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SHARP WEIGHTED RELLICH AND UNCERTAINTY PRINCIPLE INEQUALITIES ON CARNOT GROUPS

ISMAIL KOMBE

[1]Department of Mathematics, Oklahoma City University, 2501 North Blackwelder
Oklahoma City, OK 73106-1493, USA
E-mail: ikombe@okcu.edu

ABSTRACT. In this work we prove sharp weighted Rellich-type inequalities and their improved versions for general Carnot groups. To derive the improved Rellich-type inequalities we have established new weighted Hardy-type inequalities with remainder terms. We also prove new sharp forms of the weighted Hardy-Poincaré and uncertainty principle inequalities for polarizable Carnot groups.

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1. INTRODUCTION

The classical Rellich inequality [26] states that for $n \geq 5$, for all $\phi \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |\Delta\phi(x)|^2 dx \geq \frac{n^2(n-4)^2}{16} \int_{\mathbb{R}^n} \frac{|\phi(x)|^2}{|x|^4} dx. \tag{1.1}$$

It is well-known that the constant $\frac{n^2(n-4)^2}{16}$ in inequality (1.1) is sharp. In a recent paper, Tertikas and Zographopoulos [28] obtained the following Rellich-type inequality that connects first to second-order derivatives:

$$\int_{\mathbb{R}^n} |\Delta\phi(x)|^2 dx \geq \frac{n^2}{4} \int_{\mathbb{R}^n} \frac{|\nabla\phi(x)|^2}{|x|^2} dx, \tag{1.2}$$

where $\phi \in C_0^\infty(\mathbb{R}^n)$, $n \geq 5$ and the constant $\frac{n^2}{4}$ is sharp. There has been considerable amount work on the Rellich-type inequalities in Euclidean spaces and Riemannian manifolds, e.g., [14], [5], [28], [24], [20] and the references therein. However, Rellich-type inequalities have not been established for general Carnot groups. Our main contribution in this direction is to find sharp weighted Rellich-type inequalities and their improved versions for general Carnot groups.

The Rellich inequality (1.1) is the first generalization of Hardy's inequality

$$\int_{\mathbb{R}^n} |\nabla\phi(x)|^2 dx \geq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{|\phi(x)|^2}{|x|^2} dx \tag{1.3}$$

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to the higher order derivatives and they are intimately related. For example, the Rellich inequality (1.1) is an easy consequence of (1.2) and the weighted Hardy inequality:

$$\int_{\mathbb{R}^n} \frac{|\nabla\phi(x)|^2}{|x|^2}dx \geq \frac{(n-4)^2}{4}\int_{\mathbb{R}^n} \frac{|\phi(x)|^2}{|x|^4}dx, \tag{1.4}$$

where $\phi \in C_0^\infty(\mathbb{R}^n)$, $n \geq 5$ and the constant $\frac{(n-4)^2}{4}$ is sharp.

It is well-known that Hardy and Rellich inequalities as well as their improved versions play important roles in many questions from spectral theory, harmonic analysis and analysis of linear and nonlinear partial differential equations. A striking example where the sharp Hardy inequality (1.3) plays a major role is the following linear heat equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + \frac{c}{|x|^2}u & \text{in} \quad \mathbb{R}^n \times (0,T), \\ u(x,0) = u_0(x) \geq 0 & \text{in} \quad \mathbb{R}^n. \end{cases} \tag{1.5}$$

In their classical paper Baras and Goldstein [4] proved that the initial value problem (1.5) has no nonnegative solutions except $u \equiv 0$ if $c > C^*(n) = (\frac{n-2}{2})^2$. Moreover, all positive solutions blow up instantaneously in the sense that if u_n is the solution of the same problem with the potential $c/|x|^2$ replaced by $V_n = \min\{c/|x|^2, n\}$, then $\lim_{n\longrightarrow\infty} u_n(x,t) = \infty$ for all $x \in \mathbb{R}^n$ and $t > 0$. If $c \leq C^*(n) = (\frac{n-2}{2})^2$, positive weak solutions do exist.

Note that the above inequalities are strict unless ϕ is identically equal to 0. Therefore it is natural to expect some extra term might be added on the right hand side of the inequalities (1.1), (1.2), (1.3) and (1.4). A remarkable result in this direction has been obtained by Brezis and Vázquez [8]. They have discovered the following sharp improved Hardy inequalities for a bounded domain $\Omega \subset \mathbb{R}^n$

$$\int_\Omega |\nabla\phi(x)|^2 dx \geq \Big(\frac{n-2}{2}\Big)^2 \int_\Omega \frac{|\phi(x)|^2}{|x|^2}dx + \mu\big(\frac{\omega_n}{|\Omega|}\big)^{2/n}\int_\Omega \phi^2 dx, \tag{1.6}$$

$$\int_\Omega |\nabla\phi(x)|^2 dx \geq \Big(\frac{n-2}{2}\Big)^2 \int_\Omega \frac{|\phi(x)|^2}{|x|^2}dx + C\Big(\int_\Omega \phi^q dx\Big)^{\frac{2}{q}}, \tag{1.7}$$

where $\phi \in H_0^1(\Omega)$, $C = C(\Omega, n) > 0$, ω_n and $|\Omega|$ denote the n-dimensional Lebesgue measure of the unit ball $B \subset \mathbb{R}^n$ and the domain Ω respectively. Here μ is the first eigenvalue of the Laplace operator in the two dimensional unit disk, and it is optimal when Ω is a ball centered at the origin. In (1.7) we assume that $2 \leq q < \frac{2n}{n-2}$ and the critical Sobolev exponent $q = 2^* = \frac{2n}{n-2}$ is not included. The work of Brezis and Vazquez [8] has been a continuous source of inspiration and a lot of progress has been made to find further improvement of the inequalities (1.1), (1.2), (1.3), (1.4), (1.6) and (1.7) in the various settings e.g., [30], [2], [6], [31], [1], [5], [19], [28], [24], [20], [23] and the references therein.

The connection between weighted Hardy and Rellich inequalities, and the importance of Hardy's inequality in analysis and partial differential equations motivates us to establish weighted Hardy-type inequalities and their improved versions on Carnot groups. Indeed, in our earlier paper, we found some sharp weighted Hardy-type inequalities and their improved versions for L^2-norms of gradients on Carnot groups [23]. In the present paper we first prove a new (non standard) form of weighted L^p-Hardy-type inequality with a sharp constant and then derive new weighted L^2-Hardy-type inequalities with remainder terms for bounded domains in Carnot groups. We stress here that weighted Hardy-type inequalities and their improved versions are the main tools for establishing weighted Rellich-type inequalities and their improved versions, respectively.

We should mention that Hardy-type inequalities have been the target of investigation in Carnot-Carathéodory spaces since the work of Garofalo and Lanconelli [18], and there has been a continuously growing literature in this direction. We refer to the recent papers by Danielli, Garofalo and Phuc [13], and Goldstein and Kombe [21], and the monograph by Capogna et al. [11] and the references therein.

It is known that Hardy and Sobolev inequalities are closely related to the Heisenberg uncertainty principle in quantum mechanics. The Heisenberg uncertainty principle says that the position and momentum of a particle cannot be determined exactly at the same time but only with an "uncertainty". More precisely, the uncertainty principle on the Euclidean space $\mathbb{R}^n$ can be stated in the following way:

$$\Big(\int_{\mathbb{R}^n} |x|^2|f(x)|^2dx\Big)\Big(\int_{\mathbb{R}^n} |\nabla f(x)|^2dx\Big) \geq \frac{n^2}{4}\Big(\int_{\mathbb{R}^n} |f(x)|^2dx\Big)^2 \tag{1.8}$$

for all $f \in L^2(\mathbb{R}^n)$. It is well known that equality is attained in the above if and only if f is a Gaussian function (i.e. $f(x) = Ae^{-\alpha|x|^2}$ for some $A \in \mathbb{R}, \alpha > 0$). There exists large literature devoted to deriving various uncertainty principle type inequalities in the Euclidean and other settings (see [16], [23] and the references therein). However, much less is known about sharp uncertainty principle inequalities on Carnot groups. In [23] we obtained the following uncertainty principle-type inequality:

$$\Big(\int_{\mathbb{G}} N^2|\nabla_{\mathbb{G}}N|^2\phi^2dx\Big)\Big(\int_{\mathbb{G}} |\nabla_{\mathbb{G}}\phi|^2dx\Big) \geq \Big(\frac{Q-2}{2}\Big)^2\Big(\int_{\mathbb{G}} |\nabla_{\mathbb{G}}N|^2\phi^2dx\Big)^2, \tag{1.9}$$

where $N = u^{1/(2-Q)}$ is the homogeneous norm associated to Folland's fundamental solution u for the sub-Laplacian $\Delta_{\mathbb{G}}$ and Q is the homogeneous dimension of $\mathbb{G}$. It is clear that this inequality is not sharp. In this paper, motivated by a result of Balogh and Tyson [3], we prove a sharp analog of the uncertainty principle inequality (1.8) for polarizable Carnot groups.

In order to state and prove our theorems, we first recall the basic properties of Carnot group $\mathbb{G}$ and some well-known results that will be used in the sequel. Further information can be found in [3], [7], [10], [12], [15], [17], [25], [27], [29].

2. PRELIMINARIES

A Carnot group is a connected, simply connected, nilpotent Lie group $\mathbb{G}$ whose Lie algebra $\mathcal{G}$ admits a stratification. That is, there exist linear subspaces $V_1, \ldots, V_k$ of $\mathcal{G}$ such that

$$\mathcal{G} = V_1 \oplus \cdots \oplus V_k, \quad [V_1, V_i] = V_{i+1}, \quad \text{for} \quad i = 1, 2, \ldots, k-1 \quad \text{and} \quad [V_1, V_k] = 0 \tag{2.1}$$

where $[V_1, V_i]$ is the subspace of $\mathcal{G}$ generated by the elements $[X, Y]$ with $X \in V_1$ and $Y \in V_i$. This defines a k-step Carnot group and integer $k \geq 1$ is called the step of $\mathbb{G}$.

Via the exponential map, it is possible to induce on $\mathbb{G}$ a family of automorphisms of the group, called dilations, $\delta_\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^n (\lambda > 0)$ such that

$$\delta_\lambda(x_1, \ldots, x_n) = (\lambda^{\alpha_1} x_1, \ldots, \lambda^{\alpha_n} x_n)$$

where $1 = \alpha_1 = \cdots = \alpha_m < \alpha_{m+1} \leq \cdots \leq \alpha_n$ are integers and $m = \dim(V_1)$.

The group law can be written in the following form

$$x \cdot y = x + y + P(x, y), \quad x, y \in \mathbb{R}^n \tag{2.2}$$

where $P : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ has polynomial components and $P_1 = \cdots = P_m = 0$. Note that the inverse x^{-1} of an element $x \in \mathbb{G}$ has the form $x^{-1} = -x = (-x_1, \ldots, -x_n)$.

Let $X_1, \ldots, X_m$ be a family of left invariant vector fields which form an orthonormal basis of $V_1 \equiv \mathbb{R}^m$ at the origin, that is, $X_1(0) = \partial_{x_1}, \ldots, X_m(0) = \partial_{x_m}$. The vector fields X_j have polynomial coefficients and can be assumed to be of the form

$$X_j(x) = \partial_j + \sum_{i=j+1}^{n} a_{ij}(x)\partial_i, \quad X_j(0) = \partial_j, j = 1, \ldots, m,$$

where each polynomial a_{ij} is homogeneous with respect to the dilations of the group, that is $a_{ij}(\delta_\lambda(x)) = \lambda^{\alpha_i - \alpha_j} a_{ij}(x)$. The horizontal gradient on Carnot group $\mathbb{G}$ is the vector valued operator

$$\nabla_{\mathbb{G}} = (X_1, \ldots, X_m)$$

where $X_1, \ldots, X_m$ are the generators of $\mathbb{G}$. The sub-Laplacian is the second-order partial differential operator on $\mathbb{G}$ given by

$$\Delta_{\mathbb{G}} = \sum_{j=1}^{m} X_j^2.$$

The fundamental solution u for $\Delta_{\mathbb{G}}$ is defined to be a weak solution to the equation

$$-\Delta_{\mathbb{G}} u = \delta \tag{2.3}$$

where δ denotes the Dirac distribution with singularity at the neutral element 0 of $\mathbb{G}$. In [15], Folland proved that in any Carnot group $\mathbb{G}$, there exists a homogeneous norm N such that

$$u = N^{2-Q}$$

is harmonic in $\mathbb{G}\setminus\{0\}$. Furthermore there exists a constant $c_2 > 0$ so that $c_2 u$ satisfies (2.3) in the sense of distributions. The number Q, called the homogeneous dimension of $\mathbb{G}$, is defined by

$$Q = \sum_{j=1}^{k} j(\text{dim}V_j)$$

and plays an important role in in the analysis of Carnot groups.

We now set

$$N(x) := \begin{cases} u^{\frac{1}{2-Q}} & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases} \tag{2.4}$$

We recall that a homogeneous norm on $\mathbb{G}$ is a continuous function $N : \mathbb{G} \longrightarrow [0,\infty)$ smooth away from the origin which satisfies the conditions : $N(\delta_\lambda(x)) = \lambda N(x)$, $N(x^{-1}) = N(x)$ and $N(x) = 0$ iff $x = 0$.

A Carnot group $\mathbb{G}$ is said to be polarizable if the homogeneous norm $N = u^{1/(2-Q)}$ satisfies the following ∞-sub-Laplace equation,

$$\Delta_{\mathbb{G},\infty} N := \frac{1}{2}\langle \nabla_{\mathbb{G}}(|\nabla_{\mathbb{G}} N|^2), \nabla_{\mathbb{G}} N\rangle = 0, \qquad \text{in} \quad \mathbb{G}\setminus\{0\}. \tag{2.5}$$

This class of groups were introduced by Balogh and Tyson [3] and admit the analogue of polar coordinates. It is known that Euclidean space, the Heisenberg group and the Kaplan's H-type group [22] are polarizable Carnot groups (see [12], [3]).

In [3], Balogh and Tyson proved that the homogeneous norm $N = u^{1/(2-Q)}$, associated to Folland's solution u for the sub-Laplacian $\Delta_{\mathbb{G}}$, enters also in the expression of the fundamental solution of the sub-elliptic p-Laplacian:

$$\Delta_{\mathbb{G},p} u = \sum_{i=1}^{m} X_i(|Xu|^{p-2} X_i u), \quad 1 < p < \infty, \tag{2.6}$$

on polarizable Carnot groups. More precisely, they proved that for every $1 < p < \infty, p = Q$

$$u_p = \begin{cases} N^{\frac{p-Q}{p-1}}, & \text{if} \quad p \neq Q, \\ -\log N, & \text{if} \quad p = Q. \end{cases} \tag{2.7}$$

is p-harmonic in $\mathbb{G}\setminus\{0\}$. Furthermore there exists a constant c_p so that $c_p u_p$ satisfies

$$-\Delta_{\mathbb{G},p}(c_p u_p) = \delta$$

in the sense of distributions. In the setting of H-type groups, explicit formulas for the fundamental solutions of the sub-elliptic p-Laplacian has been found by Capogna, Danielli and Garofalo [10].

The following formula:

$$\nabla_{\mathbb{G}} \cdot \Big(\frac{N}{|\nabla_{\mathbb{G}} N|^2} \cdot \nabla_{\mathbb{G}} N\Big) = Q \quad \text{in} \quad \mathbb{G}\setminus\mathcal{Z} \tag{2.8}$$

was proved by Balogh and Tyson [3]. Here $\mathcal{Z} := \{0\} \cup \{x \in \mathbb{G} \setminus \{0\} : \nabla_{\mathbb{G}} N(x) = 0\}$ has Haar measure zero and $\nabla_{\mathbb{G}} N \neq 0$ for a.e. $x \in \mathbb{G}$.

The curve $\gamma : [a, b] \subset \mathbb{R} \longrightarrow \mathbb{G}$ is called horizontal if its tangents lie in V_1, i.e, $\gamma'(t) \in$ *span*$\{X_1, \ldots, X_m\}$ for all t. Then, the Carnot-Carathéodory distance $d_{CC}(x, y)$ between two points $x, y \in \mathbb{G}$ is defined to be the infimum of all horizontal lengths $\int_a^b \langle \gamma'(t), \gamma'(t)\rangle^{1/2} dt$ over all horizontal curves $\gamma : [a, b] \longrightarrow \mathbb{G}$ such that $\gamma(a) = x$ and $\gamma(b) = y$. Notice that d_{cc} is a homogeneous distance and satisfies the invariance property

$$d_{cc}(z \cdot x, z \cdot y) = d_{cc}(x, y), \quad \text{for all } x, y, z \in \mathbb{G},$$

and is homogeneous of degree one with respect to the dilation δ_λ, i.e.

$$d_{cc}(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d_{cc}(x, y), \quad \text{for all } x, y, z \in \mathbb{G}, \text{for all } \lambda > 0.$$

The Carnot-Carathéodory balls are defined by $B(y, R) = \{x \in \mathbb{G} | d_{cc}(y, x) < R\}$. By left-translation and dilation, it is easy to see that the Haar measure of $B(y, R)$ is proportional by R^Q. More precisely

$$|B(y, R)| = R^Q |B(y, 1)| = R^Q |B(0, 1)|.$$

We now set

$$B_\varrho := B(0, R) = \{x \in \mathbb{G} : \varrho(x) < R\}$$

where $\varrho := d_{cc}(0, x)$ is the Carnot-Carathéodory distance of x from the origin. Note that ϱ is a homogeneous norm and equivalent to other homogeneous norm on $\mathbb{G}$. At this point we remark that $|\nabla_{\mathbb{G}} N|$ is uniformly bounded and $N : (\mathbb{G}, d_{cc}) \longrightarrow \mathbb{R}$ is Lipschitz (see [3]).

We now recall the following integration formula in polar coordinates on $\mathbb{G}$

$$\int_{\mathbb{G}} f(x) dx = \int_0^\infty \int_S f(\delta_\lambda u) \lambda^{Q-1} d\sigma(u) d\lambda$$

which is valid for all $f \in L^1(\mathbb{G})$. Here $S = \{N = 1\}$ is the unit sphere with respect to the homogeneous norm N and $d\sigma$ is a Radon measure on S (see [17], [3], [25], [7]). Now it is clear the radial function ρ^α (ϱ is any homogeneous norm on $\mathbb{G}$) is locally integrable if $\alpha > -Q$.

3. SHARP WEIGHTED HARDY TYPE INEQUALITIES

In this section we prove various weighted Hardy-type inequalities and their improved versions. We begin this section by proving a new form of the weighted Hardy-Poincaré-type inequality with a sharp constant.

Theorem 3.1. *Let $\mathbb{G}$ be a polarizable Carnot group with homogeneous dimension $Q \geq 3$ and let $\phi \in C_0^\infty(\mathbb{G})$, $1 < p < Q$ and $\alpha > -Q$. Then the following inequality is valid :*

$$\int_{\mathbb{G}} N^{\alpha+p} \frac{|\nabla_{\mathbb{G}} N \cdot \nabla_{\mathbb{G}} \phi|^p}{|\nabla_{\mathbb{G}} N|^{2p}} dx \geq \Big(\frac{Q + \alpha}{p}\Big)^p \int_{\mathbb{G}} N^\alpha |\phi|^p dx. \tag{3.1}$$

Furthermore, the constant $(\frac{Q+\alpha}{p})^p$ is sharp.

Proof. Using the volume growth condition formula (2.8) and integration by parts, we get

$$(Q+\alpha)\int_{\mathbb{G}} N^{\alpha}|\phi|^p dx = -p\int_{\mathbb{G}} \frac{|\phi|^{p-2}\phi N^{\alpha+1}}{|\nabla_{\mathbb{G}}N|^2}\nabla_{\mathbb{G}}N\cdot\nabla_{\mathbb{G}}\phi dx.$$

An application of Hölder's and Young's inequality yields

$$\begin{aligned}(Q+\alpha)\int_{\mathbb{G}} N^{\alpha}|\phi|^p dx &\le p\Big(\int_{\mathbb{G}} N^{\alpha}|\phi|^p dx\Big)^{(p-1)/p}\Big(\int_{\mathbb{G}}\frac{N^{\alpha+p}|\nabla_{\mathbb{G}}N\cdot\nabla_{\mathbb{G}}\phi|^p}{|\nabla_{\mathbb{G}}N|^{2p}}dx\Big)^{1/p}\\ &\le (p-1)\epsilon^{-p/(p-1)}\int_{\mathbb{G}} N^{\alpha}|\phi|^p dx + \epsilon^p\int_{\mathbb{G}}\frac{N^{\alpha+p}|\nabla_{\mathbb{G}}N\cdot\nabla_{\mathbb{G}}\phi|^p}{|\nabla_{\mathbb{G}}N|^{2p}}dx\end{aligned}$$

for any $\epsilon>0$. Therefore

$$\int_{\mathbb{G}}\frac{N^{\alpha+p}|\nabla_{\mathbb{G}}N\cdot\nabla_{\mathbb{G}}\phi|^p}{|\nabla_{\mathbb{G}}N|^{2p}}dx \ge \epsilon^{-p}\Big(Q+\alpha-(p-1)\epsilon^{-p/(p-1)}\Big)\int_{\mathbb{G}} N^{\alpha}|\phi|^p dx. \tag{3.2}$$

Note that the function $\epsilon \longrightarrow \epsilon^{-p}\Big(Q+\alpha-(p-1)\epsilon^{-p/(p-1)}\Big)$ attains the maximum for $\epsilon^{p/(p-1)} = \frac{p}{Q+\alpha}$, and this maximum is equal to $\Big(\frac{Q+\alpha}{p}\Big)^p$. Now we obtain the desired inequality

$$\int_{\mathbb{G}} N^{\alpha+p}\frac{|\nabla_{\mathbb{G}}N\cdot\nabla_{\mathbb{G}}\phi|^p}{|\nabla_{\mathbb{G}}N|^{2p}}dx \ge \Big(\frac{Q+\alpha}{p}\Big)^p\int_{\mathbb{G}} N^{\alpha}|\phi|^p dx.$$

Next we claim that $\big(\frac{Q+\alpha}{p}\big)^p$ is the best constant in (3.1):

$$\begin{aligned}C_H :&= \inf_{0\neq\phi\in C_0^\infty(\mathbb{G})}\frac{\int_{\mathbb{G}} N^{\alpha+p}\frac{|\nabla_{\mathbb{G}}N\cdot\nabla_{\mathbb{G}}\phi|^p}{|\nabla_{\mathbb{G}}N|^{2p}}dx}{\int_{\mathbb{G}} N^{\alpha}|\phi|^p dx},\\ &= \Big(\frac{Q+\alpha}{p}\Big)^p.\end{aligned}$$

It is clear that

$$\big(\frac{Q+\alpha}{p}\big)^p \le \frac{\int_{\mathbb{G}} N^{\alpha+p}\frac{|\nabla_{\mathbb{G}}N\cdot\nabla_{\mathbb{G}}\phi|^p}{|\nabla_{\mathbb{G}}N|^{2p}}dx}{\int_{\mathbb{G}} N^{\alpha}|\phi|^p dx} \tag{3.3}$$

holds for all $\phi\in C_0^\infty(\mathbb{G})$. If we pass to the inf in (3.3) we get that $\big(\frac{Q+\alpha}{p}\big)^p \le C_H$. We only need to show that $C_H \le \big(\frac{Q+\alpha}{p}\big)^p$ and for this we use the following family of radial functions

$$\phi_\epsilon(N) = \begin{cases} N^{\frac{Q+\alpha}{p}+\epsilon} & \text{if} \quad N\in[0,1],\\ N^{-(\frac{Q+\alpha}{p}+\epsilon)} & \text{if} \quad N>1,\end{cases} \tag{3.4}$$

where $\epsilon>0$. Notice that $\phi_\epsilon(N)$ can be approximated by smooth functions with compact support in $\mathbb{G}$.

A direct computation shows that

$$N^{\alpha+p}\frac{|\nabla_{\mathbb{G}}N\cdot\nabla_{\mathbb{G}}\phi_\epsilon|^p}{|\nabla_{\mathbb{G}}N|^{2p}} = \begin{cases}\big(\frac{Q+\alpha}{p}+\epsilon\big)^p N^{Q+2\alpha+p\epsilon} & \text{if} \quad N\in[0,1],\\ \big(\frac{Q+\alpha}{p}+\epsilon\big)^p N^{-Q-p\epsilon} & \text{if} \quad N>1.\end{cases}$$

Let us denote by $\mathbb{B}_1 = \{x \in \mathbb{G} : N(x) \leq 1\}$ the unit ball with respect to the homogeneous norm N. Hence

$$\int_{\mathbb{G}} N^{\alpha}|\phi_\epsilon|^p dx = \int_{\mathbb{B}_1} N^{Q+2\alpha+p\epsilon}dx + \int_{\mathbb{G}\setminus\mathbb{B}_1} N^{-Q-p\epsilon}dx.$$

Note that, for every $\epsilon > 0$, the weights $N^{Q+2\alpha+p\epsilon}$ and $N^{-Q-p\epsilon}$ are integrable at 0 and ∞, respectively. This implies that $\int_{\mathbb{G}} N^{\alpha}|\phi_\epsilon|^p dx$ is finite. Thus we have

$$\begin{aligned}(\frac{Q+\alpha}{p}+\epsilon)^p \int_{\mathbb{G}} N^{\alpha}|\phi_\epsilon|^p dx &= (\frac{Q+\alpha}{p}+\epsilon)^p \Big[\int_{\mathbb{B}_1} N^{Q+2\alpha+p\epsilon}dx + \int_{\mathbb{G}\setminus\mathbb{B}_1} N^{-Q-p\epsilon}dx\Big] \\ &= \int_{\mathbb{G}} N^{\alpha+p}\frac{|\nabla_{\mathbb{G}}N\cdot\nabla_{\mathbb{G}}\phi_\epsilon|^p}{|\nabla_{\mathbb{G}}N|^{2p}}dx.\end{aligned}$$

On the other hand

$$\begin{aligned}\frac{\big(\frac{Q+\alpha}{p}+\epsilon\big)^p}{C_H}\int_{\mathbb{G}} N^{\alpha+p}\frac{|\nabla_{\mathbb{G}}N\cdot\nabla_{\mathbb{G}}\phi_\epsilon|^p}{|\nabla_{\mathbb{G}}N|^{2p}}dx &\geq (\frac{Q+\alpha}{p}+\epsilon)^p\int_{\mathbb{G}} N^{\alpha}|\phi_\epsilon|^p dx \\ &= \int_{\mathbb{G}} N^{\alpha+p}\frac{|\nabla_{\mathbb{G}}N\cdot\nabla_{\mathbb{G}}\phi_\epsilon|^p}{|\nabla_{\mathbb{G}}N|^{2p}}dx.\end{aligned}$$

It is clear that $\big(\frac{Q+\alpha}{p}+\epsilon\big)^p \geq C_H$ and letting $\epsilon \longrightarrow 0$ we obtain $\big(\frac{Q+\alpha}{p}\big)^p \geq C_H$. Therefore $C_H = \big(\frac{Q+\alpha}{p}\big)^p$. $\square$

The following L^p-Hardy-type inequality is the weighted extension of Theorem 3.1 in [21] and plays important roles in the proof of Theorem 3.6 and Section 4.

Theorem 3.2. *Let $\mathbb{G}$ be a polarizable Carnot group with homogeneous dimension $Q \geq 3$ and let $\phi \in C_0^\infty(\mathbb{G})$, $\alpha \in \mathbb{R}$, $1 < p < Q$ and $Q+\alpha-p > 0$. Then the following inequality is valid :*

$$\int_{\mathbb{G}} N^{\alpha}|\nabla_{\mathbb{G}}\phi|^p dx \geq \Big(\frac{Q+\alpha-p}{p}\Big)^p \int_{\mathbb{G}} N^{\alpha}\frac{|\nabla_{\mathbb{G}}N|^p}{N^p}|\phi|^p dx. \tag{3.5}$$

Furthermore, the constant $(\frac{Q+\alpha-p}{p})^p$ is sharp.

Proof. Let $\phi \in C_0^\infty(\mathbb{G})$ and define $\psi = N^{-\gamma}\phi$ where $\gamma < 0$. A direct calculation shows that

$$|\nabla_{\mathbb{G}}\phi| = |\gamma N^{\gamma-1}\psi\nabla_{\mathbb{G}}N + N^{\gamma}\nabla_{\mathbb{G}}\psi|. \tag{3.6}$$

We now use the following convexity inequality

$$|a+b|^p - |a|^p \geq c(p)|b|^p + p|a|^{p-2}a\cdot b, \tag{3.7}$$

where $a, b \in \mathbb{R}^n$, $p \geq 2$ and $c(p) > 0$ (see [6]). In view of (3.7) we have that

$$\begin{aligned}\int_{\mathbb{G}} N^{\alpha}|\nabla_{\mathbb{G}}\phi|^p dx \geq |\gamma|^p &\int_{\mathbb{G}} N^{\alpha+\gamma p-p}|\nabla_{\mathbb{G}}N|^p|\psi|^p dx \\ &+ |\gamma|^{p-2}\gamma\int_{\mathbb{G}} N^{\alpha+\gamma p-p+1}|\nabla_{\mathbb{G}}N|^{p-2}\nabla_{\mathbb{G}}N\cdot\nabla_{\mathbb{G}}(|\psi|^p)dx \\ &+ c(p)\int_{\mathbb{G}} N^{\alpha+p\gamma}|\nabla_{\mathbb{G}}\psi|^p dx.\end{aligned}$$

Clearly

$$\int_{\mathbb{G}} N^{\alpha}|\nabla_{\mathbb{G}}\phi|^p dx \geq |\gamma|^p \int_{\mathbb{G}} N^{\alpha+\gamma p-p}|\nabla_{\mathbb{G}}N|^p|\psi|^p dx + |\gamma|^{p-2}\gamma \int_{\mathbb{G}} N^{\alpha+\gamma p-p+1}|\nabla_{\mathbb{G}}N|^{p-2}\nabla_{\mathbb{G}}N \cdot \nabla_{\mathbb{G}}(|\psi|^p)dx,$$

and integration by parts gives

$$\int_{\mathbb{G}} N^{\alpha}|\nabla_{\mathbb{G}}\phi|^p dx \geq |\gamma|^p \int_{\mathbb{G}} N^{\alpha+\gamma p-p}|\nabla_{\mathbb{G}}N|^p|\psi|^p dx - |\gamma|^{p-2}\gamma \int_{\mathbb{G}} \nabla_{\mathbb{G}} \cdot (N^{\alpha+\gamma p-p+1}|\nabla_{\mathbb{G}}N|^{p-2}\nabla_{\mathbb{G}}N)|\psi|^p dx.$$

We now choose $\gamma = \frac{p-Q-\alpha}{p}$; then we get

$$\int_{\mathbb{G}} \nabla_{\mathbb{G}} \cdot (N^{\alpha+\gamma p-p+1}|\nabla_{\mathbb{G}}N|^{p-2}\nabla_{\mathbb{G}}N)|\psi|^p dx = \int_{\mathbb{G}} \nabla_{\mathbb{G}} \cdot (N^{1-Q}|\nabla_{\mathbb{G}}N|^{p-2}\nabla_{\mathbb{G}}N)|\psi|^p dx. \tag{3.8}$$

Since u_p is the fundamental solution of sub-p-Laplacian $-\Delta_{\mathbb{G},p}$, we then have

$$\begin{aligned}\int_{\mathbb{G}} \nabla_{\mathbb{G}} \cdot (N^{1-Q}|\nabla_{\mathbb{G}}N|^{p-2}\nabla_{\mathbb{G}}N)|\psi|^p dx &= -c(\mathbb{G},p)|\phi(0)|^p N^{(Q+\alpha-p)}(0) \\ &= 0\end{aligned} \tag{3.9}$$

where $c(\mathbb{G},p)$ is a positive constant (see [3]). Hence we obtain the desired inequality

$$\int_{\mathbb{G}} N^{\alpha}|\nabla_{\mathbb{G}}\phi|^p dx \geq \Big(\frac{Q+\alpha-p}{p}\Big)^p \int_{\mathbb{G}} N^{\alpha}\frac{|\nabla_{\mathbb{G}}N|^p}{N^p}|\phi|^p dx. \tag{3.10}$$

To show that the constant $\left(\frac{Q+\alpha-p}{p}\right)^p$ is sharp, we use the following family of radial functions

$$\phi_\epsilon(N) = \begin{cases} N^{\frac{Q+\alpha-p}{p}+\epsilon} & \text{if} \quad N \in [0,1], \\ N^{-(\frac{Q+\alpha-p}{p}+\epsilon)} & \text{if} \quad N > 1, \end{cases}$$

and pass to the limit as $\epsilon \longrightarrow 0$. Note that the Theorem (3.2) also holds for $1 < p < 2$ and in this case we use the following inequality:

$$|a+b|^p - |a|^p \geq c(p)\frac{|b|^2}{(|a|+|b|)^{2-p}} + p|a|^{p-2}a \cdot b \tag{3.11}$$

where $a \in \mathbb{R}^n, b \in \mathbb{R}^n$ and $c(p) > 0$ (see [6]). □

Remark 3.3. We remark that if $p = 2$ then we remove the polarizability condition and the inequality (3.5) holds in any Carnot group (see [23]).

IMPROVED HARDY TYPE INEQUALITIES. We now prove improved weighted Hardy-type inequalities and also collect other known improved weighted Hardy-type inequalities that will be used in Section 4. To motivate our discussion, let us recall the the following sharp improved Hardy inequality from the Euclidean setting:

$$\int_\Omega |x|^\alpha |\nabla\phi|^2 dx \geq \Big(\frac{n+\alpha-2}{2}\Big)^2 \int_\Omega |x|^\alpha \frac{\phi^2}{|x|^2} dx + \frac{1}{4}\int_\Omega |x|^\alpha \frac{\phi^2}{|x|^2(\ln\frac{R}{|x|})^2} dx, \qquad (3.12)$$

where Ω is abounded domain with smooth boundary, $0\in\Omega$, $\phi\in C_0^\infty(\Omega)$, $n\geq 1$, $\alpha\in\mathbb{R}$, $R\geq e\sup_\Omega |x|$ and $n+\alpha-2>0$. Furthermore, the constant $\frac{1}{4}$ is sharp and this inequality has immediate applications in partial differential equations (see [2], [31], [1] [19]). Motivated by the above results our first goal is to obtain the analog of (3.12) for bounded domains in Carnot groups.

Theorem 3.4. *Let $\mathbb{G}$ be a Carnot group with homogeneous norm $N=u^{1/(2-Q)}$ and let $\Omega\subset\mathbb{G}$ be a bounded domain with smooth boundary, $0\in\Omega$, $R\geq e\sup_\Omega N$, $\alpha\in\mathbb{R}$, $Q\geq 3$, $Q+\alpha-2>0$. Then the following inequality holds:*

$$\int_\Omega N^\alpha |\nabla_{\mathbb{G}}\phi|^2 dx \geq \Big(\frac{Q+\alpha-2}{2}\Big)^2 \int_\Omega N^\alpha \frac{|\nabla_{\mathbb{G}}N|^2}{N^2}\phi^2 dx + \frac{1}{4}\int_\Omega N^\alpha \frac{|\nabla_{\mathbb{G}}N|^2}{N^2}\frac{\phi^2}{(\ln\frac{R}{N})^2} dx \qquad (3.13)$$

for all compactly supported smooth function $\phi\in C_0^\infty(\Omega)$.

Proof. Let $\phi\in C_0^\infty(\Omega)$ and define $\psi=N^{-\beta}\phi$ where $\beta<0$. A direct calculation shows that

$$|\nabla_{\mathbb{G}}\phi|^2 = \beta^2 N^{2\beta-2}|\nabla_{\mathbb{G}}N|^2\psi^2 + 2\beta N^{2\beta-1}\psi\nabla_{\mathbb{G}}N\cdot\nabla_{\mathbb{G}}\psi + N^{2\beta}|\nabla_{\mathbb{G}}\psi|^2. \qquad (3.14)$$

Multiplying both sides of (3.14) by the N^α and applying integration by parts over Ω gives

$$\begin{aligned}\int_\Omega N^\alpha |\nabla_{\mathbb{G}}\phi|^2 dx = \beta^2\int_\Omega N^{\alpha+2\beta-2}|\nabla_{\mathbb{G}}N|^2\psi^2 dx - \frac{\beta}{\alpha+2\beta}\int_\Omega \Delta_{\mathbb{G}}(N^{\alpha+2\beta})\psi^2 dx \\ + \int_\Omega N^{\alpha+2\beta}|\nabla_{\mathbb{G}}\psi|^2 dx.\end{aligned} \qquad (3.15)$$

We can easily show that

$$-\frac{\beta}{\alpha+2\beta}\Delta_{\mathbb{G}}(N^{\alpha+2\beta}) = -\beta(\alpha+2\beta+Q-2)N^{\alpha+2\beta-2}|\nabla_{\mathbb{G}}N|^2 - \frac{\beta}{2-Q}N^{\alpha+2\beta+Q-2}\Delta_{\mathbb{G}}u. \qquad (3.16)$$

Substituting (3.16) into (3.15) and using the fact that $\psi^2=N^{-2\beta}\phi^2$ yield

$$\begin{aligned}\int_\Omega N^\alpha |\nabla_{\mathbb{G}}\phi|^2 dx = (-\beta^2-\beta(\alpha+Q-2)\int_\Omega N^\alpha\frac{|\nabla_{\mathbb{G}}N|^2}{N^2}\phi^2 dx \\ - \frac{\beta}{2-Q}\int_\Omega (\Delta_{\mathbb{G}}u)N^{\alpha+Q-2}\phi^2 dx \\ + \int_\Omega N^{\alpha+2\beta}|\nabla_{\mathbb{G}}\psi|^2 dx.\end{aligned}$$

The middle integral vanishes since u is the fundamental solution of sub-Laplacian $\Delta_{\mathbb{G}}$, therefore we have

$$\int_{\Omega} N^{\alpha}|\nabla_{\mathbb{G}}\phi|^2dx = \big(-\beta^2-\beta(\alpha+Q-2)\big)\int_{\Omega} N^{\alpha}\frac{|\nabla_{\mathbb{G}}N|^2}{N^2}\phi^2dx + \int_{\Omega} N^{\alpha+2\beta}|\nabla_{\mathbb{G}}\psi|^2dx.$$

Note that the quadratic function $-\beta^2-\beta(\alpha+Q-2)$ attains maximum for $\beta=\frac{2-Q-\alpha}{2}$ and this maximum equal to $(\frac{Q+\alpha-2}{2})^2$. Therefore

$$\int_{\Omega} N^{\alpha}|\nabla_{\mathbb{G}}\phi|^2dx = \Big(\frac{Q+\alpha-2}{2}\Big)^2\int_{\Omega} N^{\alpha}\frac{|\nabla_{\mathbb{G}}N|^2}{N^2}\phi^2dx + \int_{\Omega} N^{2-Q}|\nabla_{\mathbb{G}}\psi|^2dx. \tag{3.17}$$

Let us define $\varphi(x)=(\ln\frac{R}{N})^{-\frac{1}{2}}\psi(x)$ where N is the homogeneous norm which is defined as in (2.4). A direct calculation shows that

$$\begin{aligned}\int_{\Omega} N^{2-Q}|\nabla_{\mathbb{G}}\psi|^2dx &= \frac{1}{4}\int_{\Omega} N^{-Q}|\nabla_{\mathbb{G}}N|^2(\ln(\frac{R}{N})^{-1}\varphi^2 + \int_{\Omega} N^{2-Q}\ln(\frac{R}{N})|\nabla_{\mathbb{G}}\varphi|^2dx\\ &\quad -\frac{1}{2(2-Q)}\int_{\Omega}\Delta_{\mathbb{G}}(N^{2-Q})\varphi^2dx.\end{aligned}$$

It is clear that the last integral term vanishes. Therefore we have

$$\begin{aligned}\int_{\Omega} N^{2-Q}|\nabla_{\mathbb{G}}\psi|^2dx &\geq \frac{1}{4}\int_{\Omega} N^{-Q}|\nabla_{\mathbb{G}}N|^2(\ln(\frac{R}{N})^{-1}\varphi^2\\ &= \frac{1}{4}\int_{\Omega} N^{-Q}|\nabla_{\mathbb{G}}N|^2\frac{\psi^2}{(\ln\frac{R}{N})^2}dx\\ &= \frac{1}{4}\int_{\Omega} N^{\alpha-2}|\nabla_{\mathbb{G}}N|^2\frac{\phi^2}{(\ln\frac{R}{N})^2}dx.\end{aligned} \tag{3.18}$$

Substituting (3.18) into (3.17) which yields the desired inequality (3.13). □

One of the advantages of our approach is that it automatically yields a remainder term and then using a suitable functional change lead us to obtain an explicit remainder term as in the Theorem 3.3. On the other hand, there are other techniques that we can use to obtain explicit remainder term. In our earlier paper [23] we have used weighted Sobolev-Poincare inequalities and obtained the following improved weighted Hardy-type inequalities.

Theorem 3.5. *(*[23]*) Let $\mathbb{G}$ be a Carnot group with homogeneous norm $N=u^{1/(2-Q)}$, and let $\phi\in C_0^{\infty}(B_{\varrho})$, $\alpha\in\mathbb{R}$, $Q\geq 3$ and $Q+\alpha-2>0$. Then the following inequality is valid:*

$$\int_{B_{\varrho}} N^{\alpha}|\nabla_{\mathbb{G}}\phi|^2dx \geq \Big(\frac{Q+\alpha-2}{2}\Big)^2\int_{B_{\varrho}} N^{\alpha}\frac{|\nabla_{\mathbb{G}}N|^2}{N^2}\phi^2dx + \frac{1}{C^2R^2}\int_{B_{\varrho}} N^{\alpha}\phi^2dx, \tag{3.19}$$

where C is a positive constant and R is the radius of the ball B_{ϱ}.

Theorem 3.6. *(*[23]*) Let $\mathbb{G}$ be a Carnot group with homogeneous norm $N=u^{1/(2-Q)}$ and let $\phi\in C_0^{\infty}(B_{\varrho})$, $\alpha\in\mathbb{R}$, $Q\geq 3$, $Q+\alpha-2>0$ and $q>2$. Then the following inequality is valid:*

$$\int_{B_{\varrho}} N^{\alpha}|\nabla_{\mathbb{G}}\phi|^2dx \geq \Big(\frac{Q+\alpha-2}{2}\Big)^2\int_{B_{\varrho}} N^{\alpha}\frac{|\nabla_{\mathbb{G}}N|^2}{N^2}\phi^2dx + \frac{K}{C^2R^2}\Big(\int_{B_{\varrho}} N^{\sigma}\phi^q dx\Big)^{2/q}, \tag{3.20}$$

where $C>0$, R is the radius of the ball B_ϱ, $\sigma=\frac{(2-Q)(2-q)+\alpha q}{2}$ and $K=\left(\int_{B_\varrho} N^{2-Q}dx\right)^{\frac{q-2}{q}}$.

Notice that the remainder terms in Theorems 3.4 and 3.5 contain functions of the homogeneous norm N and ϕ. Motivated by the recent work of Abdellaoui, Colorado and Peral [1] we have following inequality (which is a weighted version of the inequality (3.4) in [23]) so that remainder term contains functions of N and $|\nabla_{\mathbb{G}}\phi|$.

Theorem 3.7. *Let $\mathbb{G}$ be a polarizable Carnot group with homogeneous dimension $Q\geq 3$ and let Ω be a bounded domain with smooth boundary which contains the origin, $\alpha\in\mathbb{R}$, $Q+\alpha-2>0$, and $1<q<2$. Then there exists a positive constant $C=C(Q,q,\Omega)$ such that the following inequality holds:*

$$\int_\Omega N^\alpha|\nabla_{\mathbb{G}}\phi|^2dx\geq\Big(\frac{Q+\alpha-2}{2}\Big)^2\int_\Omega N^\alpha\frac{|\nabla_{\mathbb{G}}N|^2}{N^2}\phi^2dx+C\Big(\int_\Omega N^{\frac{\alpha q}{2}}|\nabla_{\mathbb{G}}\phi|^q dx\Big)^{2/q} \tag{3.21}$$

for all compactly supported smooth function $\phi\in C_0^\infty(\Omega)$.

Proof. The proof is similar to the proof Theorem in [23] (see also [24]). We only need to use the weighted L^p-Hardy-type inequality (3.5). □

4. SHARP WEIGHTED RELLICH TYPE INEQUALITIES AND THEIR IMPROVED VERSIONS

Our main goal in this section is to obtain weighted analogues of the Rellich inequality (1.1) and (1.2) for general Carnot groups. Furthermore, we shall also obtain their improved versions for bounded domains. The following is the first result of this section.

Theorem 4.1. *Let $\mathbb{G}$ be a Carnot group with homogeneous norm $N=u^{1/(2-Q)}$ and let $\phi\in C_0^\infty(\mathbb{G})$, $\alpha\in\mathbb{R}$, $Q\geq 3$, $Q+\alpha-4>0$. Then the following inequality is valid:*

$$\int_{\mathbb{G}}\frac{N^\alpha}{|\nabla_{\mathbb{G}}N|^2}|\Delta_{\mathbb{G}}\phi|^2dx\geq\frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}\int_{\mathbb{G}}N^\alpha\frac{|\nabla_{\mathbb{G}}N|^2}{N^4}\phi^2dx. \tag{4.1}$$

Furthermore, the constant $\frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}$ is sharp.

Proof. A straightforward computation shows that

$$\Delta_{\mathbb{G}}N^{\alpha-2}=(Q+\alpha-4)(\alpha-2)N^{\alpha-4}|\nabla_{\mathbb{G}}N|^2+\frac{\alpha-2}{2-Q}N^{Q+\alpha-4}\Delta u. \tag{4.2}$$

Multiplying both sides of (4.2) by ϕ^2 and integrating over the domain $\mathbb{G}$, we obtain

$$\int_{\mathbb{G}}\phi^2\Delta_{\mathbb{G}}N^{\alpha-2}dx=\int_{\mathbb{G}}N^{\alpha-2}(2\phi\Delta_{\mathbb{G}}\phi+2|\nabla_{\mathbb{G}}\phi|^2)dx.$$

Since u is the fundamental solution of $\Delta_{\mathbb{G}}$ and $Q+\alpha-4>0$ we obtain

$$\int_{\mathbb{G}}\phi^2\Delta_{\mathbb{G}}N^{\alpha-2}dx=(Q+\alpha-4)(\alpha-2)\int_{\mathbb{G}}N^{\alpha-4}|\nabla_{\mathbb{G}}N|^2\phi^2dx.$$

Therefore

$$(Q+\alpha-4)(\alpha-2)\int_{\mathbb{G}} N^{\alpha-4}|\nabla_{\mathbb{G}}N|^2\phi^2dx - 2\int_{\mathbb{G}} N^{\alpha-2}\phi\Delta_{\mathbb{G}}\phi dx = 2\int_{\mathbb{G}} N^{\alpha-2}|\nabla_{\mathbb{G}}\phi|^2dx. \tag{4.3}$$

Applying the weighted Hardy inequality (3.5) on the right hand side of (4.3), we get

$$\begin{aligned}&(Q+\alpha-4)(\alpha-2)\int_{\mathbb{G}} N^{\alpha-4}|\nabla_{\mathbb{G}}N|^2\phi^2dx - 2\int_{\mathbb{G}} N^{\alpha-2}\phi\Delta_{\mathbb{G}}\phi dx\\ &\geq 2(\frac{Q+\alpha-4}{2})^2\int_{\mathbb{G}} N^{\alpha-4}|\nabla_{\mathbb{G}}N|^2\phi^2dx.\end{aligned}$$

Now it is clear that,

$$-\int_{\mathbb{G}} N^{\alpha-2}\phi\Delta_{\mathbb{G}}\phi dx \geq (\frac{Q+\alpha-4}{2})(\frac{Q-\alpha}{2})\int_{\mathbb{G}} N^{\alpha-4}|\nabla_{\mathbb{G}}N|^2\phi^2dx. \tag{4.4}$$

Next, we apply the Cauchy-Schwarz inequality to the integrand $-\int_{\mathbb{G}} N^{\alpha-2}\phi\Delta\phi dx$ and we obtain

$$-\int_{\mathbb{G}} N^{\alpha-2}\phi\Delta_{\mathbb{G}}\phi dx \leq \Big(\int_{\mathbb{G}} N^{\alpha-4}|\nabla_{\mathbb{G}}N|^2\phi^2dx\Big)^{1/2}\Big(\int_{\mathbb{G}}\frac{|\Delta_{\mathbb{G}}\phi|^2}{|\nabla_{\mathbb{G}}N|^2}N^{\alpha}dx\Big)^{1/2}. \tag{4.5}$$

Combining (4.5) and (4.4), we obtain the inequality (4.1).

Now we prove that the constant $C(Q,\alpha)=\frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}$ is the best constant for the Rellich-type inequality (4.1), that is

$$\mathcal{C}_{\mathcal{R}} := \inf_{0\neq f\in C_0^{\infty}(\mathbb{G})}\frac{\int_{\mathbb{G}} N^{\alpha}\frac{|\Delta_{\mathbb{G}}f|^2}{|\nabla_{\mathbb{G}}N|^2}dx}{\int_{\mathbb{G}} N^{\alpha}\frac{|\nabla_{\mathbb{G}}N|^2}{N^4}f^2dx} = \frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}.$$

It is clear that

$$\frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16} \leq \frac{\int_{\mathbb{G}} N^{\alpha}\frac{|\Delta_{\mathbb{G}}f|^2}{|\nabla_{\mathbb{G}}N|^2}dx}{\int_{\mathbb{G}} N^{\alpha}\frac{|\nabla_{\mathbb{G}}N|^2}{N^4}f^2dx}. \tag{4.6}$$

If we pass to the infimum in (4.6) we get that $\frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}\leq\mathcal{C}_{\mathcal{R}}$. We only need to show that $\mathcal{C}_{\mathcal{R}}\leq\frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}$. Given $\epsilon>0$, we define the function $\phi_\epsilon(N)$ by

$$\phi_\epsilon(N) = \begin{cases} -(\frac{Q+\alpha-4}{2}+\epsilon)\big(N-1\big)+1 & \text{if}\quad N\in[0,1],\\ N^{-(\frac{Q+\alpha-4}{2}+\epsilon)} & \text{if}\quad N>1.\end{cases} \tag{4.7}$$

Notice that $\phi_\epsilon(N)$ can be well approximated by smooth functions with compact support in $\mathbb{G}$. By direct computation we get

$$|\Delta_{\mathbb{G}}\phi_\epsilon|^2 = \begin{cases}(\frac{Q+\alpha-4}{2}+\epsilon)^2|\nabla_{\mathbb{G}}N|^4\frac{(Q-1)^2}{N^2} & \text{if}\quad N\leq 1,\\ (\frac{Q+\alpha-4}{2}+\epsilon)^2(\frac{Q-\alpha}{2}-\epsilon)^2N^{-Q-\alpha-2\epsilon}|\nabla_{\mathbb{G}}N|^4 & \text{if}\quad N>1.\end{cases}$$

Let us denote by $\mathbb{B}_1=\{x\in\mathbb{G}: N\leq 1\}$ the unit ball with respect to the homogeneous norm N. Hence

$$\int_{\mathbb{G}} N^{\alpha}\frac{|\Delta_{\mathbb{G}}\phi_\epsilon|^2}{|\nabla_{\mathbb{G}}N|^2}dx = A(Q,\alpha,\epsilon)\int_{\mathbb{B}_1} N^{\alpha-2}|\nabla_{\mathbb{G}}N|^2dx + B(Q,\alpha,\epsilon)\int_{\mathbb{G}\setminus\mathbb{B}_1} N^{-Q-2\epsilon}|\nabla_{\mathbb{G}}N|^2dx$$

where $A(Q,\alpha,\epsilon) = (Q-1)^2(\frac{Q+\alpha-4}{2}+\epsilon)^2$ and $B(Q,\alpha,\epsilon) = (\frac{Q+\alpha-4}{2}+\epsilon)^2(\frac{Q-\alpha}{2}-\epsilon)^2$. Note that the integrand $\int_{\mathbb{B}_1} N^{\alpha-2}|\nabla_{\mathbb{G}}N|^2dx$ is finite because $|\nabla_{\mathbb{G}}N|$ is uniformly bounded and $Q+\alpha-4>0$. Therefore

$$\int_{\mathbb{G}} N^{\alpha}\frac{|\Delta_{\mathbb{G}}\phi_\epsilon|^2}{|\nabla_{\mathbb{G}}N|^2}dx = B(Q,\alpha,\epsilon)\int_{\mathbb{G}\backslash\mathbb{B}_1} N^{-Q-2\epsilon}|\nabla_{\mathbb{G}}N|^2dx + O(1). \tag{4.8}$$

Next,

$$\int_{\mathbb{G}} N^{\alpha}\frac{|\nabla_{\mathbb{G}}N|^2}{N^4}\phi_\epsilon^2dx = \int_{\mathbb{B}_1} N^{\alpha}\frac{|\nabla_{\mathbb{G}}N|^2}{N^4}\phi_\epsilon^2dx + \int_{\mathbb{G}\backslash\mathbb{B}_1} N^{\alpha}\frac{|\nabla_{\mathbb{G}}N|^2}{N^4}\phi_\epsilon^2dx.$$

It is clear that the first integrand $\int_{\mathbb{B}_1} N^{\alpha}\frac{|\nabla_{\mathbb{G}}N|^2}{N^4}\phi_\epsilon^2dx$ is finite and we get

$$\int_{\mathbb{G}} N^{\alpha}\frac{|\nabla_{\mathbb{G}}N|^2}{N^4}\phi_\epsilon^2dx = \int_{\mathbb{G}\backslash\mathbb{B}_1} N^{-Q-2\epsilon}|\nabla_{\mathbb{G}}N|^2dx + O(1). \tag{4.9}$$

Taking the limit as $\epsilon \longrightarrow 0$ and noting that

$$\int_{\mathbb{G}\backslash\mathbb{B}_1} N^{-Q-2\epsilon}|\nabla_{\mathbb{G}}N|^2dx \longrightarrow \infty$$

we get

$$\frac{\int_{\mathbb{G}} N^{\alpha}\frac{|\Delta_{\mathbb{G}}\phi_\epsilon|^2}{|\nabla_{\mathbb{G}}N|^2}dx}{\int_{\mathbb{G}} N^{\alpha}\frac{|\nabla_{\mathbb{G}}N|^2}{N^4}\phi_\epsilon^2dx} \leq \frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}.$$

Therefore $\mathcal{C}_{\mathcal{R}} = \frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}$. □

Remark 4.2. In the Abelian case, when $\mathbb{G} = \mathbb{R}^n$ with the ordinary dilations, one has $\mathcal{G} = V_1 = \mathbb{R}^n$ so that $Q = n$. It is clear that the inequality (4.1) with the homogeneous norm $N(x) = |x|$ and $\alpha = 0$ reduces the Rellich inequality (1.1).

IMPROVED RELLICH TYPE INEQUALITIES. In this subsection we obtain various improved versions of the weighted Rellich-type inequality (4.1) for smooth bounded domains. One virtue of our approach is that, one can obtain as many as improved weighted Rellich-type inequalities as one can construct improved weighted L^2-Hardy-type inequalities. The following theorem is the first result in this direction.

Theorem 4.3. *Let $\mathbb{G}$ be a Carnot group with homogeneous norm $N = u^{1/(2-Q)}$ and let $\Omega \subset \mathbb{G}$ be a bounded domain with smooth boundary, $0 \in \Omega$, $Q \geq 3$, $4-Q < \alpha < Q$ and $R \geq e\sup_\Omega N$. Then the following inequality holds:*

$$\begin{aligned}\int_{\Omega} \frac{N^{\alpha}}{|\nabla_{\mathbb{G}}N|^2}|\Delta_{\mathbb{G}}\phi|^2dx \geq{}& \frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}\int_{\Omega} N^{\alpha}\frac{|\nabla_{\mathbb{G}}N|^2}{N^4}\phi^2dx \\ &+ \frac{(Q+\alpha-4)(Q-\alpha)}{8}\int_{\Omega} N^{\alpha}\frac{|\nabla_{\mathbb{G}}N|^2}{N^4}\frac{\phi^2}{(\ln\frac{R}{N})^2}dx\end{aligned} \tag{4.10}$$

for all compactly supported functions $\phi \in C_0^\infty(\Omega)$.

Proof. The proof of Theorem 4.2 is similar to that of Theorem 4.1. Let $\phi \in C_0^\infty(\Omega)$ and using the same argument as in Theorem 4.1, we have the following identity:

$$(Q+\alpha-4)(\alpha-2)\int_\Omega N^{\alpha-4}|\nabla_{\mathbb{G}}N|^2\phi^2dx - 2\int_\Omega N^{\alpha-2}\phi\Delta_{\mathbb{G}}\phi dx = 2\int_\Omega N^{\alpha-2}|\nabla_{\mathbb{G}}\phi|^2dx. \tag{4.11}$$

We now apply improved weighted Hardy-type inequality (3.13) to the right hand side of (4.11):

$$\begin{aligned}(Q+\alpha-4)(\alpha-2)&\int_\Omega N^{\alpha-4}|\nabla_{\mathbb{G}}N|^2\phi^2dx - 2\int_\Omega N^{\alpha-2}\phi\Delta_{\mathbb{G}}\phi dx\\ &\geq 2\Big[\big(\frac{Q+\alpha-4}{2}\big)^2\int_\Omega N^{\alpha-4}|\nabla_{\mathbb{G}}N|^2\phi^2dx + \frac{1}{4}\int_\Omega N^{\alpha-4}|\nabla_{\mathbb{G}}N|^2\frac{\phi^2}{(\ln\frac{R}{N})^2}dx.\Big]\end{aligned}$$

Now it is clear that

$$\begin{aligned}-\int_\Omega N^{\alpha-2}\phi\Delta_{\mathbb{G}}\phi dx &\geq \big(\frac{Q+\alpha-4}{2}\big)\big(\frac{Q-\alpha}{2}\big)\int_\Omega N^{\alpha-4}|\nabla_{\mathbb{G}}N|^2\phi^2dx\\ &\quad+\frac{1}{4}\int_\Omega N^{\alpha-4}|\nabla_{\mathbb{G}}N|^2\frac{\phi^2}{(\ln\frac{R}{N})^2}dx.\end{aligned} \tag{4.12}$$

On the other hand we have, by the Young's inequality,

$$-\int_B N^{\alpha-2}\phi\Delta_{\mathbb{G}}\phi dx \leq \epsilon\int_B N^{\alpha-4}|\nabla_{\mathbb{G}}N|^2\phi^2dx + \frac{1}{4\epsilon}\int_B\frac{N^\alpha}{|\nabla_{\mathbb{G}}N|^2}|\Delta_{\mathbb{G}}\phi|^2dx, \tag{4.13}$$

where $\epsilon > 0$ and will be chosen later. Substituting (4.13) into (4.12) we obtain

$$\begin{aligned}\int_B\frac{N^\alpha}{|\nabla_{\mathbb{G}}N|^2}|\Delta_{\mathbb{G}}\phi|^2dx &\geq \big(-4\epsilon^2+(Q+\alpha-4)(Q-\alpha)\epsilon\big)\int_B N^{\alpha-4}|\nabla_{\mathbb{G}}N|^2\phi^2dx\\ &\quad+\epsilon\int_\Omega N^{\alpha-4}|\nabla_{\mathbb{G}}N|^2\frac{\phi^2}{(\ln\frac{R}{N})^2}dx.\end{aligned}$$

It is clear that the quadratic function $-4\epsilon^2+(Q+\alpha-4)(Q-\alpha)\epsilon$ attains the maximum for $\epsilon = \frac{(Q+\alpha-4)(Q-\alpha)}{8}$ and this maximum is equal to $\frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}$. Hence we obtain the desired inequality:

$$\begin{aligned}\int_B\frac{N^\alpha}{|\nabla_{\mathbb{G}}N|^2}|\Delta_{\mathbb{G}}\phi|^2dx &\geq \frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}\int_B N^\alpha\frac{|\nabla_{\mathbb{G}}N|^2}{N^4}\phi^2dx\\ &\quad+\frac{(Q+\alpha-4)(Q-\alpha)}{8}\int_\Omega N^\alpha\frac{|\nabla_{\mathbb{G}}N|^2}{N^4}\frac{\phi^2}{(\ln\frac{R}{N})^2}dx.\end{aligned}$$

□

Using the same arguments as in Theorem 4.2 and improved Hardy-type inequalities (3.19) and (3.20) we obtain the following improved Rellich-type inequalities on a metric ball, respectively.

Theorem 4.4. *Let $\mathbb{G}$ be a Carnot group with homogeneous norm $N = u^{1/(2-Q)}$ and let $B_\varrho \subset \mathbb{G}$ be a ϱ-ball in $\mathbb{G}$, $Q \geq 3$, $\alpha \in \mathbb{R}$ and $4 - Q < \alpha < Q$. Then the following inequality holds:*

$$\begin{aligned}\int_{B_\varrho} \frac{N^\alpha}{|\nabla_{\mathbb{G}} N|^2} |\Delta_{\mathbb{G}} \phi|^2 dx \geq \frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16} \int_{B_\varrho} N^\alpha \frac{|\nabla_{\mathbb{G}} N|^2}{N^4} \phi^2 dx \\ + \frac{(Q+\alpha-4)(Q-\alpha)}{2c^2 r^2} \int_{B_\varrho} N^{\alpha-2} \phi^2 dx\end{aligned} \tag{4.14}$$

for all compactly supported smooth functions $\phi \in C_0^\infty(B_\varrho)$.

Theorem 4.5. *Let $\mathbb{G}$ be a Carnot group with homogeneous norm $N = u^{1/(2-Q)}$ and let $B_\varrho \subset \mathbb{G}$ be a ϱ-ball in $\mathbb{G}$, $\phi \in C_0^\infty(B_\varrho)$, $Q \geq 3$, $\alpha \in \mathbb{R}$, $4 - Q < \alpha < Q$ and $q > 2$. Then the following inequality is valid:*

$$\begin{aligned}\int_{B_\varrho} \frac{N^\alpha}{|\nabla_{\mathbb{G}} N|^2} |\Delta_{\mathbb{G}} \phi|^2 dx \geq \frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16} \int_{B_\varrho} N^\alpha \frac{|\nabla_{\mathbb{G}} N|^2}{N^4} \phi^2 dx \\ + \frac{(Q+\alpha-4)(Q-\alpha)}{2c^2 r^2} K \Big(\int_{B_\varrho} N^{\sigma'} \phi^q dx \Big)^{2/q},\end{aligned} \tag{4.15}$$

where c is a positive constant, $\sigma' = \frac{(2-Q)(2-q)+(\alpha-2)q}{2}$ and $K = \Big(\int_{B_\varrho} N^{2-Q} dx \Big)^{\frac{q-2}{q}}$.

The following improved Rellich-type inequality holds for bounded domains in polarizable Carnot groups.

Theorem 4.6. *Let $\mathbb{G}$ be a polarizable Carnot group and let $\Omega \subset \mathbb{G}$ be a bounded domain with smooth boundary, $0 \in \Omega$, $\alpha \in \mathbb{R}$, $Q \geq 3$ and $4 - Q < \alpha < Q$. Then the following inequality holds:*

$$\begin{aligned}\int_{\Omega} \frac{N^\alpha}{|\nabla_{\mathbb{G}} N|^2} |\Delta_{\mathbb{G}} \phi|^2 dx \geq \frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16} \int_{\Omega} N^\alpha \frac{|\nabla_{\mathbb{G}} N|^2}{N^4} \phi^2 dx \\ + \frac{C(Q+\alpha-4)(Q-\alpha)}{2} \big(\int_{\Omega} |\nabla_{\mathbb{G}} \phi|^q N^{\frac{(\alpha-2)q}{2}} dx \big)^{2/q}\end{aligned} \tag{4.16}$$

for all compactly supported smooth functions $\phi \in C_0^\infty(\Omega)$.

Proof. The proof is similar to the proof of Theorem 4.2. We only need to use the improved Hardy-type inequality (3.21). ☐

WEIGHTED RELLICH TYPE INEQUALITY II. We now turn our attention to another Rellich-type inequality that connects first to second order derivatives. The following theorem is first result in this direction.

Theorem 4.7 (Weighted Rellich-type inequality II)**.** *Let $\mathbb{G}$ be a Carnot group with homogeneous norm $N = u^{1/(2-Q)}$ and let $\phi \in C_0^\infty(\mathbb{G})$, $Q \geq 3$ and $\frac{8-Q}{3} < \alpha < Q$. Then the following inequality is valid:*

$$\int_{\mathbb{G}} N^\alpha \frac{|\Delta_{\mathbb{G}} \phi|^2}{|\nabla_{\mathbb{G}} N|^2} dx \geq \frac{(Q-\alpha)^2}{4} \int_{\mathbb{G}} N^\alpha \frac{|\nabla_{\mathbb{G}} \phi|^2}{N^2} dx. \tag{4.17}$$

Furthermore, the constant $C(Q,\alpha)=\left(\frac{Q-\alpha}{2}\right)^2$ *is sharp.*

Proof. Our starting point is the identity

$$\begin{aligned}\int_{\mathbb{G}} N^{\alpha-2}|\nabla_{\mathbb{G}}\phi|^2dx=\frac{(Q+\alpha-4)(\alpha-2)}{2}\int_{\mathbb{G}} N^{\alpha-4}|\nabla_{\mathbb{G}}N|^2\phi^2dx\\ -\int_{\mathbb{G}} N^{\alpha-2}\phi\Delta_{\mathbb{G}}\phi dx\end{aligned}\tag{4.18}$$

valid for all $\phi\in C_0^\infty(\mathbb{G})$ and $Q+\alpha-4>0$ (see (4.3)).

By applying Cauchy's inequality we obtain

$$-\int_{\mathbb{G}} N^{\alpha-2}\phi\Delta_{\mathbb{G}}\phi dx\le\epsilon\int_{\mathbb{G}} N^{\alpha-4}|\nabla_{\mathbb{G}}N|^2\phi^2dx+\frac{1}{4\epsilon}\int_{\mathbb{G}} N^{\alpha}\frac{|\Delta_{\mathbb{G}}\phi|^2}{|\nabla_{\mathbb{G}}N|^2}dx,\tag{4.19}$$

where $\epsilon>0$ and will be chosen later. Combining (4.19) and (4.18), we get

$$\begin{aligned}\int_{\mathbb{G}} N^{\alpha-2}|\nabla_{\mathbb{G}}\phi|^2dx\le\Big(\frac{(Q+\alpha-4)(\alpha-2)}{2}+\epsilon\Big)\int_{\mathbb{G}} N^{\alpha-4}|\nabla_{\mathbb{G}}N|^2\phi^2dx\\ +\frac{1}{4\epsilon}\int_{\mathbb{G}} N^{\alpha}\frac{|\Delta_{\mathbb{G}}\phi|^2}{|\nabla_{\mathbb{G}}N|^2}dx.\end{aligned}\tag{4.20}$$

We only consider the case $\frac{(Q+\alpha-4)(\alpha-2)}{2}+\epsilon>0$ because other cases do not allow us to obtain sharp weighted Rellich-type inequality that connects first to second-order derivatives. We now apply the Rellich-type inequality (4.1) to the first integral term on the right hand side of (4.20) and get

$$\int_{\mathbb{G}} N^{\alpha}\frac{|\nabla_{\mathbb{G}}\phi|^2}{N^2}dx\le\Big[\frac{16\epsilon}{(Q+\alpha-4)^2(Q-\alpha)^2}+\frac{8(\alpha-2)}{(Q+\alpha-4)(Q-\alpha)^2}+\frac{1}{4\epsilon}\Big]\int_{\mathbb{G}}\frac{|\Delta_{\mathbb{G}}\phi|^2}{|\nabla_{\mathbb{G}}N|^2}dx.$$

Note that the function $\epsilon\longrightarrow\frac{16\epsilon}{(Q+\alpha-4)^2(Q-\alpha)^2}+\frac{8(\alpha-2)}{(Q+\alpha-4)(Q-\alpha)^2}+\frac{1}{4\epsilon}$ attains the minimum for $\epsilon=\frac{(Q+\alpha-4)(Q-\alpha)}{8}$, and this minimum is equal to $\frac{4}{(Q-\alpha)^2}$. Therefore we obtain the desired inequality:

$$\int_{\mathbb{G}} N^{\alpha}\frac{|\Delta_{\mathbb{G}}\phi|^2}{|\nabla_{\mathbb{G}}N|^2}dx\ge\frac{(Q-\alpha)^2}{4}\int_{\mathbb{G}} N^{\alpha}\frac{|\nabla_{\mathbb{G}}\phi|^2}{N^2}dx.\tag{4.21}$$

To show that constant $\left(\frac{Q-\alpha}{2}\right)^2$ is sharp, we again use the same sequence of functions (4.7) and we get

$$\frac{\int_{\mathbb{G}} N^{\alpha}\frac{|\Delta_{\mathbb{G}}\phi_\epsilon|^2}{|\nabla_{\mathbb{G}}N|^2}dx}{\int_{\mathbb{G}} N^{\alpha}\frac{|\nabla_{\mathbb{G}}\phi_\epsilon|^2}{N^2}dx}\longrightarrow\Big(\frac{Q-\alpha}{2}\Big)^2$$

as $\epsilon\longrightarrow 0$. ☐

Remark 4.8. Note that one can also apply the weighted Hardy-type inequality (3.5) with $p=2$ to the first integral on the right hand side of (4.20) and reach the same inequality (4.21).

IMPROVED RELLICH TYPE INEQUALITY II. We now present improved versions of the Rellich-type inequality (4.17) for bounded domains. Their proofs are very similar to that of Theorem 4.6, except instead of using plain weighted Hardy-type inequality, we use improved weighted Hardy-type inequalities, (3.13), (3.19), (3.20) and (3.21), respectively.

Theorem 4.9. *Let $\mathbb{G}$ be a Carnot group with homogeneous norm $N = u^{1/(2-Q)}$ and let $\Omega \subset \mathbb{G}$ be a bounded domain with smooth boundary, $0 \in \Omega$, $Q \geq 3$, $\frac{8-Q}{3} < \alpha < Q$ and $R \geq e \sup_\Omega N$. Then the following inequality holds:*

$$\begin{aligned}\int_\Omega N^\alpha \frac{|\Delta_{\mathbb{G}}\phi|^2}{|\nabla_{\mathbb{G}} N|^2} dx \geq \frac{(Q-\alpha)^2}{4} \int_\Omega N^\alpha \frac{|\nabla_{\mathbb{G}}\phi|^2}{N^2} dx \\ + C(Q,\alpha) \int_\Omega N^{\alpha-4} |\nabla_{\mathbb{G}} N|^2 \frac{\phi^2}{(\ln \frac{R}{N})^2} dx\end{aligned} \tag{4.22}$$

for all compactly supported smooth functions $\phi \in C_0^\infty(\Omega)$. Here $C(Q,\alpha) = \frac{(Q-\alpha)(Q+3\alpha-8)}{16}$.

Theorem 4.10. *Let $\mathbb{G}$ be a Carnot group with homogeneous norm $N = u^{1/(2-Q)}$ and let B_ϱ be a ϱ-ball in $\mathbb{G}$, $\phi \in C_0^\infty(B_\varrho)$, $Q \geq 3$ and $\frac{8-Q}{3} < \alpha < Q$. Then the following inequality holds:*

$$\begin{aligned}\int_{B_\varrho} N^\alpha \frac{|\Delta_{\mathbb{G}}\phi|^2}{|\nabla_{\mathbb{G}} N|^2} dx \geq \frac{(Q-\alpha)^2}{4} \int_{B_\varrho} N^\alpha \frac{|\nabla_{\mathbb{G}}\phi|^2}{N^2} dx \\ + \frac{(Q-\alpha)(Q+3\alpha-8)}{4C^2R^2} \int_{B_\varrho} N^\alpha \frac{\phi^2}{N^2} dx,\end{aligned} \tag{4.23}$$

where $C > 0$ and R is the radius of the ball B_ϱ.

Theorem 4.11. *Let $\mathbb{G}$ be a Carnot group with homogeneous norm $N = u^{1/(2-Q)}$ and let B_ϱ be a ϱ-ball in $\mathbb{G}$, $\phi \in C_0^\infty(B_\varrho)$, $Q \geq 3$ and $\frac{8-Q}{3} < \alpha < Q$. Then the following inequality is valid:*

$$\begin{aligned}\int_{B_\varrho} N^\alpha \frac{|\Delta_{\mathbb{G}}\phi|^2}{|\nabla_{\mathbb{G}} N|^2} dx \geq \frac{(Q-\alpha)^2}{4} \int_{B_\varrho} N^\alpha \frac{|\nabla_{\mathbb{G}}\phi|^2}{N^2} dx \\ + \frac{(Q-\alpha)(Q+3\alpha-8)}{4C^2R^2} K \int_{B_\varrho} N^{\sigma'} \phi^q dx,\end{aligned} \tag{4.24}$$

where R is the radius of the ball B_ϱ, $C > 0$, $\sigma' = \frac{(2-Q)(2-q)+(\alpha-2)q}{2}$ and
$K = \left(\int_{B_\varrho} N^{2-Q} dx \right)^{\frac{q-2}{q}}$.

Theorem 4.12. *Let $\mathbb{G}$ be a polarizable Carnot group with homogeneous norm $N = u^{1/(2-Q)}$ and let Ω be a bounded domain with smooth boundary, $0 \in \Omega$, $Q \geq 3$ and $\frac{8-Q}{3} < \alpha < Q$. Then the following inequality holds:*

$$\int_\Omega N^\alpha \frac{|\Delta_{\mathbb{G}}\phi|^2}{|\nabla_{\mathbb{G}} N|^2} dx \geq \frac{(Q-\alpha)^2}{4} \int_\Omega N^\alpha \frac{|\nabla_{\mathbb{G}}\phi|^2}{N^2} dx + \tilde{C} \Big(\int_\Omega |\nabla_{\mathbb{G}}\phi|^q N^{\frac{(\alpha-2)q}{2}} dx \Big)^{2/q} \tag{4.25}$$

for all compactly supported smooth functions $\phi \in C_0^\infty(\Omega)$. Here $\tilde{C} = \frac{C(Q-\alpha)(Q+3\alpha-8)}{4}$ and $C > 0$.

5. UNCERTAINTY PRINCIPLE INEQUALITY

In [23] we obtained the following uncertainty principle-type inequality for general Carnot groups:

$$\Big(\int_{\mathbb{G}} N^2|\nabla_{\mathbb{G}}N|^2\phi^2 dx\Big)\Big(\int_{\mathbb{G}}|\nabla_{\mathbb{G}}\phi|^2 dx\Big) \geq \Big(\frac{Q-2}{2}\Big)^2\Big(\int_{\mathbb{G}}|\nabla_{\mathbb{G}}N|^2\phi^2 dx\Big)^2, \tag{5.1}$$

where $\phi \in C_0^\infty(\mathbb{G})$. It is clear that this inequality does not recover the Euclidean uncertainty principle inequality (1.8). As we pointed out before one of the main goal of this paper is to establish a sharp uncertainty principle inequality for Carnot groups and the following theorem is the main result of this section.

Theorem 5.1. *Let $\mathbb{G}$ be a polarizable Carnot group with homogeneous norm $N = u^{1/(2-Q)}$ and let $Q \geq 3$ and $\phi \in C_0^\infty(\mathbb{G})$. Then the following inequality is valid:*

$$\Big(\int_{\mathbb{G}} N^2\phi^2 dx\Big)\Big(\int_{\mathbb{G}}\frac{|\nabla_{\mathbb{G}}\phi|^2}{|\nabla_{\mathbb{G}}N|^2} dx\Big) \geq \frac{Q^2}{4}\Big(\int_{\mathbb{G}}\phi^2 dx\Big)^2. \tag{5.2}$$

Proof. By the volume growth formula (2.8) and integration by parts, we get

$$\int_{\mathbb{G}} Q\phi^2 dx = -2\int_{\mathbb{G}}\Big(\frac{\phi N}{|\nabla_{\mathbb{G}}N|^2}\nabla_{\mathbb{G}}\phi\cdot\nabla_{\mathbb{G}}N\Big)dx. \tag{5.3}$$

Applying Cauchy-Schwarz inequality to the right hand-side of (5.3) gives the desired inequality:

$$\Big(\int_{\mathbb{G}} N^2\phi^2 dx\Big)\Big(\int_{\mathbb{G}}\frac{|\nabla_{\mathbb{G}}\phi|^2}{|\nabla_{\mathbb{G}}N|^2} dx\Big) \geq \frac{Q^2}{4}\Big(\int_{\mathbb{G}}\phi^2 dx\Big)^2.$$

It is easy to verify that the equality is attained in Theorem 5.1 by the functions $\phi = Ae^{-\beta N^2}$ for some $A \in \mathbb{R}$, $\beta > 0$. ☐

Remark 5.2. In the Abelian case, when $\mathbb{G} = \mathbb{R}^n$ with the ordinary dilations, one has $\mathcal{G} = V_1 = \mathbb{R}^n$ so that $Q = n$. It is clear that the inequality (5.2) with the homogeneous norm $N(x) = |x|$ recover the uncertainty principle inequality (1.8).

In connection with uncertainty principle inequality we now present the following Caffarelli-Kohn-Nirenberg [9]-type inequality for polarizable Carnot groups. It is clear that this inequality reduces to the uncertainty principle inequality (5.2) for $\alpha = 0$ and $p = q = 2$.

Theorem 5.3. *Let $\mathbb{G}$ be a polarizable Carnot group with homogeneous norm $N = u^{1/(2-Q)}$ and let $Q \geq 3$, $\alpha > -Q$, $p > 1$, $q = \frac{p}{p-1}$ and $\phi \in C_0^\infty(\mathbb{G})$. Then the following inequality is valid:*

$$\Big(\int_{\mathbb{G}} N^{\frac{q}{q-1}}|\phi|^{(\frac{p-1}{q-1})q} dx\Big)^{\frac{q-1}{q}}\Big(\int_{\mathbb{G}}\frac{|\nabla_{\mathbb{G}}\phi|^q}{|\nabla_{\mathbb{G}}N|^q}N^{\alpha q} dx\Big)^{1/q} \geq \big(\frac{Q+\alpha}{p}\big)\int_{\mathbb{G}} N^\alpha|\phi|^p dx. \tag{5.4}$$

Proof. The proof is similar to the proof of Theorem 3.1. We omit the details. ☐

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