $\tau_1 \geq \tau^* > 0$, $u^* \geq \tau_1\phi_L$, and so from (4.4),

$$u^* = \mathcal{T}u^* + \tau^*\phi_L \geq \mu_L L u^* + \tau^*\phi_L \geq (\tau_1 + \tau^*)\phi_L,$$

which contradicts the definition of τ_1. Hence, (4.5) holds. By Lemma 2.6, we have

$$i(\mathcal{T}, K \cap \Omega_1, K) = 0. \tag{4.6}$$

For $i = 1, \ldots, n$, since $F_{i,0} < \mu_L$, there exist $0 < \epsilon < 1$ and $0 < R_2 < R_1$ such that

$$f_i(x) \leq (1-\epsilon)\mu_L |x| = (1-\epsilon)\mu_L \sum_{j=1}^n x_j \tag{4.7}$$

for any $x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$ with $|x| = \sum_{j=1}^n x_j \leq R_2$. Let

$$\Omega_2 = \{u \in X \ : \ ||u|| < R_2\}.$$

For $u = (u_1, \ldots, u_n) \in K \cap \partial\Omega_2$ and $t \in \mathbb{R}$, from (4.7),

$$f_i(g(u(t))) \leq (1-\epsilon)\mu_L \sum_{j=1}^n u_j(t-\tau_j(t)).$$

Hence,

$$\begin{aligned} \mathcal{T}_i u(t) &\leq (1-\epsilon)\mu_L \int_t^{t+T} G_i(t,s) b_i(s) \sum_{j=1}^n u_j(s-\tau_j(s))ds \\ &= (1-\epsilon)\mu_L L_i u(t), \end{aligned}$$

i.e.,

$$\mathcal{T}u(t) \leq (1-\epsilon)\mu_L L u(t). \tag{4.8}$$

We now claim that

$$u \neq \tau \mathcal{T} u \quad \text{for all } u \in K \cap \partial\Omega_2 \text{ and } \tau \in [0,1]. \tag{4.9}$$

For otherwise, there exist $u^* = (u_1^*, \ldots, u_n^*) \in K \cap \partial\Omega_2$ and $\tau^* \in [0,1]$ such that $u^* = \tau^* \mathcal{T} u^*$. Since $u^* = (u_1^*, \ldots, u_n^*) \in K \cap \partial\Omega_2$, we have that

$$||u^*|| = \sum_{i=1}^{n} ||u_i^*||_\infty = R_2 > 0. \tag{4.10}$$

Note that

$$u_i^*(t) \geq \sigma ||u_i^*||_\infty \quad \text{for } i = 1, \ldots, n \text{ and } \ t \in \mathbb{R}. \tag{4.11}$$

Then, from (4.10) and (4.11), there exists at least one $i_0 \in \{1, \ldots, n\}$ such that

$$u_{i_0}^*(t) > 0 \quad \text{on } \mathbb{R}. \tag{4.12}$$

Moreover, from (4.11), we also have the observation that for $i \in \{1, \ldots, n\} \setminus \{i_0\}$

$$\text{either} \quad u_i^*(t) \equiv 0 \quad \text{or} \quad u_i^*(t) > 0 \quad \text{on } \mathbb{R}. \tag{4.13}$$

Let

$$I = \{i \in \{1, \ldots, n\} \ : \ u_i^*(t) > 0 \quad \text{on } \mathbb{R}\}. \tag{4.14}$$

Then $i_0 \in I$ (and hence $I \neq \emptyset$) and $I \subseteq \{1, \ldots, n\}$. From (4.13), it is also obvious that if I is a proper subset of $\{1, \ldots, n\}$, then $u_i^*(t) \equiv 0$ on $\mathbb{R}$ for $t \in \{1, \ldots, n\} \setminus I$.

Note that

$$\phi_L = (\phi_{L,1}, \ldots, \phi_{L,n})$$

and

$$\phi_{L,i}(t) = \mu_L \int_t^{t+T} G_i(t,s) b_i(s) \left(\sum_{j=1}^{n} \phi_{L,j}(s - \tau_j(s)) \right), \ i = 1, \ldots, n.$$

Then, from the facts that $\phi_L \geq (0, \ldots, 0)$ is nontrivial and $G_i(t,s) \geq c_i > 0$ on $\mathbb{R}$, it follows that

$$\phi_{L,i}(t) > 0 \quad \text{for all } i = 1, \ldots, n \text{ and } t \in \mathbb{R}. \tag{4.15}$$

For $i \in \{1, \ldots, n\}$, let

$$k_i = \max_{t \in \mathbb{R}} \frac{u_i^*(t)}{\phi_{L,i}(t)} = \max_{t \in [0,T]} \frac{u_i^*(t)}{\phi_{L,i}(t)}.$$

Then, in view of (4.14) and (4.15), k_i is well defined, $k_i > 0$ for $i \in I$, and

$$0 < u_i^*(t) \leq k_i \phi_{L,i}(t) \quad \text{for } i \in I \text{ and } t \in \mathbb{R}. \tag{4.16}$$

Let the set S be defined by

$$S = \{\tau \ : \ u^* \leq \tau \phi_L\},$$

i.e.,

$$S = \{\tau \ : \ (u_1^*, \ldots, u_n^*) \leq \tau(\phi_{L,1}, \ldots, \phi_{L,n})\}.$$

Then, from (4.16), we see that $\max_{i\in I} k_i \in S$ (In fact, any number larger than or equal to $\max_{i\in I} k_i \in S$ is in S.), so, $S \neq \emptyset$. Define

$$\tau_2 = \inf S,$$

i.e.,

$$\tau_2 = \inf\{\tau \ : \ (u_1^*, \dots, u_n^*) \leq \tau(\phi_{L,1}, \dots, \phi_{L,n})\}.$$

Then, from (4.13), (4.14), and (4.16), it is clear that τ_2 is well defined, $u^* \leq \tau_2\phi_L$, and

$$\tau_2 = \max_{i\in I} k_i \geq k_{i_0} > 0.$$

Thus,

$$\mu_L L u^* \leq \mu_L L(\tau_2\phi_L) = \tau_2\mu_L L\phi_L = \tau_2\phi_L.$$

This, together with (4.8), implies that

$$u^* = \tau^*\mathcal{T}u^* \leq \mathcal{T}u^* \leq (1-\epsilon)\tau_2\phi_L,$$

which contradicts the definition of τ_2. Therefore, (4.9) holds. By Lemma 2.7, it follows that

$$i(\mathcal{T}, K \cap \Omega_2, K) = 1. \tag{4.17}$$

By (4.6), (4.17), and the additivity property of the fixed point index, we obtain that

$$i(\mathcal{T}, K \cap (\Omega_1 \setminus \overline{\Omega}_2), K) = -1.$$

Thus, from the solution property of the fixed point index, $\mathcal{T}$ has at least one fixed point $u = (u_1, \dots, u_n)$ in $K \cap (\Omega_1 \setminus \overline{\Omega}_2)$, which is a T-periodic solution of system (1.1). Since $||u|| = \sum_{i=1}^n ||u_i||_\infty > R_2$ and $u_i(t) \geq ||u_i||_\infty$ on $\mathbb{R}$ for $i = 1, \dots, n$, at least one component of u is positive on $\mathbb{R}$, i.e., $u(t)$ is a positive solution.

Now, we assume (3.2) holds. For $i = 1, \dots, n$, since $f_{i,0} > \mu_L$, there exists $R_3 > 0$ such that

$$f_i(x) \geq \mu_L|x| = \mu_L\sum_{j=1}^n x_j$$

for any $x = (x_1, \dots, x_n) \in \mathbb{R}_+^n$ with $|x| = \sum_{j=1}^n x_j < R_3$. Let

$$\Omega_3 = \{u \in X \ : \ ||u|| < R_3\}.$$

Then, for $u \in K \cap \partial\Omega_3$ and $t \in \mathbb{R}$, we have

$$f_i(g(u(t))) \geq \mu_L\sum_{j=1}^n u_j(t - \tau_j(t)). \tag{4.18}$$

From (2.8), (4.1), and (4.18), we see that

$$\mathcal{T}_i u(t) \geq \mu_L\int_t^{t+T} G_i(t,s)b_i(s)\sum_{j=1}^n u_j(s - \tau_j(s))ds = \mu_L L_i u(t) > 0.$$

Thus,

$$\mathcal{T}u(t) \geq \mu_L L u(t) > 0.$$

Now, by an argument similar to the one used in proving that (4.5) holds, we can show that

$$u - \mathcal{T}u \neq \tau\phi_L \quad \text{for all } u \in K \cap \partial\Omega_3 \text{ and } \tau \geq 0.$$

Then, by Lemma 2.6,

$$i(\mathcal{T}, K \cap \Omega_3, K) = 0. \tag{4.19}$$

For $i = 1, \ldots, n$, since $F_{i,\infty} < \mu_L$, there exist $0 < \epsilon < 1$ and $R_4 > R_3$ such that

$$f_i(x) \leq (1-\epsilon)\mu_L|x| = (1-\epsilon)\mu_L \sum_{j=1}^{n} x_j \tag{4.20}$$

for any $x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$ with $|x| = \sum_{j=1}^n x_j \geq \sigma R_4$. Let

$$\Omega_4 = \{u \in X \ : \ ||u|| < R_4\}.$$

For any $u = (u_1, \ldots, u_n) \in K \cap \partial\Omega_4$ and $t \in \mathbb{R}$, we have

$$\sum_{j=1}^{\infty} u_j(t - \tau_j(t)) \geq \sigma \sum_{j=1}^{n} ||u_j||_\infty = \sigma||u|| \geq \sigma R_4.$$

From (4.20), it follows that

$$f_i(g(u(t))) \leq (1-\epsilon)\mu_L \sum_{j=1}^{n} u_j(t - \tau_j(t)).$$

Then, $u(t)$ satisfies (4.8), Now, as in verifying that (4.9) holds, we obtain

$$u \neq \tau\mathcal{T}u \quad \text{for all } u \in K \cap \partial\Omega_4 \text{ and } \tau \in [0,1].$$

Lemma 2.7 then implies

$$i(\mathcal{T}, K \cap \Omega_4, K) = 1. \tag{4.21}$$

By (4.19), (4.21), and the additivity property of the fixed point index, we obtain

$$i(\mathcal{T}, K \cap (\Omega_4 \setminus \overline{\Omega}_3), K) = 1.$$

Thus, from the solution property of the fixed point index, $\mathcal{T}$ has at least one fixed point v in $K \cap (\Omega_4 \setminus \overline{\Omega}_3)$, which is a T-periodic solution of system (1.1). As in the previous case, $u(t)$ is positive. This completes the proof of the theorem. □

Proof of Corollary 3.2. The conclusion follows from Lemma 2.5 and Theorem 3.1. □

Proof of Corollary 3.3. The conclusion readily follows from Corollary 3.2. □

Proof of Theorem 3.4. We first assume (B1) holds. For $i = 1, \ldots, n$, since $F_{i,0} < \mu_L$, there exist $0 < \epsilon < 1$ and $0 < R_1 < p_2$ such that

$$f_i(x) \leq (1-\epsilon)\mu_L|x| = (1-\epsilon)\mu_L \sum_{j=1}^{n} x_j$$

for any $x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$ with $|x| = \sum_{j=1}^n x_j \leq R_1$. Let

$$\Omega_1 = \{u \in X \ : \ ||u|| < R_1\}. \tag{4.22}$$

Using a similar argument as in verifying (4.9), we obtain

$$u \neq \tau \mathcal{T} u \quad \text{for all } u \in K \cap \partial\Omega_1 \text{ and } \tau \in [0,1].$$

By Lemma 2.7,

$$i(\mathcal{T}, K \cap \Omega_1, K) = 1. \tag{4.23}$$

Let

$$\Omega_2 = \{u \in X \ : \ ||u|| < p_1\}. \tag{4.24}$$

Then, for $i = 1, \dots, n$, $u \in K \cap \partial\Omega_2$, and $t \in \mathbb{R}$, from (A1), we see that

$$0 \leq \mathcal{T}_i u(t) \leq d_i \int_0^T b_i(s) f_i(g(u(s)))ds < p_1 \xi d_i \int_0^T b_i(s)ds.$$

Thus,

$$||\mathcal{T}_i u||_\infty < p_1 \xi d_i \int_0^T b_i(s)ds.$$

Hence, in view of (2.11), we have

$$||\mathcal{T} u|| = \sum_{i=1}^{\infty} ||\mathcal{T}_i u||_\infty < p_1 \xi \sum_{i=1}^{\infty} d_i \int_0^T b_i(s)ds = p_1 = ||u||.$$

By Lemma 2.7, it follows that

$$i(\mathcal{T}, K \cap \Omega_2, K) = 1. \tag{4.25}$$

Let

$$\Omega_3 = \{u \in X \ : \ ||u|| < p_2\}. \tag{4.26}$$

For $u = (u_1, \dots, u_n) \in K \cap \partial\Omega_3$ and $t \in \mathbb{R}$, we have

$$\sigma p_1 = \sigma ||u|| = \sigma \sum_{j=1}^{n} ||u_j||_\infty \leq \sum_{j=1}^{n} u_j(t - \tau_j(t)) \leq \sum_{j=1}^{n} ||u_j||_\infty = ||u|| = p_1.$$

Then, for $i = 1, \dots, n$, from (A2),

$$\mathcal{T}_i u(t) \geq c_i \int_0^T b_i(s) f_i(g(u(s)))ds > \sigma p_2 \eta c_i \int_0^T b_i(s)ds,$$

which implies that

$$||\mathcal{T}_i u||_\infty > \sigma p_2 \eta c_i \int_0^T b_i(s)ds.$$

Thus, in view of (2.11), we see that

$$||\mathcal{T} u|| = \sum_{i=1}^{\infty} ||\mathcal{T}_i u||_\infty > \sigma p_2 \eta \sum_{i=1}^{\infty} c_i \int_0^T b_i(s)ds = p_2 = ||u||.$$

By Lemma 2.6,

$$i(\mathcal{T}, K \cap \Omega_3, K) = 0. \tag{4.27}$$

Since $p_1 > p_2 > R_1$, from (4.23), (4.25), and (4.27), we reach the conclusions that

$$i(\mathcal{T}, K \cap (\Omega_3 \setminus \overline{\Omega}_1), K) = -1 \tag{4.28}$$

and

$$i(\mathcal{T}, K \cap (\Omega_2 \setminus \overline{\Omega}_3), K) = 1. \tag{4.29}$$

Hence, $\mathcal{T}$ has at least two fixed points, $u_1 \in K \cap (\Omega_3 \setminus \overline{\Omega}_1)$ and $u_2 \in K \cap (\Omega_2 \setminus \overline{\Omega}_3)$. As in the proof of Theorem 3.1, it is clear that $u_1(t)$ and $u_2(t)$ are two positive T-periodic solutions of system (1.1).

Now, we assume (B2) holds. For $i = 1, \ldots, n$, since $f_{i,\infty} > \mu_L$, there exists $R_2 > p_1$ such that

$$f_i(x) \geq \mu_L |x| = \mu_L \sum_{j=1}^{n} x_j$$

for any $x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$ with $|x| = \sum_{j=1}^n x_j > \sigma R_2$. Let

$$\Omega_4 = \{u \in X \ : \ ||u|| < R_2\}. \tag{4.30}$$

As in demonstrating (4.5), we have

$$u - \mathcal{T}u \neq \tau \phi_L \quad \text{for all } u \in K \cap \partial\Omega_4 \text{ and } \tau \geq 0.$$

By Lemma 2.6,

$$i(\mathcal{T}, K \cap \Omega_4, K) = 0. \tag{4.31}$$

Note that (4.25) and (4.27) still hold in this case. Then, in view of the fact that $R_2 > p_1 > p_2$ and from (4.25), (4.27), and (4.31), it follows that (4.29) holds and

$$i(\mathcal{T}, K \cap (\Omega_4 \setminus \overline{\Omega}_2), K) = -1. \tag{4.32}$$

Therefore, $\mathcal{T}$ has at least two fixed points, $u_1 \in K \cap (\Omega_2 \setminus \overline{\Omega}_3)$ and $u_2 \in K \cap (\Omega_4 \setminus \overline{\Omega}_2)$, which are two positive T-periodic solutions of system (1.1).

The cases where (B3) and (B4) hold can be proved similarly to cases (B1) and (B2). We omit the details here.

Next, we assume (B5) holds. Since (A1) holds and $f_{i,\infty} > \mu_L$ for $i = 1, \ldots, n$, there exists $R_2 > p_1$ such that (4.25) and (4.31) hold, where Ω_2 and Ω_4 are defined by (4.24) and (4.30) with p_1 and the above R_2. For $i = 1, \ldots, n$, since $f_{i,0} > \mu_L$, using a similar argument as in verifying (4.19), there exists $R_3 < p_1$ such that

$$i(\mathcal{T}, K \cap \Omega_5, K) = 0, \tag{4.33}$$

where

$$\Omega_5 = \{u \in X \ : \ ||u|| < R_3\}.$$

Since $R_2 > p_1 > R_3$, from (4.25), (4.31), and (4.33), we see that

$$i(\mathcal{T}, K \cap (\Omega_2 \setminus \overline{\Omega}_5), K) = 1$$

and

$$i(\mathcal{T}, K \cap (\Omega_4 \setminus \overline{\Omega}_2), K) = -1.$$

Hence, $\mathcal{T}$ has at least two fixed points, $u_1 \in K \cap (\Omega_2 \setminus \overline{\Omega}_5)$ and $u_2 \in K \cap (\Omega_4 \setminus \overline{\Omega}_2)$, which are two positive T-periodic solutions of system (1.1).

The proof when (B6) holds is similar to that of case (B5) and so we omit the details.

Finally, we prove the "moreover" part of the theorem when both (B1) and (B2) hold. The proof when both (B3) and (B4) hold is similar. As before, we derive (4.23), (4.25), (4.27), and (4.31). Hence, (4.28), (4.29), and (4.32) hold. Thus, $\mathcal{T}$ has at least three fixed points $u_1 \in K \cap (\Omega_3 \setminus \overline{\Omega}_1)$, $u_2 \in K \cap (\Omega_2 \setminus \overline{\Omega}_3)$, and $u_3 \in K \cap (\Omega_4 \setminus \overline{\Omega}_2)$, which are three positive T-periodic solutions of system (1.1). This completes the proof of the theorem. $\square$

Proof of Corollary 3.5. The conclusion follows from Lemma 2.5 and Theorem 3.4. $\square$

Proof of Corollary 3.6. We first assume (3.8) holds. For ξ and η given in (2.11), by a simple computation, we see that $\xi = C$ and $\eta = D$, where C and D are defined in (3.9). Then, from (3.7), it follows that

$$f_{i,0} = \lim_{|x|\to 0^+} \frac{f_i(x)}{|x|} = \infty > \eta \quad \text{and} \quad f_{i,\infty} = \liminf_{|x|\to\infty} \frac{f_i(x)}{|x|} \geq \nu_1 D > \eta.$$

Moreover, for any p_1 satisfying

$$\max_{1\leq i\leq n} (l_{i,1}C^{-1})^{1/(1-k_{i,1})} < p_1 \leq m \max_{1\leq i\leq n} (l_{i,1}C^{-1})^{1/(1-k_{i,1})},$$

we have

$$f_i(x) \leq l_{i,1} p_1^{k_{i,1}} < p_1 C = p_1 \xi \quad \text{if } 0 \leq |x| \leq p_1,$$

i.e., (A1) holds. Thus, (C5) of Corollary 3.5 holds. By a similar argument, we can show that (C6) of Corollary 3.5 holds if f_i satisfies (3.8). The conclusion then follows from Corollary 3.5. $\square$

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