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LARGE TIME BEHAVIOR OF MULTIDIMENSIONAL NONLINEAR LATTICES WITH NONLINEAR DAMPING

J. C. OLIVEIRA[1], J. M. PEREIRA[2], AND G. PERLA MENZALA[3]

[1,2]Department of Mathematics, Federal University of Santa Catarina
Florianópolis, SC 88040-900 Brasil
E-mails: jauber@mtm.ufsc.br, jardel@mtm.ufsc.br

[3]National Laboratory of Scientific Computation (LNCC/MCT)
Petrópolis, RJ, 25651-070, Brasil and
and Federal University of Rio de Janeiro,
Rio de Janeiro, 68530, Brasil
E-mail: perla@lncc.br

ABSTRACT. In this paper we study the asymptotic behavior of solutions of multidimensional nonlinear lattices subject to cyclic boundary conditions under the effect of a nonlinear dissipation. We establish the existence of a global attractor.

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1. INTRODUCTION

This work is devoted to study the nature of vibrations which arise in a lattice structure. More precisely we will consider a system of point masses which in their state of rest have periodic distributions. These point masses have interactions which are coupled in a nonlinear way. Besides, a nonlinear damping mechanism is present in the model.

Nonlinear lattices, also called nonlinear networks are infinite systems of ordinary differential equations and appear very often in chemical reactions, image processing, biology, cellular automata among many other important areas in science and technology, see [4], [7], [15] and the references therein. In the last 30 or 40 years research on the so-called theory of deterministic chaos has influenced the study of global attractors and possible upper bounds for the Hausdorff dimension of such sets. This may be a difficult task. Therefore, the study of global attractors for nonlinear lattices seems to be a good alternative in order to "approximate" the global attractor for the partial differential equation modeling the continuous phenomenon.

In this article we are interested in the long time behavior of multidimensional nonlinear lattices under the effect of nonlinear damping. In order to simplify notation we assume that all "spatial" variables have the same period N. But, all our discussion could be easily

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rewritten to include lattices in which each "spatial" variable has a different period N_j with $1 \leq j \leq d$. In other words, the closure conditions are valid

$$u_{n_1,n_2,\ldots,n_d} = u_{n_1+N_1,n_2,\ldots,n_d} = u_{n_1+N_1,n_2+N_2,n_3,\ldots,n_d} = \cdots = u_{n_1+N_1,n_2+N_2,\ldots,n_d+N_d}$$

for any integers n_j, $j = 1, 2, \ldots, d$, where N_j are given positive integers. There is a quite large number of contributions in the literature where authors studied relevant properties of nonlinear lattices but almost all treated only the one-dimensional case ([2, 5, 6, 9, 10, 11, 14, 18] and the references therein). Only in recent years more attention on relevant multidimensional discrete models were given ([8], [12] and the references therein).

This article is organized as follows: In Section 2 we give the notation and assumptions to obtain our main results. Then we state our conclusions. Section 3 is devoted to prove global well-posedness of the nonlinear multidimensional lattice with nonlinear internal damping. In Section 4 we prove the existence of a global attractor. In the Appendix, we deduce the expression of the Green's function associated to the linear lattice using the discrete Fourier Transform as well as a proof of Poincaré's inequality when $d > 1$.

2. NOTATION, ASSUMPTIONS AND MAIN RESULTS

We denote by $\mathbb{Z}$ the set of integers. Let $d \in \mathbb{Z}^+$ and write $\mathbf{n} = (n_1, n_2, n_3, \ldots, n_d)$, where $n_j \in \mathbb{Z}$, $1 \leq j \leq d$. We write $\mathbb{Z}^d = \mathbb{Z} \times \mathbb{Z} \times \cdots \mathbb{Z}$ (d times). We consider a fixed positive integer N (the period) and denote by ℓ_{per} the set of sequences

$$\ell_{\text{per}} = \Big\{ a = \{a_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d};\ a_{\mathbf{n}} \in \mathbb{R},\ \sum_{n_1=1}^{N}\sum_{n_2=1}^{N}\cdots\sum_{n_d=1}^{N} a_{n_1,n_2,\ldots,n_d} = 0$$

and

$$a_{\mathbf{n}} = a_{n_1+N,n_2,\ldots,n_d} = a_{n_1,n_2+N,n_3,\ldots,n_d} = \cdots = a_{n_1,n_2,\ldots,n_{d-1},n_d+N},\ \forall \mathbf{n} \Big\}. \tag{2.1}$$

From now on $\sum\limits_{n_1=1}^{N}\sum\limits_{n_2=1}^{N}\cdots\sum\limits_{n_d=1}^{N} a_{n_1,n_2,\ldots,n_d}$ will be written as $\sum\limits_{\mathbf{n}=1}^{N} a_{\mathbf{n}}$. We will denote by $\|\ \|$ the norm of ℓ_{per}, which is given by $\|a\| = \left(\sum\limits_{\mathbf{n}=1}^{N} a_{\mathbf{n}}^2\right)^{1/2}$, whenever $a = \{a_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ belongs to ℓ_{per}. Next, we define the discrete operators $\partial_j^+, \partial_j^-, \nabla^+, \nabla^-$ and the discrete multidimensional Laplace operator Δ: Let $u_{\mathbf{n}} = u_{n_1,n_2,\ldots,n_d}$ and let us define for $1 \leq j \leq d$:

$$\begin{aligned}
\partial_j^+ u_{\mathbf{n}} &= u_{n_1,n_2,\ldots,n_{j-1},n_j+1,n_{j+1},\ldots,n_j} - u_{n_1,n_2,\ldots,n_j,n_{j+1},\ldots,n_d},\\
\partial_j^- u_{\mathbf{n}} &= u_{n_1,n_2,\ldots,n_j,n_{j+1},\ldots,n_d} - u_{n_1,n_2,\ldots,n_{j-1},n_j-1,n_{j+1},\ldots,n_d},\\
\nabla^+ u_{\mathbf{n}} &= (\partial_1^+ u_{\mathbf{n}}, \partial_2^+ u_{\mathbf{n}}, \ldots, \partial_d^+ u_{\mathbf{n}}),\\
\nabla^- u_{\mathbf{n}} &= (\partial_1^- u_{\mathbf{n}}, \partial_2^- u_{\mathbf{n}}, \ldots, \partial_d^- u_{\mathbf{n}}),
\end{aligned} \tag{2.2}$$

$$\begin{aligned}\Delta u_{\mathbf{n}} = & u_{n_1,n_2,\ldots,n_d+1} + u_{n_1,n_2,\ldots,n_{d-1}+1,n_d} + \cdots + \\ & + u_{n_1,n_2+1,n_3,\ldots,n_d} + u_{n_1+1,n_2,\ldots,n_d} \\ & + u_{n_1,n_2,\ldots,n_d-1} + u_{n_1,n_2,\ldots,n_{d-1}-1,n_d} + \cdots \\ & + u_{n_1,n_2-1,\ldots,n_d} + u_{n_1-1,n_2,\ldots,n_d} - 2du_{\mathbf{n}}.\end{aligned}$$

Thus, $\Delta = \sum_{j=1}^{d} \partial_j^- \, \partial_j^+ = \sum_{j=1}^{d} \left(\partial_j^+ - \partial_j^-\right)$.

Next, we describe our assumptions on the nonlinear terms of the model: Let h and g be vector-valued functions from $\mathbb{R}^d \mapsto \mathbb{R}^d$ such that:

1) h and g are functions of class C^1.
2) $h(x_1, x_2, \ldots, x_d) = (h_1(x_1), h_2(x_2), \ldots, h_d(x_d))$
   $g(x_1, x_2, \ldots, x_d) = (g_1(x_1), g_2(x_2), \ldots, g_d(x_d))$,
   for some functions $h_j(s)$, $g_j(s)$, $1 \le j \le d$.
3) h_j and g_j satisfy $h_j(0) = 0 = g_j(0)$ for any $1 \le j \le d$ and g is not identically zero.
4) $\tilde{h}_j(s) = \int_0^s h_j(\tau)d\tau \ge 0 \;\; \forall\; s \in \mathbb{R}, \;\; \forall\; 1 \le j \le d$.

The set of assumptions 1) up to 4) will be referred as (H1).

Let $f = \{f_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}}$, $a = \{a_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}}$, $b = \{b_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}}$ given in ℓ_{per}. We consider the following problem: Find a unique solution $u = \{u_{\mathbf{n}}(t)\}_{\mathbf{n}\in\mathbb{Z}}$, $t \ge 0$, such that

$$\begin{cases} \ddot{u}_{\mathbf{n}} - \Delta u_{\mathbf{n}} = \operatorname{div} h(\nabla u_{\mathbf{n}}) + \operatorname{div} g(\nabla \dot{u}_{\mathbf{n}}) + f_{\mathbf{n}} \\ u_{\mathbf{n}}(0) = a_{\mathbf{n}}, \;\; \dot{u}_{\mathbf{n}}(0) = b_{\mathbf{n}} \\ u_{\mathbf{n}}(t) \in \ell_{\text{per}} \quad \text{for all } t \ge 0 \text{ and any } \mathbf{n} \in \mathbb{Z}^d \end{cases} \tag{2.3}$$

where

$$\dot{u}_{\mathbf{n}} = \frac{du_{\mathbf{n}}}{dt}, \quad \ddot{u}_{\mathbf{n}} = \frac{d^2u_{\mathbf{n}}}{dt^2},$$

$$\operatorname{div} h(\nabla u_{\mathbf{n}}) = \sum_{j=1}^{d} [h_j(\partial_j^+ u_{\mathbf{n}}) - h_j(\partial_j^- u_{\mathbf{n}})]$$

and

$$\operatorname{div} g(\nabla \dot{u}_{\mathbf{n}}) = \sum_{j=1}^{d} [g_j(\partial_j^+ \dot{u}_{\mathbf{n}}) - g_j(\partial_j^- \dot{u}_{\mathbf{n}})].$$

Remark 1. Problem (2.3) could be consider as a semi-discrete version of the (continuous) model

$$u_{tt} - \Delta u = \operatorname{div} h(\nabla u) + \operatorname{div} g(\nabla u_t) + f.$$

Remark 2. Several authors define $\operatorname{div} h(\nabla u_{\mathbf{n}})$ and $\operatorname{div} g(\nabla \dot{u}_{\mathbf{n}})$ as $\sum_{j=1}^{d} [h_j(\partial_j^+ u_{\mathbf{n}}) + h_j(\partial_j^- u_{\mathbf{n}})]$ and $\sum_{j=1}^{d} [g_j(\partial_j^+ \dot{u}_{\mathbf{n}}) + g_j(\partial_j^- \dot{u}_{\mathbf{n}})]$, respectively. If we use their definition, we would require that $h_j, \; g_j$ are odd functions and not all g_j are identically zero.

In the next two sections we will prove the following results.

Theorem 2.1. (Global existence and uniqueness) *Let* $\{a_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$, $\{b_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$, $\{f_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ *be given in* ℓ_{per} *and (H1) holds. In addition, assume that* $sg_j(s) \geq 0, \forall\, s \in \mathbb{R}$ *and* $1 \leq j \leq d$*. Then, there exists a unique solution* $u(t) = \{u_{\mathbf{n}}(t)\}_{\mathbf{n}\in\mathbb{Z}^d}$ *of problem* (2.3) *and* $u_{\mathbf{n}}(t) \in C^2([0, +\infty; \mathbb{R}))$ *for any* $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$*.*

Theorem 2.2. (Existence of global attractor) *Under the assumptions of Theorem 2.1 suppose that there exist positive constants* k_0, k_1 *and* $p \geq 0$ *such that*

$$sh_j(s) \geq k_0\tilde{h}_j(s), \quad sg_j(s) \geq k_1|s|^{p+2},$$

for all $s \in \mathbb{R}$, $\forall\, 1 \leq j \leq d$*. Then, the semigroup of operators*

$$S(t)(a, b) = (u(t), \dot{u}(t)), \quad t \geq 0$$

associated to the solution $u(t)$ *of* (2.3) *has a global attractor* $\mathcal{A}$ *in the Hilbert space* $H = \ell_{\text{per}} \times \ell_{\text{per}}$ *with the norm* $||(a, b)||_H = \left(\sum\limits_{\mathbf{n}=1}^{N}(a_{\mathbf{n}}^2 + b_{\mathbf{n}}^2)\right)^{1/2}$, $\mathbf{n} = (n_1, n_2, \ldots, n_d)$*. The set* $\mathcal{A}$ *is compact and "attracts" every bounded set of* H *(in the topology of the norm in* H*).*

3. GLOBAL WELL POSEDNESS

Let $a = \{a_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ and $b = \{b_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ belong to ℓ_{per} and consider the linear problem associated with (2.3)

$$\begin{cases} \ddot{v}_{\mathbf{n}} - \Delta v_{\mathbf{n}} = 0 \\ v_{\mathbf{n}}(0) = a_{\mathbf{n}}, \quad \dot{v}_{\mathbf{n}}(0) = b_{\mathbf{n}} \\ v_{\mathbf{n}}(t) \in \ell_{\text{per}} \text{ for all } t \geq 0 \text{ and any } \mathbf{n} \in \mathbb{Z}^d. \end{cases} \tag{3.1}$$

The explicit solution of (3.1) is given by

$$v_{\mathbf{n}}(t) = \sum_{\mathbf{m}=1}^{N}[\dot{G}(\mathbf{n} - \mathbf{m}, t)a_{\mathbf{m}} + G(\mathbf{n} - \mathbf{m}, t)b_{\mathbf{m}}], \tag{3.2}$$

where $G(\mathbf{n}, t)$ is the discrete Green's function associated with (3.1). For purposes of self-containess we deduce $G(\mathbf{n}, t)$ in Appendix. Some results valid for $v_{\mathbf{n}}(t)$ are given in the Lemma 3.3.

Lemma 3.1. *Let* $b = \{b_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ *and* $c = \{c_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ *belong to* ℓ_{per}*. Then*

1) $\sum\limits_{\mathbf{n}=1}^{N} \Delta b_{\mathbf{n}} = 0,$ 2) $\sum\limits_{\mathbf{n}=1}^{N} b_{\mathbf{n}}\Delta c_{\mathbf{n}} = -\sum\limits_{\mathbf{n}=1}^{N} \nabla^+ b_{\mathbf{n}} \cdot \nabla^+ c_{\mathbf{n}},$

where $\nabla^+ b_{\mathbf{n}} \cdot \nabla^+ c_{\mathbf{n}} = \sum\limits_{j=1}^{d} \partial_j^+ b_{\mathbf{n}} \partial_j^+ c_{\mathbf{n}}.$

Proof: The proofs are straightforward calculations using the periodicity. □

Next, we need a discrete version of Poincaré's inequality.

Lemma 3.2. *Let $a = \{a_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d} \in \ell_{\text{per}}$. Then, there exists a positive constant $C = C(N, d)$ such that*

$$\sum_{n=1}^{N} a_{\mathbf{n}}^2 \leq C(N, d) \sum_{n=1}^{N} |\nabla^+ a_{\mathbf{n}}|^2,$$

where $|\nabla^+ a_{\mathbf{n}}|^2 = \sum_{j=1}^{d} \left(\partial_j^+ a_{\mathbf{n}}\right)^2$. The positive constant $C(N, d)$ can be taken to be $4dN^{2d}$.

Proof: The proof for $d = 1$ is given in Agarwal [1], page 860. Another proof was given by Konotop and Perla Menzala in [9]. We did not find a proof in the literature in case $d > 1$. Thus, we provide one in the Appendix. $\square$

Lemma 3.3. *Let $a = \{a_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ and $b = \{b_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ belong to ℓ_{per}. Let $v(t) = \{v_{\mathbf{n}}(t)\}_{\mathbf{n}\in\mathbb{Z}^d}$ be the solution of* (3.1)*. Then*

a) $\sum_{\mathbf{n}=1}^{N} v_{\mathbf{n}}(t) = 0.$

b) $\sum_{\mathbf{n}=1}^{N} v_{\mathbf{n}}^2(t) \leq C(N, d) \sum_{\mathbf{n}=1}^{N} |\nabla^+ v_{\mathbf{n}}(t)|^2.$

c) $\sum_{\mathbf{n}=1}^{N} (\dot{v}_{\mathbf{n}}(t))^2 \leq C(N, d) \sum_{\mathbf{n}=1}^{N} |\nabla^+ \dot{v}_{\mathbf{n}}(t)|^2$

for any $t \geq 0$ and $\mathbf{n} = (n_1, n_2, \ldots, n_d)$.

Proof: Using (3.1) and Lemma 3.1 we know that

$$\frac{d^2}{dt^2} \sum_{\mathbf{n}=1}^{N} v_{\mathbf{n}}(t) = \sum_{\mathbf{n}=1}^{N} \Delta v_{\mathbf{n}}(t) = 0, \quad \forall\ t \geq 0.$$

Therefore,

$$\sum_{\mathbf{n}=1}^{N} v_{\mathbf{n}}(t) = c_1 + c_2 t, \quad \forall\ t \geq 0,$$

where c_1 and c_2 are constants. The initial conditions in (3.1) imply that $c_1 = c_2 = 0$. This proves a). Items b) and c) follow from Lemma 3.2 and a). $\square$

Let T be a fixed positive number. We define the space $X(T)$ consisting of all vector-valued functions $u(t) = \{u_{\mathbf{n}}(t)\}_{\mathbf{n}\in\mathbb{Z}^d}$ such that

i) $u_{n_1+N, n_2, n_3, \ldots, n_d}(t) = \cdots = u_{n_1, n_2, \ldots, n_d+N}(t) = u_{\mathbf{n}}(t), \ \ \forall\ t \in [0, T).$

ii) $u_{\mathbf{n}} \in C^1([0, T); \mathbb{R})$ for any $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$

and

iii)

$$\sup_{0\le t<T}\sum_{\mathbf{n}=1}^{N}[(\dot u_{\mathbf{n}}(t))^2+|\nabla^+u_{\mathbf{n}}(t)|^2+u_{\mathbf{n}}^2(t)]<+\infty. \tag{3.3}$$

The norm in the linear space $X(T)$ is given by

$$||u||_{X(T)}^2=\sup_{0\le t<T}\sum_{\mathbf{n}=1}^{N}[(\dot u_{\mathbf{n}}(t))^2+|\nabla^+u_{\mathbf{n}}(t)|^2+u_{\mathbf{n}}^2(t)].$$

Clearly $(X(T),||\cdot||_{X(T)})$ is a Banach space. Observe that the solution $v(t)=\{v_{\mathbf{n}}(t)\}_{\mathbf{n}\in\mathbb{Z}^d}$ of (3.1) belongs to $X(T)$ for any $T>0$.

Let us use the notation

$$F(u_{\mathbf{n}},\dot u_{\mathbf{n}})=\operatorname{div}h(\nabla^+u_{\mathbf{n}})+\operatorname{div}g(\nabla^+\dot u_{\mathbf{n}}), \tag{3.4}$$

where h and g are as in Section 2.

Lemma 3.4. *Let $\{a_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$, $\{b_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ and $\{f_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ in ℓ_{per}. There exist $T_0>0$ and a unique element $u(t)=\{u_{\mathbf{n}}(t)\}_{\mathbf{n}\in\mathbb{Z}^d}$ in $X(T_0)$ such that*

$$u_{\mathbf{n}}(t)=v_{\mathbf{n}}(t)+\sum_{\mathbf{m}=1}^{N}\int_0^t G(\mathbf{n}-\mathbf{m},t-s)[F(u_{\mathbf{m}}(s),\dot u_{\mathbf{m}}(s))+f_{\mathbf{m}}]ds,$$

for any $\mathbf{n}=(n_1,n_2,\dots,n_d)\in\mathbb{Z}^d$. Here, $\mathbf{m}=(m_1,m_2,\dots,m_d)$ and $\mathbf{n}-\mathbf{m}=(n_1-m_1,n_2-m_2,\dots,n_d-m_d)$. The function G is given in the Appendix.

Proof: The conclusion follows using Banach fixed point theorem and the properties of $G(n,t)$. □

Proposition 3.1. *Let $\{a_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$, $\{b_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}}$ and $\{f_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ belonging to ℓ_{per}. Then, there exist $T_0>0$ and a unique element $u=\{u_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ in $X(T_0)$ such that $u_{\mathbf{n}}$ is a solution of* (2.3) *and $u_{\mathbf{n}}\in C^2([0,T_0);\mathbb{R})$ for any $\mathbf{n}\in\mathbb{Z}^d$.*

Proof: Let $T_0>0$ as in Lemma 3.4. Applying the operator $\frac{\partial^2}{\partial t^2}-\Delta$ (where $\Delta=\sum\limits_{j=1}^{d}\partial_j^-\partial_j^+$) to the solution $u_{\mathbf{n}}$ of the integral equation it follows that $\ddot u_{\mathbf{n}}-\Delta u_{\mathbf{n}}=F(u_{\mathbf{n}},\dot u_{\mathbf{n}})+f_{\mathbf{n}}$ for any $\mathbf{n}\in\mathbb{Z}^d$. Obviously $u_{\mathbf{n}}(0)=a_{\mathbf{n}}$ and $\dot u_{\mathbf{n}}(0)=b_{\mathbf{n}}$. Uniqueness follows using Gronwall's inequality. Since the procedure is standard we will omit the details here. □

Proof of Theorem 2.1. Let $[0,T_{\max})$ be the maximum interval of existence of problem (2.3) obtained in Proposition 3.1. Consider the total energy $E_N(t)$ given by

$$E_N(t)=\frac{1}{2}\sum_{\mathbf{n}=1}^{N}[(\dot u_{\mathbf{n}})^2+|\nabla^+u_{\mathbf{n}}|^2]+\sum_{\mathbf{n}=1}^{N}\sum_{j=1}^{d}\tilde h_j(\partial_j^+u_{\mathbf{n}})-\sum_{\mathbf{n}=1}^{N}f_{\mathbf{n}}u_{\mathbf{n}}. \tag{3.5}$$

Using equation (2.3) we obtain

$$\sum_{\mathbf{n}=1}^{N}\dot u_{\mathbf{n}}\ddot u_{\mathbf{n}}=\sum_{\mathbf{n}=1}^{N}\dot u_{\mathbf{n}}\Delta u_{\mathbf{n}}+\sum_{\mathbf{n}=1}^{N}\dot u_{\mathbf{n}}\operatorname{div}h(\nabla u_{\mathbf{n}})+\sum_{\mathbf{n}=1}^{N}\dot u_{\mathbf{n}}\operatorname{div}g(\nabla\dot u_{\mathbf{n}})+\sum_{\mathbf{n}=1}^{N}f_{\mathbf{n}}\dot u_{\mathbf{n}}. \tag{3.6}$$

Since

$$\sum_{\mathbf{n}=1}^{N} \dot{u}_{\mathbf{n}} h_j(\partial_j^- u_{\mathbf{n}}) = -\sum_{\mathbf{n}=1}^{N} h_j(\partial_j^+ u_{\mathbf{n}})\partial_j^+ \dot{u}_{\mathbf{n}}$$

then

$$\sum_{\mathbf{n}=1}^{N} \dot{u}_{\mathbf{n}} \operatorname{div} h(\nabla u_{\mathbf{n}}) = -\sum_{\mathbf{n}=1}^{N}\sum_{j=1}^{d} h_j \ \left(\partial_j^+ \dot{u}_{\mathbf{n}}\right) \ \partial_j^+ \dot{u}_{\mathbf{n}}. \tag{3.7}$$

Similarly

$$\sum_{\mathbf{n}=1}^{N} \dot{u}_{\mathbf{n}} \operatorname{div} g(\nabla \dot{u}_{\mathbf{n}}) = -\sum_{\mathbf{n}=1}^{N}\sum_{j=1}^{d} g_j(\partial_j^+ \dot{u}_{\mathbf{n}})\partial_j^+ \dot{u}_{\mathbf{n}}. \tag{3.8}$$

Next, we find the derivative of $E_N(t)$, replace the identities (3.6)–(3.8) and use Lemma 3.1, item 2) to obtain

$$\frac{d}{dt}E_N(t) = -\sum_{\mathbf{n}=1}^{N}\sum_{j=1}^{d} g_j(\partial_j^+ \dot{u}_{\mathbf{n}})\partial_j^+ \dot{u}_{\mathbf{n}} \leq 0 \tag{3.9}$$

for any $0 \leq t < T_{\max}$. Consequently, $E_N(t) \leq E_N(0)$ for any $0 \leq t < T_{\max}$. Now, we claim that

$$||u||^2_{X(T)} \leq \tilde{c}_1 E_N(0) + \tilde{c}_2 ||f||^2, \tag{3.10}$$

for some positive constants $\tilde{c}_1, \tilde{c}_2$ and any $T > 0$ such that $0 \leq t \leq T < T_{\max}$. First, we will prove that $\sum\limits_{\mathbf{n}=1}^{N} u_{\mathbf{n}}(t) = 0$ for any $0 \leq t < T_{\max}$. Due to periodicity we know that

$$\begin{aligned}
\sum_{\mathbf{n}=1}^{N} \operatorname{div} h(\nabla u_{\mathbf{n}}) &= \sum_{\mathbf{n}=1}^{N}\sum_{j=1}^{d} [h_j(\partial_j^+ u_{\mathbf{n}}) - h_j(\partial_j^- u_{\mathbf{n}})] \\
&= \sum_{j=1}^{d} \left\{ \sum_{\mathbf{n}=1}^{N} [h_j(\partial_j^+ u_{\mathbf{n}}) - h_j(\partial_j^- u_{\mathbf{n}})] \right\} = 0
\end{aligned}$$

and $\sum\limits_{\mathbf{n}=1}^{N} \operatorname{div} g(\nabla \dot{u}_{\mathbf{n}})=0$. Then, using (2.3) and Lemma 3.1 we deduce that $\sum\limits_{\mathbf{n}=1}^{N} \ddot{u}_{\mathbf{n}}(t)=0$. Consequently, $\sum\limits_{\mathbf{n}=1}^{N} u_{\mathbf{n}}(t) = \sum\limits_{\mathbf{n}=1}^{N} a_{\mathbf{n}} + t\sum\limits_{\mathbf{n}=1}^{N} b_{\mathbf{n}} = 0$ for any $0 \leq t < T_{\max}$.

Now, for any $\alpha > 0$

$$\left|\sum_{\mathbf{n}=1}^{N} f_{\mathbf{n}} u_{\mathbf{n}}\right| \leq \frac{1}{2\alpha}\sum_{\mathbf{n}=1}^{N} |u_{\mathbf{n}}|^2 + \frac{\alpha}{2}||f||^2$$

holds. We use Lemma 3.2 and $\alpha = 2\, C(N,d)$ to obtain

$$\frac{1}{4}\sum_{\mathbf{n}=1}^{N} \{(\dot{u}_{\mathbf{n}})^2 + |\nabla^+ u_{\mathbf{n}}|^2\} \leq E_N(t) + 4\, C(N,d)||f||^2, \tag{3.11}$$

for any $0 \leq t < T_{\max}$. Again, using Lemma 3.2 we conclude that

$$||u||^2_{X(T)} \leq 4\big(1 + C(N,d)\big)E_N(0) + 16\, C(N,d)\big(1 + C(N,d)\big)||f||^2,$$

for any $0 \leq t \leq T$. This proves (3.10). Therefore $T_{\max} = +\infty$. Uniqueness follows using Gronwall's inequality. □

4. GLOBAL ATTRACTOR

Let us consider the space $H = \ell_{\text{per}} \times \ell_{\text{per}}$ where ℓ_{per} is as in (2.1). The norm in H is given by

$$||(a,b)||_H = \big(||a||^2 + ||b||^2\big)^{1/2} = \left(\sum_{\mathbf{n}=1}^{N}(a_{\mathbf{n}}^2 + b_{\mathbf{n}}^2)\right)^{1/2},$$

whenever $a = \{a_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ and $b = \{b_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ belong to ℓ_{per}. Given a, b and $f = \{f_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ in ℓ_{per} then by Theorem 2.1 we have a unique global solution $u(t) = \{u_{\mathbf{n}}(t)\}_{\mathbf{n}\in\mathbb{Z}^d}$ of problem (2.3).

Thus we can consider the map $S(t)\colon H \mapsto H$ given by $S(t)(a,b) = \big(u(t), \dot{u}(t)\big)$, $t \geq 0$. $\{S(t)\}_{t\geq 0}$ is a semigroup of operators from H into H.

Lemma 4.1. *Assume (H1) and $sg_j(s) \geq 0$ for any $s \in \mathbb{R}$ and $1 \leq j \leq d$. Let a, b and f in ℓ_{per}. Then, for each $T > 0$, the map $R\colon H \mapsto C([0,T]; H)$ defined by $R(a,b) = \big(u(t), \dot{u}(t)\big)$ is continuous. Here $u = u(t) = \{u_{\mathbf{n}}(t)\}_{\mathbf{n}\in\mathbb{Z}^d}$ is the global solution of problem* (2.3).

Proof: Let $u = \{u_{\mathbf{n}}(t)\}_{\mathbf{n}\in\mathbb{Z}^d}$ and $w = \{w_{\mathbf{n}}(t)\}_{\mathbf{n}\in\mathbb{Z}^d}$ be global solutions of (2.3) with initial data $a = \{a_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$, $b = \{b_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ and $\tilde{a} = \{\tilde{a}_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$, $\tilde{b} = \{\tilde{b}_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ in ℓ_{per}, respectively. Let $z = \{z_{\mathbf{n}}(t)\}_{\mathbf{n}\in\mathbb{Z}^d}$ where $z_{\mathbf{n}}(t) = u_{\mathbf{n}}(t) - w_{\mathbf{n}}(t)$ for any $t \geq 0$ and $\mathbf{n} = (n_1, n_2, \dots, n_d) \in \mathbb{Z}^d$. Thus, $\{z_{\mathbf{n}}(t)\}_{\mathbf{n}\in\mathbb{Z}^d}$ satisfies

$$\ddot{z}_{\mathbf{n}} - \Delta z_{\mathbf{n}} = F(u_{\mathbf{n}}, \dot{u}_{\mathbf{n}}) - F(w_{\mathbf{n}}, \dot{w}_{\mathbf{n}}). \tag{4.1}$$

Multiplying (4.1) by $\dot{z}_{\mathbf{n}}$ and adding we obtain

$$\frac{1}{2}\frac{d}{dt}\sum_{\mathbf{n}=1}^{N}[(\dot{z}_{\mathbf{n}})^2 + |\nabla^+ z_{\mathbf{n}}|^2] = \chi_N(t), \tag{4.2}$$

where

$$\begin{aligned}\chi_N(t) = \sum_{\mathbf{n}=1}^{N}\Big\{\sum_{j=1}^{d}\Big[&h_j(\partial_j^+ u_{\mathbf{n}}) - h_j(\partial_j^+ w_{\mathbf{n}}) + h_j(\partial_j^- w_{\mathbf{n}}) - h_j(\partial_j^- u_{\mathbf{n}})\\ &+ g_j(\partial_j^+ \dot{u}_{\mathbf{n}}) - g_j(\partial_j^+ \dot{w}_{\mathbf{n}}) + g_j(\partial_j^- \dot{w}_{\mathbf{n}}) - g_j(\partial_j^- \dot{u}_{\mathbf{n}})\Big]\Big\}\dot{z}_{\mathbf{n}}.\end{aligned} \tag{4.3}$$

We can find estimates for $\chi_N(t)$ using the same arguments as we did in the proof of Lemma 3.4 to get

$$\begin{aligned}|\chi_N(t)| \leq \sum_{\mathbf{n}=1}^{N}\Big\{\sum_{j=1}^{d}\Big[&2M_1(|\partial_j^+ z_{\mathbf{n}}| + |\partial_j^- z_{\mathbf{n}}|)\\ &+ 2M_2(|\partial_j^+ \dot{z}_{\mathbf{n}}| + |\partial_j^- \dot{z}_{\mathbf{n}}|)\Big]\Big\}|\dot{z}_{\mathbf{n}}|,\end{aligned} \tag{4.4}$$

where

$$M_1 = \sum_{j=1}^{d}\max_{|s|\leq 2s_1}|h_j^{(1)}(s)|, \quad M_2 = \sum_{j=1}^{d}\max_{|s|\leq 2s_1}|g_j^{(1)}(s)|$$

and $s_1 = ||u||_{X(T)} + ||w||_{X(T)}$. Using Schwarz's inequality and the periodicity we obtain from (4.4) the estimate

$$|\chi_N(t)| \leq 2^{\frac{d+4}{2}} M_0 \left\{ \frac{1}{2} \sum_{\mathbf{n}=1}^{N} (\dot{z}_{\mathbf{n}})^2 + \sum_{\mathbf{n}=1}^{N} |\nabla^+ z_{\mathbf{n}}|^2 + 2d \sum_{\mathbf{n}=1}^{N} (\dot{z}_{\mathbf{n}})^2 \right\},$$

where $M_0 = \max\{M_1, M_2\}$. From (4.2) and the above estimate it follows

$$\frac{d}{dt} \sum_{\mathbf{n}=1}^{N} [(\dot{z}_{\mathbf{n}})^2 + |\nabla^+ z_{\mathbf{n}}|^2] \leq 2^{\frac{d+4}{2}} M_0 (1+4d) \sum_{\mathbf{n}=1}^{N} [(\dot{z}_{\mathbf{n}})^2 + |\nabla^+ z_{\mathbf{n}}|^2]. \tag{4.5}$$

Finally, using Gronwall's inequality and Lemma 3.2 we deduce from (4.5) the estimate

$$\sum_{\mathbf{n}=1}^{N} \{(\dot{z}_{\mathbf{n}})^2 + z_{\mathbf{n}}^2\} \leq C_0 \sum_{\mathbf{n}=1}^{N} \{(\dot{z}_{\mathbf{n}}(0)^2 + (z_{\mathbf{n}}(0))^2\}, \tag{4.6}$$

where $C_0 = 4d[1 + C(N,d)] \exp(2^{\frac{d+4}{2}} M_0 (1+4d)T)$. From (4.6) it follows that

$$\sup_{0 \leq t \leq T} ||(u, \dot{u}) - (w, \dot{w})||_H \leq C_0 ||(a,b) - (\tilde{a}, \tilde{b})||. \tag{4.7}$$

In order to prove the continuity of R remains to estimate the quantity M_0. Observe the estimate below (3.11). On the right hand side, it will be enough to get an appropriate bound for $E_N(0)$. Using (3.5) at $t = 0$ we find that

$$E_N(0) \leq \big(2d + \frac{1}{2}\big) ||(a,b)||_H^2 + N^d \tilde{M}_1 + \frac{1}{2} ||f||^2,$$

where $\tilde{M}_1 = \sum_{j=1}^{d} \max_{|s| \leq 2||a||} |\tilde{h}_j(s)|$. Therefore,

$$||u||_{X(T)}^2 \leq 4\big(1 + C(N,d)\big) \left\{ \big(2d + \frac{1}{2}\big) ||(a,b)||_H^2 + N^d \tilde{M}_1 + \big(4\, C(N,d) + \frac{1}{2}\big) ||f||^2 \right\}.$$

Similarly,

$$||w||_{X(T)}^2 \leq 4\big(1 + C(N,d)\big) \left\{ \big(2d + \frac{1}{2}\big) ||(\tilde{a}, \tilde{b})||_H^2 + N^d \tilde{M}_2 + \big(4\, C(N,d) + \frac{1}{2}\big) ||f||^2 \right\},$$

where $\tilde{M}_2 = \sum_{j=1}^{d} \max_{|s| \leq 2||\tilde{a}||} |\tilde{h}_j(s)|$. The above estimates allow us to conclude from (4.7) that the continuity of R follows from the continuity of the functions $h_j(s), g_j(s)$ and $\tilde{h}_j(s)$, $j = 1, \ldots, d$. $\square$

The proof of Theorem 2.2 will follow from the next five Lemmas.

Lemma 4.2. *Under the assumptions of Theorem* 2.2*, there exist a positive constant* α *depending only on* N, p, d, k_0, k_1 *and* $\max_{|s| \leq 1} \left\{ \sum_{j=1}^{d} |g_j^{(1)}(s)| \right\}$ *such that*

$$\sup_{t \leq s \leq t+1} \tilde{E}_N(s) \leq \alpha \left[F^{\frac{4}{p+2}}(t) + F^2(t) + ||f||^2 \right],$$

for any $t \geq 0$*, where* $\tilde{E}_N(t) = E_N(t) + 4\, C(N,d) ||f||^2$*,* $F^2(t) = E_N(t) - E_N(t+1)$ *and* $E_N(t)$ *is given by* (3.5).

Proof: Let

$$J(s)=\sum_{\mathbf{n}=1}^{N}\sum_{j=1}^{d}\partial_j^+\dot u_{\mathbf{n}}(s)g_j(\partial_j^+\dot u_{\mathbf{n}}(s)).$$

Integration of (3.9) over $[t,t+1]$ give us

$$F^2(t)=\int_t^{t+1}J(s)\,ds. \tag{4.8}$$

Using the mean value theorem for integrals we obtain the existence of $t_1\in[t,t+\frac14]$ and $t_2\in[t+\frac34,t+1]$ such that

$$J(t_1)=4\int_t^{t+\frac14}J(s)\,ds \quad \text{and} \quad J(t_2)=4\int_{t+\frac34}^{t+1}J(s)\,ds. \tag{4.9}$$

Using (4.8) and (4.9) we get

$$J(t_1)+J(t_2)\le 4F^2(t). \tag{4.10}$$

Multiplying equation (2.3) by $u_{\mathbf{n}}$, adding in $\{1,2,\ldots,N\}^d$, using periodicity and integrating the result over $[t_1,t_2]$ we obtain

$$\begin{aligned}
&\sum_{\mathbf{n}=1}^{N}\int_{t_1}^{t_2}|\nabla^+u_{\mathbf{n}}|^2ds+\sum_{\mathbf{n}=1}^{N}\sum_{j=1}^{d}\int_{t_1}^{t_2}\partial_j^+u_{\mathbf{n}}(s)h_j(\partial_j^+u_{\mathbf{n}}(s))ds\\
&=-\sum_{\mathbf{n}=1}^{N}\{u_{\mathbf{n}}(t_2)\dot u_{\mathbf{n}}(t_2)-u_{\mathbf{n}}(t_1)\dot u_{\mathbf{n}}(t_1)\}+\sum_{\mathbf{n}=1}^{N}\int_{t_1}^{t_2}(\dot u_{\mathbf{n}}(s))^2ds\\
&\quad-\sum_{\mathbf{n}=1}^{N}\sum_{j=1}^{d}\int_{t_1}^{t_2}\partial_j^+u_{\mathbf{n}}(s)g_j(\partial_j^+\dot u_{\mathbf{n}}(s))ds+\sum_{\mathbf{n}=1}^{N}\int_{t_1}^{t_2}f_{\mathbf{n}}u_{\mathbf{n}}(s)ds.
\end{aligned} \tag{4.11}$$

Next, we will find good estimates for each term on the right hand side of (4.11). Using Lemma 3.2 and (3.11) we obtain

$$\begin{aligned}
\left|\sum_{\mathbf{n}=1}^{N}u_{\mathbf{n}}(t_2)\dot u_{\mathbf{n}}(t_2)\right| &\le C(N,d)\left(\sum_{\mathbf{n}=1}^{N}|\nabla^+u_{\mathbf{n}}(t_2)|^2\right)^{1/2}\left(\sum_{\mathbf{n}=1}^{N}|\nabla^+\dot u_{\mathbf{n}}(t_2)|^2\right)^{1/2}\\
&\le 4\,C(N,d)\big(\tilde E_N(t_2)\big)^{1/2}\left(\sum_{\mathbf{n}=1}^{N}|\nabla^+\dot u_{\mathbf{n}}(t_2)|^2\right)^{1/2}.
\end{aligned} \tag{4.12}$$

For each $t\ge0$ we consider the sets

$$I_{j,1}(t)=\{\mathbf{n}\in\{1,2,\ldots,N\}^d \quad \text{such that} \quad |\partial_j^+\dot u_{\mathbf{n}}(t)|\le1\}$$

$$I_{j,2}(t)=\left\{1,2,\ldots,N^d\right\}\setminus I_{j,1}(t),\quad j=1,2,\ldots,d.$$

Now, using the inequality $sg_j(s)\ge k_1|s|^{p+2}$ and Hölder's inequality, we obtain

$$\begin{aligned}
\sum_{\mathbf{n}\in I_{j,2}(t_2)}\big(\partial_j^+\dot u_{\mathbf{n}}(t_2)\big)^2 &\le \sum_{\mathbf{n}\in I_{j,2}(t_2)}|\partial_j^+\dot u_{\mathbf{n}}(t_2)|^{p+2}\\
&\le k_1^{-1}\sum_{\mathbf{n}\in I_{j,2}(t_2)}\partial_j^+\dot u_{\mathbf{n}}(t_2)g_j(\partial_j^+\dot u_{\mathbf{n}}(t_2)).
\end{aligned} \tag{4.13}$$

Hölder's inequality implies

$$\sum_{\mathbf{n}\in I_{j,1}(t_2)} \left(\partial_j^+ \dot{u}_{\mathbf{n}}(t_2)\right)^2 \leq N^{\frac{pd}{p+2}} k_1^{-\frac{2}{p+2}} \left(\sum_{\mathbf{n}=1}^{N} \partial_j^+ \dot{u}_{\mathbf{n}}(t_2) g_j\left(\partial_j^+ \dot{u}_{\mathbf{n}}(t_2)\right)\right)^{\frac{2}{p+2}}. \tag{4.14}$$

Using (4.13) and (4.14) we deduce

$$\begin{aligned}
\sum_{\mathbf{n}=1}^{N} |\nabla^+ \dot{u}_{\mathbf{n}}(t_2)|^2 &= \sum_{j=1}^{d} \sum_{\mathbf{n}=1}^{N} (\partial_j^+ \dot{u}_{\mathbf{n}}(t_2))^2 \\
&= \sum_{j=1}^{d} \left\{ \sum_{\mathbf{n}\in I_{j,1}(t_2)} |\partial_j^+ \dot{u}_{\mathbf{n}}(t_2)|^2 + \sum_{\mathbf{n}\in I_{j,2}(t_2)} |\partial_j^+ \dot{u}_{\mathbf{n}}(t_2)|^2 \right\} \\
&\leq \sum_{j=1}^{d} \left\{ k_1^{-1} \sum_{\mathbf{n}=1}^{N} \partial_j^+ \dot{u}_{\mathbf{n}}(t_2) g_j(\partial_j^+ \dot{u}_{\mathbf{n}}(t_2)) \right. \\
&\qquad \left. + N^{\frac{pd}{p+2}} k_1^{-\frac{2}{p+2}} \left(\sum_{\mathbf{n}=1}^{N} \partial_j^+ \dot{u}_{\mathbf{n}}(t_2) g_j(\partial_j^+ \dot{u}_{\mathbf{n}}(t_2))\right)^{\frac{2}{p+2}} \right\}.
\end{aligned} \tag{4.15}$$

Using (4.10) and the inequality $\sum\limits_{j=1}^{d} a_j^r \leq d \left(\sum\limits_{j=1}^{d} a_j\right)^r$ valid whenever $a_j > 0$ and $r > 0$, it follows from (4.15) the estimate

$$\left(\sum_{\mathbf{n}=1}^{N} |\nabla^+ \dot{u}_{\mathbf{n}}(t_2)|^2\right)^{1/2} \leq 2 k_1^{-1/2} F(t) + d^{\frac{1}{2}} \; 4^{\frac{1}{p+2}} N^{\frac{dp}{2(p+2)}} k_1^{-\frac{1}{p+2}} \left(F(t)\right)^{\frac{2}{p+2}}. \tag{4.16}$$

Combining (4.12) with (4.16) we proved the estimate

$$\left|\sum_{\mathbf{n}=1}^{N} u_{\mathbf{n}}(t_2) \dot{u}_{\mathbf{n}}(t_2)\right| \leq 4\, C(N,d) \left(\tilde{E}_N(t_2)\right)^{1/2} \left\{\beta F(t) + \gamma F^{\frac{2}{p+2}}(t)\right\},$$

where β and γ are positive constants. Similarly, we can obtain an estimate for $\left|\sum\limits_{\mathbf{n}=1}^{N} u_{\mathbf{n}}(t_1) \dot{u}_{\mathbf{n}}(t_1)\right|$. Therefore, we have the inequality

$$\left|\sum_{\mathbf{n}=1}^{N} \left\{u_{\mathbf{n}}(t_2) \dot{u}_{\mathbf{n}}(t_2) - u_{\mathbf{n}}(t_1) \dot{u}_{\mathbf{n}}(t_1)\right\}\right| \leq C_1 \sup_{t \leq s \leq t+1} (\tilde{E}_N(s))^{1/2} \left[F(t) + F^{\frac{2}{p+2}}(t)\right]. \tag{4.17}$$

Next, we obtain an estimate for the term

$$\sum_{j=1}^{d} \sum_{\mathbf{n}=1}^{N} \partial_j^+ u_{\mathbf{n}}(s) g_j(\partial_j^+ \dot{u}_{\mathbf{n}}(s)).$$

Let $k_2 = \max_{|s|\le 1} \left\{ \sum_{j=1}^{d} |g_j^{(1)}(s)| \right\}$. Then, we have $|g_j(s)| \le k_2 |s|$ for any $|s| \le 1$. Clearly

$$\sum_{\mathbf{n}\in I_{j,1}(s)} |\partial_j^+ u_{\mathbf{n}}(s) g_j(\partial_j^+ \dot u_{\mathbf{n}}(s))| \le \frac{1}{2}\sum_{\mathbf{n}=1}^{N} |\partial_j^+ u_{\mathbf{n}}(s)|^2 + \frac{k_2}{2}\sum_{\mathbf{n}\in I_{j,1}(s)} |\partial_j^+ \dot u_{\mathbf{n}}(s)||g_j(\partial_j^+ \dot u_{\mathbf{n}}(s))|. \tag{4.18}$$

Also

$$\begin{aligned}\sum_{\mathbf{n}\in I_{j,2}(s)} |\partial_j^+ u_{\mathbf{n}}(s) g_j(\partial_j^+ \dot u_{\mathbf{n}}(s))| &\le \left(\sum_{\mathbf{n}=1}^{N} |\partial_j^+ u_{\mathbf{n}}(s)|^2\right)^{1/2} \sum_{\mathbf{n}\in I_{j,2}(s)} |g_j(\partial_j^+ \dot u_{\mathbf{n}}(s))| \\ &\le \left(\sum_{\mathbf{n}=1}^{N} |\nabla^+ u_{\mathbf{n}}(s)|^2\right)^{1/2} \sum_{\mathbf{n}=1}^{N} |\partial_j^+ \dot u_{\mathbf{n}}(s)||g_j(\partial_j^+ \dot u_{\mathbf{n}}(s))|.\end{aligned} \tag{4.19}$$

Using (4.18), (4.19), adding over $j = 1, 2, \ldots, d$, using (3.11) and integrating over the interval $[t_1, t_2]$ we obtain

$$\left|\sum_{\mathbf{n}=1}^{N}\sum_{j=1}^{d}\int_{t_1}^{t_2} \partial_j^+ u_{\mathbf{n}}(s) g_j(\partial_j^+ \dot u_{\mathbf{n}}(s))\, ds\right| \le \frac{1}{2}\sum_{\mathbf{n}=1}^{N}\int_{t_1}^{t_2} |\nabla^+ u_{\mathbf{n}}(s)|^2 ds + \frac{k_2}{2}\int_{t_1}^{t_2} J(s) ds + \left(2 \sup_{t\le s\le t+1} \left(\tilde E_N(s)\right)^{1/2}\right)\int_{t_1}^{t_2} J(s)\, ds. \tag{4.20}$$

Next, to estimate $\sum_{\mathbf{n}=1}^{N} \left(\dot u_{\mathbf{n}}(s)\right)^2$ we use Lemma 3.2 and proceed as we did from (4.13) up to (4.16) to obtain

$$\begin{aligned}\sum_{\mathbf{n}=1}^{N} \left(\dot u_{\mathbf{n}}(s)\right)^2 &\le C(N,d)\sum_{\mathbf{n}=1}^{N} |\nabla^+ \dot u_{\mathbf{n}}(s)|^2 \\ &\le C(N,d)\left\{k_1^{-1} J(s) + d\, N^{\frac{pd}{p+2}} k_1^{-\frac{2}{p+2}} J(s)^{\frac{2}{p+2}}\right\}.\end{aligned}$$

Consequently,

$$\sum_{\mathbf{n}=1}^{N}\int_{t_1}^{t_2} \left(\dot u_{\mathbf{n}}(s)\right)^2 ds \le C_2\left[F^2(t) + (F(t))^{\frac{4}{p+2}}\right], \tag{4.21}$$

where $C_2 = C_2(N, d, p, k_1) > 0$. Finally, using Lemma 3.2 and (3.11) we deduce

$$\begin{aligned}\left|\sum_{\mathbf{n}=1}^{N}\int_{t_1}^{t_2} f_{\mathbf{n}} u_{\mathbf{n}}(s) ds\right| &\le ||f|| C(N,d)\int_{t_1}^{t_2} \left(\sum_{\mathbf{n}=1}^{N} |\nabla^+ u_{\mathbf{n}}(s)|^2\right)^{1/2} ds \\ &\le 2C(N,d)||f|| \sup_{t\le s\le t+1} \left(\tilde E_N(s)\right)^{1/2}.\end{aligned} \tag{4.22}$$

Substitution of (4.17), (4.20), (4.21) and (4.22) into (4.11) give us

$$\begin{aligned}\frac{1}{2}\sum_{\mathbf{n}=1}^{N}\int_{t_1}^{t_2}|\nabla^+ u_{\mathbf{n}}|^2 ds + \sum_{\mathbf{n}=1}^{N}\sum_{j=1}^{d}\int_{t_1}^{t_2}\partial_j^+ u_{\mathbf{n}}(s) h_j(\partial_j^+ u_{\mathbf{n}}(s))ds \\ \leq C_3\left[\left(F^{\frac{4}{2+p}}(t)\right) + F^2(t)\right] + C_4\Big[\left(F^{\frac{2}{p+2}}(t)\right) + F(t) \\ + F^2(t) + ||f||\Big] \sup_{t\leq s\leq t+1}\left(\tilde{E}_N(s)\right)^{1/2},\end{aligned} \tag{4.23}$$

for some positive constants C_3 and C_4 which depend only on N, p, d, k_1 and k_2. Since $\tilde{E}_N(s) = E_N(t) + 4C(N,d)||f||^2$, using assumptions of Theorem 2.2 (on h_j) we obtain

$$\begin{aligned}\tilde{E}_N(t) \leq & \frac{1}{2}\sum_{\mathbf{n}=1}^{N}[(\dot{u}_{\mathbf{n}})^2 + |\nabla^+ u_{\mathbf{n}}|^2] + k_0^{-1}\sum_{\mathbf{n}=1}^{N}\sum_{j=1}^{d}\partial_j^+(u_{\mathbf{n}})h_j(\partial_j^+ u_{\mathbf{n}}) \\ & - \sum_{\mathbf{n}=1}^{N} f_{\mathbf{n}} u_{\mathbf{n}} + 4C(N,d)||f||^2,\end{aligned} \tag{4.24}$$

Integration of (4.24) over $[t_1, t_2]$ and using (4.23) and (4.21) give us

$$\begin{aligned}\int_{t_1}^{t_2}\tilde{E}_N(s)ds \leq C_5 \sup_{t\leq s\leq t+1}\left(\tilde{E}_N(s)\right)^{1/2}\left[\left(F^{\frac{2}{p+2}}(t)\right) + F^2(t) + F(t) + ||f||\right] \\ + C_6\left[\left(F^{\frac{4}{p+2}}(t)\right) + F^2(t)\right] + 4\,C(N,d)||f||^2,\end{aligned} \tag{4.25}$$

where C_5 and C_6 are positive constants depending only on N, p, d, k_0, k_1 and k_2. Using the mean value theorem for integrals, there exists $t^* \in [t_1, t_2]$ such that

$$\frac{1}{2}\tilde{E}_N(t^*) \leq (t_2 - t_1)\tilde{E}_N(t^*) = \int_{t_1}^{t_2}\tilde{E}_N(s)ds. \tag{4.26}$$

Clearly, for any $\tilde{t} \in [t, t^*]$ we have

$$\tilde{E}_N(\tilde{t}) = \tilde{E}_N(t^*) + \int_{\tilde{t}}^{t^*} J(s)ds \leq \tilde{E}_N(t^*) + F^2(t)$$

due to (3.9) and (4.8). Consequently, whenever $t_1 \leq s \leq t+1$ we have $\tilde{E}_N(s) \leq \tilde{E}_N(t_1) \leq \tilde{E}_N(t^*) + F^2(t)$. Similarly, if $t \leq s \leq t_1$ the same estimate holds. Therefore, $\tilde{E}_N(s) \leq \tilde{E}_N(t^*) + F^2(t)$ for any $t \leq s \leq t+1$. Thus, from (4.26) and (4.25) we conclude the proof of Lemma 4.2. □

Lemma 4.3. *Let $\rho > 0$, a, b and f in ℓ_{per}. Under the assumptions of Theorem 2.2, whenever $||(a,b)||_H \leq \rho$, there exist positive constants α_0 and α_1 such that*

$$\sup_{t\leq s\leq t+1}\left(\tilde{E}_N(s)\right)^{\frac{p+2}{2}} \leq \alpha_0[\tilde{E}_N(t) - \tilde{E}_N(t+1)] + \alpha_1||f||^{p+2},$$

for any $t \geq 0$. Here $\alpha_0 = \alpha_0(\rho, ||f||, \alpha, N, d)$ and $\alpha_1 = \alpha_1(\rho, \alpha)$, where $\alpha > 0$ is as in Lemma 4.2.

Proof: Since $F^2(t) = E_N(t) - E_N(t+1) = \tilde{E}_N(t) - \tilde{E}_N(t+1) \le 2\tilde{E}_N(0)$ holds, we use Lemma 4.2 to obtain

$$\sup_{t\le s\le t+1} \tilde{E}_N(s) \le \alpha \left[1 + 2^{\frac{2}{p+2}}\big(\tilde{E}_N(0)\big)^{\frac{p}{p+2}} + 2^{\frac{2(p+1)}{p+2}}\big(\tilde{E}_N(0)\big)^{\frac{2(p+1)}{p+2}}\right] F^{\frac{4}{p+2}}(t) + \alpha||f||^2. \tag{4.27}$$

Since $|\partial_j^+ a_{\mathbf{n}}| \le 2||a|| \le 2\rho$ then we can easily verify the estimates

$$\begin{aligned}\tilde{E}_N(0) &\le \frac{1}{2}||b||^2 + 2d||a||^2 + \frac{1}{2}||a||^2 + \big(4C(N,d) + \frac{1}{2}\big)||f||^2 + N^d \sum_{j=1}^{d} \max_{|s|\le 2\rho} |\tilde{h}_j(s)| \\ &\le \big(2d + \frac{1}{2}\big)\rho^2 + \big(4C(N,d) + \frac{1}{2}\big)||f||^2 + N^d \sum_{j=1}^{d} \max_{|s|\le 2\rho} |\tilde{h}_j(s)|. \end{aligned} \tag{4.28}$$

Using (4.28) into (4.27) proves Lemma 4.3. $\square$

We will use the following Lemma due to M. Nakao [13].

Lemma 4.4. *Let $\Phi(t)$ be a nonnegative continuous function on $[0,T]$, $T > 1$, possibly $T = +\infty$ satisfying*

$$\sup_{t\le s\le t+1} \Phi(s)^{1+\gamma} \le C[\Phi(t) - \Phi(t+1)] + K \tag{4.29}$$

for any $0 \le t < T-1$, some $C > 0$, $K > 0$ and $\gamma > 0$. Then, we have

$$\Phi(t) \le \left[C^{-1}\gamma(t-1)^+ + (\sup_{0\le s\le 1} \Phi(s))^{-\gamma}\right]^{-1/\gamma} + K^{1/1+\gamma}$$

for $0 \le t < T$. If (4.29) holds with $\gamma = 0$, then we have

$$\Phi(t) \le \sup_{0\le s\le 1} \Phi(s) \left(\frac{C}{1+C}\right)^{[t]} + K, \quad 0 \le t < T.$$

Here $\beta^+ = \max\{\beta, 0\}$ and $[t]$ is the biggest integer less than or equal to t.

Lemma 4.5. *Let $\rho > 0$, a, b and f in ℓ_{per}. Under the assumptions of Theorem 2.2, whenever $||(a,b)||_H \le \rho$, there exist positive constants β_j $(j = 0,1,2,3)$ such that*

(i) *If $p > 0$ then $\tilde{E}_N(t) \le \beta_0(1+t)^{-\frac{2}{p}} + \beta_1||f||^2$, $\forall\, t \ge 0$.*

(ii) *If $p = 0$ then $\tilde{E}_N(t) \le \beta_2 \exp(-\beta_3 t) + \alpha_1||f||^2$, $\forall\, t \ge 0$.*

Here $\beta_3 =$ log $(\frac{1+\alpha_0}{\alpha_0})$, β_0 and β_2 depend on α_0 and p, β_1 depends on α_1 and p. The constants α_0 and α_1 are as in Lemma 4.3.

Proof: Let us prove i). Using Lemmas 4.3 and 4.4 with $\Phi = \tilde{E}_N$, $\gamma = \frac{p}{2}$, $K = \alpha_1||f||^{p+2}$ and $C = \alpha_0$ we obtain

$$\tilde{E}_N(t) \le \left[\frac{p}{2\alpha_0}(t-1) + \big(\tilde{E}_N(0)\big)^{-\frac{p}{2}}\right]^{-\frac{2}{p}} + \alpha_1^{\frac{2}{p+2}}||f||^2, \tag{4.30}$$

for any $t > 1$.

Let $\mu_1 = \frac{p}{2\alpha_0}$ and $\mu_2 = (2d + \frac{1}{2})\rho^2 + (4C(N,d) + \frac{1}{2})||f||^2 + N^d \sum_{j=1}^{d} \max_{|s|\le 2\rho} |\tilde{h}_j(s)|$. Using

(4.30) and (4.28) it follows that

$$\tilde{E}_N(t) \le \left[\mu_1(t-1) + \mu_2^{-p/2}\right]^{-\frac{2}{p}} + \alpha_1^{\frac{2}{p+2}}||f||^2, \quad \forall\, t > 1. \tag{4.31}$$

Since $t^{-2/p} \le 2^{2/p}(1+t)^{-2/p}$ for any $t > 1$ and

$$\left[\mu_1(t-1) + \mu_2^{-p/2}\right]^{-2/p} \le \min\left\{\mu_1, \mu_2^{-p/2}\right\}^{-2/p} t^{-2/p}, \quad \forall\, t > 1,$$

then it follows from (4.31) that

$$\tilde{E}_N(t) \le \tilde{C}(1+t)^{-2/p} + \alpha_1^{2/p+2}||f||^2, \quad \forall\, t > 1,$$

where $\tilde{C} = 2^{2/p}\min\{\mu_1, \mu_2^{-p/2}\}^{-2/p}$. If $0 \le t \le 1$, then

$$\begin{aligned}\tilde{E}_N(t) &\le \tilde{E}_N(0) \le \tilde{E}_N(0)2^{2/p}(1+t)^{-2/p}\\ &\le 2^{2/p}\mu_2(1+t)^{-2/p},\end{aligned}$$

which together with (4.31) proves i).

Next, we prove item ii). Using Lemmas 4.3 and 4.4 we deduce

$$\begin{aligned}\tilde{E}_N(t) &\le \sup_{0\le s\le 1} \tilde{E}_N(s)\left(\frac{\alpha_0}{1+\alpha_0}\right)^{[t]} + \alpha_1||f||^2\\ &\le \tilde{E}_N(0)\exp\left((t-1)\log\left(\frac{\alpha_0}{1+\alpha_0}\right)\right) + \alpha_1||f||^2\\ &= \tilde{E}_N(0)\left(\frac{1+\alpha_0}{\alpha_0}\right)\exp(-\beta_3 t) + \alpha_1||f||^2,\end{aligned}$$

which proves ii). □

Lemma 4.6. *There exists $\rho_0 > 0$ such that the ball $B = \{(u,w) \in H;\ ||(u,w)||_H \le \rho_0\}$ is an absorbing set for the semigroup $\{S(t)\}_{t\ge 0}$ in H.*

Proof: Let U be a bounded set in H. Choose $\rho = \rho(U) > 0$ such that $||(a,b)||_H \le \rho$ for any (a,b) in U. Using Lemmas 3.2 and 4.5 together with (3.11) we deduce

$$\begin{aligned}||S(t)(a,b)||_H^2 = ||(u,\dot{u})||_H^2 &= \sum_{\mathbf{n}=1}^{N}[(\dot{u}_{\mathbf{n}})^2 + u_{\mathbf{n}}^2]\\ &\le C(N,d)\sum_{\mathbf{n}=1}^{N}[(\dot{u}_{\mathbf{n}})^2 + |\nabla^+ u_{\mathbf{n}}|^2] \le 4C(N,d)\tilde{E}_N(t)\\ &\le 4C(N,d)[\beta_0(1+t)^{-2/p} + \beta_1||f||^2],\end{aligned}$$

for any $t \ge 0$ if $p > 0$. If $p = 0$ we have

$$||S(t)(a,b)||_H^2 \le 4C(N,d)[\beta_2\exp(-\beta_3 t) + \alpha_1||f||^2],$$

for any $t \ge 0$. Choosing $\rho_0^2 = 8C(N,d)\beta_1||f||^2$ and $T_0 = \max\left\{\left(\frac{\beta_0}{\beta_1||f||^2}\right)^{p/2} - 1, 0\right\}$ in case $p > 0$ or $\rho_0^2 = 8C(N,d)\alpha_1||f||^2$ and $T_0 = \max\left\{\beta_3^{-1}\log\left(\frac{\beta_2}{\alpha_1||f||^2}\right), 0\right\}$ if $p = 0$ then, it follows that $||S(t)(a,b)||_H \le \rho_0$ for any $t \ge T_0$, which proves Lemma 4.6. □

Proof of Theorem 2.2. Due to the estimates in Lemma 4.6 it follows that the semigroup $\{S(t)\}_{t\geq 0}$ takes bounded sets of H into bounded sets of H for any $T \geq 0$. Since H has finite dimension then each $S(t)$ is a compact operator in H. Thus, the conclusion of Theorem 2.2 follows from Lemma 4.6 and a classical result given in the book of R. Teman [16] (Chapter 1). □

5. GLOBAL ATTRACTOR FOR THE MULTIDIMENSIONAL SINE-GORDON EQUATION

In this section we briefly indicate how we can use most of the calculations in the previous sections in order to obtain the existence of a global attractor for the Sine-Gordon type model. An extensive work has been done on global attractors for the continuous as well as for the discrete case [16, 17].

We consider model (2.3) with hypothesis (H1) together with the following assumptions (H2): There exists $\beta > 0$ such that $|h_j(s)| \leq \beta$, for any $s \in \mathbb{R}$, $j = 1, 2, \ldots, d$ and there exist $k_1 > 0$ and $p > 0$ such that $sg_j(s) \geq k_1|s|^{p+2}$, for any $s \in \mathbb{R}$, $j = 1, 2, \ldots, d$.

Global existence, uniqueness and continuous dependence on the initial data follow by Theorem 2.1 and Lemma 4.1. Going back to identity (4.11) in Lemma 4.2 the term $\sum\limits_{j=1}^{d}\sum\limits_{\mathbf{n}=1}^{N} \partial_j^+ u_{\mathbf{n}}(s) h_j(\partial_j^+ u_{\mathbf{n}}(s))$ can be bound using Schwarz's inequality and (H2) as follows

$$
\begin{aligned}
\left|\sum_{j=1}^{d}\sum_{\mathbf{n}=1}^{N} \partial_j^+ u_{\mathbf{n}}(s) h_j(\partial_j^+ u_{\mathbf{n}}(s))\right| &\leq \sum_{j=1}^{d}\left(\sum_{\mathbf{n}=1}^{N}|\nabla^+ u_{\mathbf{n}}|^2\right)^{1/2}\left(\sum_{\mathbf{n}=1}^{N}|h_j(\partial_j^+ u_{\mathbf{n}})|^2\right)^{1/2} \\
&\leq \sum_{j=1}^{d}\left(4\tilde{E}_N(s)\right)^{1/2}(N^d\beta^2)^{1/2} \\
&\leq 2dN^{d/2}\beta \sup_{t\leq s\leq t+1}\left(\tilde{E}_N(s)\right)^{1/2}.
\end{aligned}
$$

Therefore

$$
\left|\sum_{j=1}^{d}\sum_{\mathbf{n}=1}^{N}\int_{t_1}^{t_2} \partial_j^+ u_{\mathbf{n}} h_j(\partial_j^+ u_{\mathbf{n}}) ds\right| \leq 2dN^{d/2}\beta \sup_{t\leq s\leq t+1}\left(\tilde{E}_N(s)\right)^{1/2}.
$$

Also

$$
\left|\sum_{j=1}^{d}\sum_{\mathbf{n}=1}^{N}\int_{t_1}^{t_2} \tilde{h}_j(\partial_j^+ u_{\mathbf{n}}(s)) ds\right| \leq 2dN^{d/2}\beta \sup_{t\leq s\leq t+1}\left(\tilde{E}_N(s)\right)^{1/2}, \tag{5.1}
$$

because $|\tilde{h}_j(s)| \leq \beta|s|$, for all $s \in \mathbb{R}$. Using all the other estimates we got in the proof of Lemma 4.2 together with the ones obtained above we get after integrating identity (4.11) over $[t_1, t_2]$

$$\frac{1}{2}\sum_{\mathbf{n}=1}^{N}\int_{t_1}^{t_2}|\nabla^+ u_{\mathbf{n}}(s)|^2 ds \le c_1 \sup_{t\le s\le t+1}\big(\tilde{E}_N(s)\big)^{1/2}\Big\{\big(F(t)\big)^{\frac{2}{p+2}} + F^2(t) + F(t)$$
$$+\,2\,C(N,d)||f|| + 2dN^{d/2}\beta\Big\} + c_2\left\{F^2(t) + \big(F(t)\big)^{4/p+2}\right\}, \qquad (5.2)$$

for some positive constants c_1 and c_2. Integration of $\tilde{E}_N(t)$ over $[t_1, t_2]$ and using (5.1) and (5.2) give us

$$\int_{t_1}^{t_2}\tilde{E}_N(s)ds \le c_3 \sup_{t\le s\le t+1}\big(\tilde{E}_N(s)\big)^{1/2}\Big[\big(F(t)\big)^{\frac{2}{p+2}} + F^2(t) + F(t)$$
$$+\,||f|| + \beta\Big] + c_4\Big[\big(F(t)\big)^{\frac{4}{p+2}} + F^2(t)\Big] + 4\,C(N,d)||f||^2,$$

for some positive constants c_3 and c_4. Next, we proceed as in Lemma 4.3 to obtain

$$\sup_{t\le s\le t+1}\big(\tilde{E}_N(s)\big)^{\frac{p+2}{p}} \le \tilde{\alpha}_0[\tilde{E}_N(t) - \tilde{E}_N(t+1)] + \tilde{\alpha}_1\big(||f||^2 + \beta^2\big)^{\frac{p+2}{p}},$$

for any $t \ge 0$. Again, using Nakao's result [13] we obtain the corresponding assumptions to verify Lemmas 4.5 and 4.6, which proves the existence of global attractor in this case. ☐

6. APPENDIX

6.1 Green's function for the semi-discrete linear d-dimensional problem.

We use the same notation as in Section 2. We want to find $G_{\mathbf{n}}(t) = G(\mathbf{n}, t)$, $\mathbf{n} = (n_1, n_2, \ldots, n_d)$ such that $\{G_{\mathbf{n}}(t)\}_{\mathbf{n}\in\mathbb{Z}^d} \in \ell_{\text{per}}$ and satisfies

$$\begin{cases} \ddot{G}(\mathbf{n}, t) - \Delta G(\mathbf{n}, t) \\ G(\mathbf{n}, 0) = 0, \quad \dot{G}(\mathbf{n}, t) = \delta_{\mathbf{n}} = \begin{cases} 1 \text{ if } n_1 = n_2 = \cdots n_d = N \\ 0, \text{ otherwise.} \end{cases} \end{cases} \qquad (6.1)$$

We look for a solution of the form

$$G(\mathbf{n}, t) = Re\left\{\sum_{\mathbf{k}=1}^{N} U_{\mathbf{k}}(t) W_N^{\mathbf{k}\cdot\mathbf{n}}\right\}, \qquad (6.2)$$

where

$$\mathbf{k} = (k_1, k_2, \ldots, k_d), \quad \sum_{\mathbf{k}=1}^{N} = \sum_{k_1=1}^{N}\cdots\sum_{k_d=1}^{N}, \quad \mathbf{k}\cdot\mathbf{n} = \sum_{j=1}^{d} k_j n_j$$

and $W_N = \exp(\frac{2\pi}{N} i)$. Substitution of (6.2) into (6.1) give us

$$\sum_{\mathbf{k}=1}^{N}\left\{\ddot{U}_{\mathbf{k}}(t) - U_{\mathbf{k}}(t)\left[W_N^{-k_1} + \cdots + W_N^{-k_d} - 2d + W_N^{k_1} + \cdots + W_N^{k_d}\right]\right\} W_N^{\mathbf{k}\cdot\mathbf{n}} = 0, \qquad (6.3)$$

for all $n_j = 1, 2, \ldots, N$ and $1 \le j \le d$. Therefore, to solve (6.3) it is sufficient to solve

$$\ddot{U}_{\mathbf{k}}(t) - U_{\mathbf{k}}(t)\left\{2\sum_{k_j=1}^{d}\cos\left(\frac{2\pi k_j}{N}\right) - 2d\right\} = 0 \qquad (6.4)$$

with

$$U_{\mathbf{k}}(0) = 0, \quad \dot{U}_{\mathbf{k}}(0) = \frac{1}{N^d}. \tag{6.5}$$

The solution of (6.4)–(6.5) is given by

$$U_{\mathbf{k}}(t) = \begin{cases} \frac{t}{N^d} & \text{if } k_1 = k_2 = \cdots = k_d = N \\ \frac{\sin(w_{\mathbf{k}}\, t)}{N^d w_{\mathbf{k}}}, & \text{otherwise} \end{cases} \tag{6.6}$$

where $w_{\mathbf{k}} = 2\left(\sum_{j=1}^{d} \sin^2(\frac{\pi k_j}{N})\right)^{1/2}$ is the dispersion quantity. Therefore, the Green's function $G(n,t)$ is given by

$$G(\mathbf{n},t) = \frac{1}{N^d}\left\{ \sum_{\mathbf{k}=1}^{N}{}' \frac{\sin(w_{\mathbf{k}} t)}{w_{\mathbf{k}}} \cos(\frac{2\pi}{N}\mathbf{k}\cdot\mathbf{n}) \right\} + \frac{t}{N^d}, \tag{6.7}$$

where the sum $\sum_{\mathbf{k}=1}^{N}{}'$ means that $\sum_{\mathbf{k}=1}^{N}$ does not include the term with $k_1 = k_2 = \cdots = k_d = N$. It is straightforward to verify that (6.7) satisfies (6.1) and (6.2). Observe that here we used the so-called discrete Fourier Transform (see [3]).

6.2 Discrete version of Poincaré's inequality in dimension $d > 1$.

In this appendix we present a proof of a discrete version of Poincaré's inequality in dimension $d > 1$. Proofs in case $d = 1$ were given in [1] and [9]. The techniques used there do not seem to work in higher dimensions. Let $A_N = \{1, 2, \ldots, N\}$ and $\mathcal{L}_d = \underbrace{A_N \times A_N \times \cdots \times A_N}_{d \text{ times}} \setminus \{(1, 1, \ldots, 1)\}$. Given $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathcal{L}_d$ and $1 \le j \le d$ we write

$$m_{j,1} = \begin{cases} (m_1, m_2, \ldots, m_j, 1, 1, \ldots, 1) & \text{if } 1 \le j < d \\ (m_1, m_2, \ldots, m_d) & \text{if } j = d \end{cases}.$$

Also

$$\partial^+_{j,k} a_{m_1,m_2,\ldots,m_d} = a_{m_1,m_2,\ldots,m_{j-1},k+1,m_{j+1},\ldots,m_d} - a_{m_1,m_2,\ldots,m_{j-1},k,m_{j+1},\ldots,m_d}.$$

The main idea of the proof consists in writing the difference $a_{\mathbf{m}} - a_{1,1,\ldots,1}$, with $\mathbf{m} \in \mathcal{L}_d$, as a telescopic sum and estimating all the resulting terms. We distinguish two cases.

(i) All components $m_j > 1,\ j = 1, 2, \ldots, d$.

(ii) The number 1 appears exactly in k components, where $1 \le k < d$.

In order to make clear the idea let us first do the calculation when $d = 3$. Given $\mathbf{m} \in \mathcal{L}_d$ we write in the case (i)

$$\begin{aligned} a_{m_1,m_2,m_3} - a_{1,1,1} = &\sum_{n_1=1}^{m_1-1} (a_{n_1+1,1,1} - a_{n_1,1,1}) + \sum_{n_2=1}^{m_1-1} (a_{m_1,n_2+1,1} - a_{m_1,n_2,1}) \\ &+ \sum_{n_3=1}^{m_3-1} (a_{m_1,m_2,n_3+1} - a_{m_1,m_2,n_3}) \end{aligned} \tag{6.8}$$

and in case (ii) we have the following possible identities

$$a_{m_1,m_2,1} - a_{1,1,1} = \sum_{n_1=1}^{m_1-1} (a_{n_1+1,1,1} - a_{n_1,1,1}) + \sum_{n_2=1}^{m_2-1} (a_{m_1,n_2+1,1} - a_{m_1,n_2,1}), \tag{6.9}$$

if $m_1, m_2 > 1$ and $m_3 = 1$;

$$a_{m_1,1,m_3} - a_{1,1,1} = \sum_{n_1=1}^{m_1-1} (a_{n_1+1,1,1} - a_{n_1,1,1}) + \sum_{n_3=1}^{m_3-1} (a_{m_1,1,n_3+1} - a_{m_1,1,n_3}), \tag{6.10}$$

if $m_1, m_3 > 1$ and $m_2 = 1$;

$$a_{1,m_2,m_3} - a_{1,1,1} = \sum_{n_2=1}^{m_1-1} (a_{1,n_2+1,1} - a_{1,n_2,1}) + \sum_{n_3=1}^{m_1-1} (a_{m_1,1,n_3+1} - a_{1,m_2,n_3}), \tag{6.11}$$

if $m_2, m_3 > 1$ and $m_1 = 1$;

$$a_{m_1,1,1} - a_{1,1,1} = \sum_{n_1=1}^{m_1-1} (a_{n_1+1,1,1} - a_{n_1,1,1}) \tag{6.12}$$

if $m_1 > 1$ and $m_2 = m_3 = 1$;

$$a_{1,m_2,1} - a_{1,1,1} = \sum_{n_2=1}^{m_2-1} (a_{1,n_2+1,1} - a_{1,n_2,1,1}) \tag{6.13}$$

if $m_2 > 1$ and $m_1 = m_3 = 1$, and

$$a_{1,1,m_3} - a_{1,1,1} = \sum_{n_3=1}^{m_3-1} (a_{1,1,n_3+1} - a_{1,1,n_3}) \tag{6.14}$$

if $m_3 > 1$ and $m_1 = m_2 = 1$.

We can write formula (6.8) as

$$a_{m_1,m_2,m_3} - a_{1,1,1} = \sum_{\ell=1}^{3} \sum_{n_\ell=1}^{m_\ell-1} \partial^+_{\ell,n_\ell} a_{m_\ell,1}, \tag{6.15}$$

and in the case (ii), we can write each one of formulas (6.9)-(6.13) as

$$a_{m_1,m_2,m_3} - a_{1,1,1} = \sum_{r=1}^{3-k} \sum_{n_{\ell_r}=1}^{m_{\ell_r}-1} \partial_{\ell_r,n_{\ell_r}} a_{m_{\ell_r},1}, \tag{6.16}$$

where $k \in \{1, 2\}$ is the number of components of (m_1, m_2, m_3) equal to one. For example, from (6.10) we have

$$a_{m_1,1,m_3} - a_{1,1,1} = \sum_{n_1=1}^{m_1-1} (a_{n_1+1,1,1} - a_{n_1,1,1}) \tag{6.17}$$

$$= \sum_{n_1=1}^{m_1-1} \partial_{1,n_1} a_{m_1,1,1} + \sum_{n_3=1}^{m_3-1} \partial_{3,n_3} a_{m_1,1,m_3} \tag{6.18}$$

$$= \sum_{n_1=1}^{m_1-1} \partial_{1,n_1} a_{m_1,1} + \sum_{n_3=1}^{m_3-1} \partial_{3,n_3} a_{m_3,1} \tag{6.19}$$

$$= \sum_{r=1}^{2} \sum_{n_{\ell_r}=1}^{m_{\ell_r}-1} \partial_{\ell_r,n_{\ell_r}} a_{m_{\ell_r},1}. \tag{6.20}$$

Observe that in this case we have chosen $\ell_1 = 1$ and $\ell_2 = 3$. Similarly, choosing $\ell_1 = 1$, $\ell_2 = 2$ and $\ell_1 = 2$, $\ell_2 = 3$ we obtain formulas (6.9) and (6.11), respectively. To write (6.12)-(6.14) as (6.16) we proceed in the same manner. For example, choosing $\ell_1 = 2$, from (6.13) we have

$$a_{1,m_2,1} - a_{1,1,1} = \sum_{n_2=1}^{m_2-1} (a_{1,n_2+1,1} - a_{1,n_2,1}) \tag{6.21}$$

$$= \sum_{n_2=1}^{m_2-1} \partial^+_{2,n_2} a_{1,m_2,1} = \sum_{n_2=1}^{m_2-1} \partial^+_{2,n_2} a_{m_2,1} \tag{6.22}$$

$$= \sum_{n_{l_1}=1}^{m_{l_1}-1} \partial^+_{l_1,n_{l_1}} a_{m_{l_1},1}. \tag{6.23}$$

In general, in any dimension $d > 1$, if $\mathbf{m} \in \mathcal{L}_d$, then we can write the analogous to (6.17) in case (i) as

$$a_{m_1,m_2,\ldots,m_d} - a_{1,1,\ldots,1} = \sum_{\ell=1}^{d} \sum_{n_\ell=1}^{m_\ell-1} \partial^+_{\ell,n_\ell} a_{m_\ell,1}. \tag{6.24}$$

In case (ii), for each k fixed components equal to 1, there exist indexes $\ell_r \in \{1, 2, \ldots d\}$, with $r = 1, 2, \ldots d-k$, such that $\ell_1 < \ell_2 < \cdots < \ell_{d-k}$ and $m_{\ell_r} > 1$, and we can write

$$a_{m_1,m_2,\ldots,m_d} - a_{1,1,\ldots,1} = \sum_{r=1}^{d-k} \sum_{n_{\ell_r}=1}^{m_{\ell_r}-1} \partial_{\ell_r,n_{\ell_r}} a_{m_{\ell_r},1}, \tag{6.25}$$

which is the analogous to formula (6.21). Observe that there are $\binom{d}{d-k} = \frac{d!}{k!\,(d-k)!}$ possibilities of choices for $\ell_1 < \ell_2 < \cdots < \ell_{d-k}$. Now we will estimate the right hand side of identities (6.24) and (6.25). We have

$$|a_{m_1,m_2,\ldots,m_d} - a_{1,1,\ldots,1}| \leq \sum_{\ell=1}^{d} \sum_{n_\ell=1}^{m_\ell-1} |\partial^+_{\ell,n_\ell} a_{m_\ell,1}| \leq \sum_{\ell=1}^{d} \sum_{\mathbf{n}=1}^{N} |\partial^+_\ell a_{\mathbf{n}}|,$$

in case (6.24) happens, or

$$|a_{m_1,m_2,\ldots,m_d} - a_{1,1,\ldots,1}| \le \sum_{r=1}^{d-k} \sum_{n_{\ell_r}=1}^{m_{\ell_r}-1} |\partial_{\ell_r,n_{\ell_r}} a_{m_{\ell_r},1}| \le \sum_{r=1}^{d} \sum_{\mathbf{n}=1}^{N} |\partial_r^+ a_{\mathbf{n}}|,$$

if (6.25) holds. Let $S(N,d) = \sum\limits_{r=1}^{d} \sum\limits_{\mathbf{m}=1}^{N} |\partial_r^+ a_{\mathbf{m}}|$. Then, we have

$$|a_{m_1,m_2,\ldots,m_d} - a_{1,1,\ldots,1}| \le S(N,d), \quad \forall\, (m_1,m_2,\ldots,m_d) \in \mathcal{L}_d. \tag{6.26}$$

Next step is to bound $|a_{1,1,\ldots,1}|$ by $S(N,d)$. We know by hypothesis that $\sum\limits_{\mathbf{m}=1}^{N} a_{\mathbf{m}} = 0$, $\forall\, \mathbf{m} = (m_1,m_2,\ldots,m_d)$. Then

$$N^d a_{1,1,\ldots,1} = \sum_{\mathbf{m}=1}^{N} [a_{1,1,\ldots,1} - a_{\mathbf{m}}] = -\sum_{\mathbf{m}\in\mathcal{L}_d} [a_{\mathbf{m}} - a_{1,1,\ldots,1}].$$

Estimating the above identity using (6.26) we obtain

$$N^d |a_{1,1,\ldots,1}| \le \sum_{\mathbf{m}\in\mathcal{L}_d} |a_{m_1,m_2,\ldots,m_d} - a_{1,1,\ldots,1}| \le \sum_{\mathbf{m}\in\mathcal{L}_d} S(N,d) \le N^d S(N,d). \tag{6.27}$$

It follows from (6.26) and (6.27) that

$$|a_{\mathbf{m}}| = |a_{m_1,m_2,\ldots,m_d}| \le |a_{m_1,m_2,\ldots,m_d} - a_{1,1,\ldots,1}| + |a_{1,1,\ldots,1}| \le 2S(N,d), \tag{6.28}$$

whenever $\mathbf{m} \in \mathcal{L}_d$ and $\sum\limits_{\mathbf{m}=1}^{N} a_{\mathbf{m}} = 0$. Combining (6.27) and (6.28) we conclude that $|a_{m_1,m_2,\ldots,m_d}| \le S(N,d)|$ holds for all $\mathbf{m} \in A_N^d$ such that $\sum\limits_{\mathbf{m}=1}^{N} a_{\mathbf{m}} = 0$. Finally, using Schwarz's inequality we obtain

$$\sum_{\mathbf{n}=1}^{N} a_{\mathbf{n}}^2 \le \sum_{\mathbf{n}=1}^{N} 4S^2(N,d) \le 4 \sum_{\mathbf{n}=1}^{N} \left\{ N^d \sum_{r=1}^{d} d \sum_{\mathbf{n}=1}^{N} |\partial_r^+ a_{\mathbf{n}}|^2 \right\} = 4d\, N^{2d} \sum_{\mathbf{n}=1}^{N} |\nabla^+ a_{\mathbf{n}}|^2,$$

which is the discrete Poincaré's inequality in dimension $d \ge 1$.

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REFERENCES

[1] R. V. Agarwal, *Difference equations and inequalities, theory, methods and applications*, Marcel Dekker, New York, Basel, 2000.

[2] P. W. Bates, K. Lu, B. Wang, Attractors for lattices dynamical systems, *Internat. J. Bifur. Chaos Appl. Sci. Eng.* **11** (1) (2001), 143–153.

[3] W. L. Briggs, V. E. Henson, *The DFT, an owners manual for the discrete Fourier Transform*, SIAM, Philadelphia, 1995.

[4] T. Ernaux, G. Nicolis, Propagating waves in discrete bistable reaction-diffusion systems, *Phys. D* **67** (1993), 237–244.

[5] S. Flach, C. R. Willis, Discrete breathers, *Physics Report* **295** (1998), 181–164.

[6] N. I. Karachalios, A. N. Yannacopoulos, Global existence and compact attractors for the discrete nonlinear Schrödinger equation, *J. Differential Equations*, **217** (2005), 88–123.

[7] J. P. Keener, Propagation and its failure in coupled systems of discrete excitable cells, *SIAM, J. Appl. Math.* **47** (1987), 556–572.

[8] P. G. Kevrekidis, B. A. Malomed, A. R. Bishop, D. J. Frantzeskakis, Localized vortices with a semi-integer charge in nonlinear dynamical lattices, *Physical Review E* **65**, (2001), 1–7.

[9] V. V. Konotop, G. Perla Menzala, Uniform decay rates of solutions of some nonlinear lattices, *Nonlinear Analysis* **54** (2003), 261–278.

[10] V. V. Konotop, J. E. Muñoz Rivera, G. Perla Menzala, Uniform rates of decay of solutions for a nonlinear lattice with memory, *Asymptotic Analysis*, **38** (2004), 167–185.

[11] V. V. Konotop, G. Perla Menzala, Localized solutions of a nonlinear diatomic lattice, *Quarterly of Applied Mathematics*, **LXIII** (2) (2005), 201–223.

[12] B. A. Malomed, P. G. Kevrekidis, D. J. Frantzeskakis, H. E. Nistazakis, A. N. Yamacopoulos, One- and two dimensional solitons in second-harmonic-generating lattices, *Physical Review E*, **65**, (2002), 10–12.

[13] M. Nakao, Global attractors for nonlinear wave equations with nonlinear dissipative terms, *J. Differential Equations*, **227** (2006), 204–229.

[14] J. C. Oliveira, J. M. Pereira, G. Perla Menzala, Attractors for second order lattices with nonlinear damping, *Journal of Difference Equations and Applications*, **14** (9) (2008), 899–921.

[15] A. Perez-Muñuzuri, V. Perez-Muñuzuri, V. Perez-Villar, L. O. Chua, Spiral waves on a 2-d array of nonlinear circuits, *IEEE Trans. Circuits Systems* **40** (1993), 872–877.

[16] R. Teman, *Infinite dimensional dynamical systems in Mechanics and Physics*, Springer-Verlag, 1988.

[17] Y. Yan, Attractors and dimensions for discretization of a weakly damped Schrödinger equations and a Sine-Gordon equation, *Nonlinear Analysis TMA*, **20** (1993), 1417–1452.

[18] S. Zhou, Attractors and approximations for lattice dynamical systems, *J. Differential Equations*, **200** (2004), 342–368.