# POSITIVE PERIODIC SOLUTIONS FOR FIRST ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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**ABSTRACT.** We establish conditions for the existence of at least two positive periodic solutions of the following functional differential equation of the form

$$x'(t) = a(t)x(t) - f(t, x(h(t))).$$

Applications to some ecological models are given.

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**Key Words.** Periodic Solution, Positive Solution

## 1. INTRODUCTION

In this article, we investigate the existence of two positive periodic solutions of a first order functional differential equation of the form

$$x'(t) = a(t)x(t) - f(t, x(h(t))), \tag{1.1}$$

where $a, h \in C(R, R^+)$ and $a(t + T) = a(t), h(t + T) = h(t), T > 0$ is a real number, $f : R \times R^+ \to R^+, f(t + T, x) = f(t, x), R = (-\infty, \infty)$ and $R^+ = [0, \infty)$.

Periodicity plays an important role in the problems associated with real world applications in particular ecosystem dynamics. There has been considerable contributions in recent years on the existence of periodic solutions of functional differential equations having periodic causal functions. Many authors have used fixed point theorems on cone expansion and cone compression method, upper-lower solution method, iterative technique method and continuation theorem of coincidence degree principle for the existence of at least one or two positive periodic solutions of (1.1). For instance one may see [2, 8, 9, 18, 19, 27, 28, 31, 32, 33]. On the other hand Leggett-Williams multiple fixed point theorem [11] has been used in [1, 20, 21, 22, 23] for the existence of three positive or nonnegative periodic solutions of the equation of the form

$$x'(t) = -a(t)x(t) + f(t, x(h(t))), \tag{1.2}$$

where $a, h$ and $f$ are as defined earlier. If $h(t) = t - \tau(t)$ and $\tau \in C(R, R^+)$ with $\tau(t) \leq t$, then (1.1) and (1.2) take the form

$$x'(t) = a(t)x(t) - f(t, x(t - \tau(t)))$$  (1.3)

and

$$x'(t) = -a(t)x(t) + f(t, x(t - \tau(t))).$$  (1.4)

As the existence of positive periodic solutions of (1.1) is regarded, one can find from the arguments in the succeeding sections that some similar results can be derived for (1.2). We note that the results obtained in [1, 20, 21, 22, 30] can be applied to (1.1). One may observe from the sufficient conditions assumed in the papers [1, 20, 21, 22, 23], that the function $f$ is needed to be unimodal, that is, the function $f$ first increases and then it decreases eventually. This is because of the choice of a constant $c_4$ needed in the use of a theorem of Leggett-Williams [11] for the existence of three fixed points of an operator which is equivalent to the existence of three positive periodic solutions of (1.1) or (1.2). The above choices of functions exclude many important class of growth functions arising in various mathematical models, such as:

(a)  The logistic equation of multiplicative type with several delays ([10])

$$x'(t) = x(t)[a(t) - \prod_{i=1}^{n} b_i(t)x(t - \tau_i(t))],$$  (1.5)

where $a, b_i, \tau_i \in C(R, R^+)$ are $T$-periodic functions.

(b)  The generalized Richards single species growth model ([10])

$$x'(t) = x(t)[a(t) - (\frac{x(t - \tau(t))}{E(t)})^\theta],$$  (1.6)

where $a, E, \tau \in C(R, R^+)$ are $T$-periodic functions and $\theta > 0$ is a constant.

(c)  The generalized Michaelis-Menton type single species growth model ([10, 24])

$$x'(t) = x(t)[a(t) - \sum_{i=1}^{n} \frac{b_i(t)x(t - \tau_i(t))}{1 + c_i(t)x(t - \tau_i(t))}],$$  (1.7)

where $a, b_i, c_i$ and $\tau_i \in C(R, R^+)$ are $T$-periodic functions.

In this article, we have made an attempt to study the existence of two positive $T$-periodic solutions of the Eq.(1.1). Then we shall apply our result to find out sufficient conditions for the existence of two positive $T$-periodic solutions of the models (1.5)–(1.7). To prove our results, we shall use Leggett-Williams multiple fixed point theorem ([11], see Theorem 3.5). The following open problem was proposed by Kuang [10] (open problem 9.2):

*Obtain a sufficient condition for the existence of positive periodic solutions of the following equation:*

$$x'(t) = x(t)[a(t) - b(t)x(t) - c(t)x(t - \tau(t)) - d(t)x'(t - \sigma(t))].\qquad(1.8)$$

Liu et al. [15] gave a partial answer to the above problem by using fixed point theorem for strict set-contractions. They proved that (1.8) has at least one positive $T$-periodic solution. Freedman and Wu [7] studied the existence and global attractivity of a positive periodic solution of (1.8) when $d(t) \equiv 0$. In this paper, we have used Leggett-Williams multiple fixed point theorem [11] to show that (1.8) has at least two positive $T$-periodic solutions (see Example 4.1) when $d(t) \equiv 0$.

It is well known that the Leggett-Williams multiple fixed point theorem [11] has been used by many authors for the existence of multiple periodic solutions of boundary value problems. Once the problem is transformed to an equivalent integral operator, then it is easy to study the existence of fixed points of the operator by using different fixed point theorems which is equivalent to the existence of periodic solutions of the problem. In this paper, we have used the same technique to find the existence of periodic solutions of (1.1). The results of this paper can be extended to

$$x'(t) = a(t)x(t) - f(t, x(h_1(t)), \ldots, x(h_n(t))),\qquad(1.9)$$

where $h_i(t) \geq 0$, $i = 1, \ldots, n$, $f \in C(R \times R^{n^+}, R^+)$ is periodic with respect to the first variable, $h_i(t + T) = h_i(t)$, $1 \leq i \leq n$.

This work has been divided into four sections. Section 1 is Introduction. Some preliminary results are given in Section 2. Section 3 deals with the main results of this paper. Applications of the obtained results to the mathematical models (1.5)–(1.8) are given in Section 4.

## 2. PRELIMINARIES

The following concept from Leggett-Williams multiple fixed point theorem [11] is needed. Let $X$ be a Banach space and $K$ be a cone in $X$. For $a > 0$, define $K_a = \{x \in K; \|x\| < a\}$. A mapping $\psi$ is said to be a nonnegative continuous functional on $K$ if $\psi : K \to [0, \infty)$ is continuous and

$$\psi(\mu x + (1 - \mu)y) \geq \mu\psi(x) + (1 - \mu)\psi(y), \quad x, y \in K, \mu \in [0, 1].$$

Let $b, c > 0$ be constants with $K$ and $X$ as defined above. Define

$$K(\psi, b, c) = \{x \in K; \psi(x) \geq b, \|x\| \leq c\}.$$

**Theorem 2.1** (Leggett-Williams multiple fixed point Theorem,(Theorem 3.5,[11]))**.** *Let $c_3 > 0$ be a constant. Assume that $A : \overline{K}_{c_3} \to K$ is completely continuous, there exists a concave nonnegative functional $\psi$ with $\psi(x) \leq \|x\|, x \in K$ and numbers $c_1$ and $c_2$ with $0 < c_1 < c_2 < c_3$ satisfying the following conditions:*

*(i)   $\{x \in K(\psi, c_2, c_3); \psi(x) > c_2\} \neq \phi$ and $\psi(Ax) > c_2$ if $x \in K(\psi, c_2, c_3)$;*

(ii)  $\|Ax\| < c_1$ if $x \in \overline{K}_{c_1}$;

and

(iii)  $\psi(Ax) > \frac{c_2}{c_3}\|Ax\|$ for each $x \in \overline{K}_{c_3}$ with $\|Ax\| > c_3$.

Then $A$ has at least two fixed points $x_1, x_2$ in $\overline{K}_{c_3}$. Furthermore, $\|x_1\| \leq c_1 < \|x_2\| < c_3$.

**Theorem 2.2** (Leggett-Williams multiple fixed point theorem,(Theorem 3.3, [11])).
*Let* $X = (X, \|.\|)$ *be a Banach space and* $K \subset X$ *a cone, and* $c_4 > 0$ *a constant.
Suppose there exists a concave nonnegative continuous function* $\psi$ *on* $K$ *with* $\psi(u) \leq u$
*for* $u \in \overline{K}_{c_4}$ *and let* $A : \overline{K}_{c_4} \to \overline{K}_{c_4}$ *be a continuous compact map. Assume that there
are numbers* $c_1, c_2$ *and* $c_3$ *with* $0 < c_1 < c_2 < c_3 \leq c_4$ *such that*
*(i)* $\{u \in K(\psi, c_2, c_3); \psi(u) > c_2\} \neq \phi$ *and* $\psi(Au) > c_2$ *for all* $u \in K(\psi, c_2, c_3)$;
*(ii)* $\|Au\| < c_1$ *for all* $u \in \overline{K}_{c_1}$;
*(iii)* $\psi(Au) > c_2$ *for all* $u \in K(\psi, c_2, c_4)$ *with* $\|Au\| > c_3$.
*Then* $A$ *has at least three fixed points* $u_1, u_2$ *and* $u_3$ *in* $\overline{K}_{c_4}$. *Furthermore, we have*
$u_1 \in \overline{K}_{c_1}$, $u_2 \in \{u \in K(\psi, c_2, c_4); \psi(u) > c_2\}$, $u_3 \in \overline{K}_{c_4} \backslash \{K(\psi, c_2, c_4) \cup \overline{K}_{c_1}\}$.

One may observe that (1.1) is equivalent to

$$x(t) = \int_t^{t+T} G(t, s) f(s, x(h(s))) \, ds,$$

where $G(t, s) = \frac{e^{-\int_t^s a(\theta) \, d\theta}}{1 - e^{-\int_0^T a(\theta) \, d\theta}}$ is the Green's kernel. The Green's kernel $G(t, s)$ used
in this paper is well known in the literature. As is shown in many articles, its lower
bound, being positive, is used for defining a cone. It is easy to verify that $G(t, s)$
satisfies the property

$$0 < \alpha = \frac{\delta}{1 - \delta} \leq G(t, s) \leq \frac{1}{1 - \delta} = \beta, \quad s \in [t, t + T],$$

where $\delta = e^{-\int_0^T a(\theta) \, d\theta} < 1$.

Let $X = \{x(t); x \in C(R, R), x(t) = x(t+T)\}$ with the norm $\|x\| = \sup_{t \in [0,T]} |x(t)|$,
then X is a Banach space with the norm $\|.\|$. Define a cone $K$ in $X$ by

$$K = \{x(t); x \in X, x(t) \geq \delta\|x\| \quad \forall t \in [0, T]\}$$

and an operator $A$ on $X$ by

$$(Ax)(t) = \int_t^{t+T} G(t, s) f(s, x(h(s))) \, ds. \tag{2.1}$$

It is easy to verify that $A(K) \subset K$. One may proceed as in the lines of Lemma 5
due to Han and Wang [8] that $A : K \to K$ is completely continuous. The existence
of a positive periodic solution of (1.1) is equivalent to the existence of a fixed point
problem of $A$ in $K$.

## 3. MAIN RESULTS

In this section, we shall prove the main results of the paper by using Theorem 2.1 and Theorem 2.2. Denote

$$f^\theta = \limsup_{x \to \theta} \frac{f(t,x)}{a(t)x} \quad and \quad F^\theta = \limsup_{x \to \theta} \frac{f(t,x)}{x}.$$

**Theorem 3.1.** *Assume that there exist constants $c_1$ and $c_2$ with $0 < c_1 < c_2$ such that*

$(H_1)$ $\int_0^T f(s, x(h(s)))\, ds > \frac{c_2}{\alpha}$   *for*   $c_2 \le x \le \frac{c_2}{\delta}$

*and*

$(H_2)$ $\int_0^T f(s, x(h(s)))\, ds < \frac{c_1}{\beta}$   *for*   $0 \le x \le c_1$

*hold. Then (1.1) has at least two positive $T$-periodic solutions.*

*Proof.* Define a nonnegative concave functional $\psi$ on $K$ by $\psi(x) = \min_{t \in [0,T]} x(t)$. Then $\psi(x) \le \|x\|$. Set $c_3 = \frac{c_2}{\delta}$ and $\phi_0(t) = \phi_0 = \frac{c_2 + c_3}{2}$. Then $\phi_0 \in \{x; x \in K(\psi, c_2, c_3), \psi(x) > c_2\}$. Further, for $x \in K(\psi, c_2, c_3)$ we have, by using $(H_1)$

$$\psi(Ax) = \min_{0 \le t \le T} \int_t^{t+T} G(t,s)f(s, x(h(s)))\, ds$$

$$\ge \alpha \int_0^T f(s, x(h(s)))ds > c_2.$$

Now, let $x \in \overline{K}_{c_1}$. Then, by using $(H_2)$

$$\|Ax\| = \sup_{0 \le t \le T} \int_t^{t+T} G(t,s)f(s, x(h(s)))\, ds$$

$$\le \beta \int_0^T f(s, x(h(s)))ds < c_1.$$

Next suppose that $x \in \overline{K}_{c_3}$ with $\|Ax\| > c_3$. Then

$$\psi(Ax) = \min_{0 \le t \le T} \int_t^{t+T} G(t,s)f(s, x(h(s)))\, ds$$

$$\ge \alpha \int_0^T f(s, x(h(s)))\, ds$$

and

$$c_3 < \|Ax\| \le \beta \int_0^T f(s, x(h(s)))\, ds$$

$$= \frac{\alpha}{\delta} \int_0^T f(s, x(h(s)))\, ds$$

$$\le \frac{1}{\delta}\psi(Ax)$$

imply that

$$\psi(Ax) > \frac{c_2}{c_3}\|Ax\|.$$

Hence, by Theorem 2.1, (1.1) has at least two positive $T$-periodic solutions. This completes the proof of the theorem. □

**Theorem 3.2.** *Assume that there exist constants $c_1$ and $c_2$ with $0 < c_1 < c_2$ such that*

($H_3$) $\quad f(t, x(h(t))) > \frac{c_2}{\alpha T} \quad$ *for* $\quad c_2 \leq x \leq \frac{c_2}{\delta}$

*and*

($H_4$) $\quad f(t, x(h(t))) < \frac{c_1}{\beta T} \quad$ *for* $\quad 0 \leq x \leq c_1$

*hold. Then (1.1) has at least two positive $T$-periodic solutions.*

The proof of the theorem follows from Theorem 3.1. Indeed, ($H_1$) and ($H_2$) follow from ($H_3$) and ($H_4$), respectively.

**Theorem 3.3.** *Let*

($H_5$) $\quad \min_{0 \leq t \leq T} f^\infty = \infty$

*and*

($H_6$) $\quad \max_{0 \leq t \leq T} f^0 = 0$

*hold. Then (1.1) has at least two positive $T$-periodic solutions.*

*Proof.* From ($H_5$), it follows that there exists a real $c_2 > 0$, $c_2$ large enough such that $f(t, x) \geq a(t)x$ for $c_2 \leq x \leq \frac{c_2}{\delta}$.

Set a nonnegative concave continuous functional $\psi$ as in Theorem 3.1, $c_3 = \frac{c_2}{\delta}$ and $\phi_0(t) = \phi_0 = \frac{c_2 + c_3}{2}$. Then $\phi_0 \in \{x; x \in K(\psi, c_2, c_3), \psi(x) > c_2\}$. For $x \in K(\psi, c_2, c_3)$ we have

$$
\begin{aligned}
\psi(Ax) &= \min_{0 \leq t \leq T} \int_t^{t+T} G(t, s) f(s, x(h(s))) \, ds \\
&\geq \min_{0 \leq t \leq T} \int_t^{t+T} a(s) G(t, s) x(s) \, ds \\
&\geq c_2 \min_{0 \leq t \leq T} \int_t^{t+T} a(s) G(t, s) \, ds = c_2.
\end{aligned}
$$

Next, by ($H_6$), there exists a real $\xi, 0 < \xi < c_2$ such that $f(t, x) < a(t)x$ for $0 < x < \xi$. Set $c_1 = \xi$. Then $c_1 < c_2$ and $f(t, x) < a(t)c_1$ for $0 < x < c_1$. Using this fact, we can easily prove that $\|Ax\| < c_1$ for $x \in \overline{K}_{c_1}$.

To complete the proof of the theorem, it remains to show that the condition $(iii)$ of Theorem 2.1 holds. Now

$$
c_3 < \|Ax\| \leq \beta \int_0^T f(s, x(h(s))) \, ds
$$

implies that

$$\psi(Ax) \geq \alpha \int_0^T f(s, x(h(s)))\, ds$$

$$\geq \alpha \cdot \frac{1}{\beta} \|Ax\|$$

$$\geq \delta \|Ax\| = \frac{c_2}{c_3} \|Ax\|.$$

Hence the theorem is complete. □

**Theorem 3.4.** *Suppose that there exists a constant* $\mu, 0 < \mu \leq 1$ *such that*

$(H_7)$ $\qquad f^0 < \mu$

*and*

$(H_8)$ $\qquad f^\infty > \frac{1}{\mu}$

*hold. Then there exist two positive $T$-periodic solutions of* (1.1).

**Corollary 3.5.** *If $f^0 < 1$ and $f^\infty > 1$, then* (1.1) *has at least two positive $T$-periodic solutions.*

**Theorem 3.6.** *If*

$(H_9)$ $\qquad \max_{t \in [0,T]} F^0 = \alpha_1 \in (0, \frac{1}{\beta T})$

*and there exists a constant $c_2 > 0$ such that*

$(H_{10})$ $f(t, x) > \frac{1}{\alpha \delta T} x$ *for $c_2 \leq x \leq \frac{c_2}{\delta}$,*

*then* (1.1) *has at least two positive $T$-periodic solutions.*

**Remark 3.7.** The conditions of our Theorems 3.1–3.6 improve the results in [8, 15, 31, 33].

**Theorem 3.8.** *Suppose that*

$(H_{11})$ $\qquad f$ *is nondecreasing with respect to $x$.*

*Further assume that there are constants $0 < c_1 < c_2$ such that*

$(H_{12})$ $\qquad \frac{\int_0^T f(t, c_1)\, dt}{(1-\delta)c_1} < 1 < \frac{\delta \int_0^T f(t, c_2)\, dt}{(1-\delta)c_2}$

*holds. Then* (1.1) *has at least two positive $T$-periodic solutions.*

*Proof.* Set $c_3 = \frac{c_2}{\delta}$. Define a nonnegative concave continuous functional $\psi$ as in Theorem 3.1 and $\phi_0(t) = \phi_0 = \frac{c_2 + c_3}{2}$. Then $\phi_0 \in \{x; x \in K(\psi, c_2, c_3), \psi(x) > c_2\}$. For $x \in K(\psi, c_2, c_3)$, it follows from $(H_{11})$ and $(H_{12})$ that we have

$$\psi(Ax) = \min_{0 \leq t \leq T} \int_t^{t+T} G(t, s) f(s, x(h(s)))\, ds$$

$$\geq \alpha \int_0^T f(s, x(h(s)))ds$$

$$\geq \alpha \int_0^T f(s, c_2)ds$$

$$> c_2.$$

Next, for $x \in \overline{K}_{c_1}$, it follows from $(H_{11})$ and $(H_{12})$ that we have

$$
\begin{aligned}
\|Ax\| &= \sup_{0 \le t \le T} \int_t^{t+T} G(t,s)f(s, x(h(s)))\, ds \\
&\le \beta \int_0^T f(s, \|x\|)ds \\
&\le \beta \int_0^T f(s, c_1)ds \\
&< c_1.
\end{aligned}
$$

Finally, for $x \in \overline{K}_{c_3}$ with $\|Ax\| > c_3$, we have

$$
\begin{aligned}
\psi(Ax) &= \min_{0 \le t \le T} \int_t^{t+T} G(t,s)f(s, x(h(s)))\, ds \\
&\ge \alpha \int_0^T f(s, x(h(s)))\, ds
\end{aligned}
$$

and

$$
\begin{aligned}
c_3 < \|Ax\| &\le \beta \int_0^T f(s, x(h(s)))\, ds \\
&= \frac{\alpha}{\delta} \int_0^T f(s, x(h(s)))\, ds \\
&\le \frac{1}{\delta}\psi(Ax),
\end{aligned}
$$

then

$$
\psi(Ax) > \frac{c_2}{c_3}\|Ax\|.
$$

Thus all the conditions of Theorem 2.1 are satisfied. Consequently, (1.1) has at least two positive $T$-periodic solutions. This completes the proof of the theorem. □

**Theorem 3.9.** *Suppose that* $(H_{11})$ *holds. Further assume that there are constants* $0 < c_1 < c_2$ *such that*
$(H_{13})$ $\quad \dfrac{\max_{t \in [0,T]} f(t, c_1)}{(1-\delta)c_1} < \dfrac{1}{T} < \dfrac{\delta \min_{t \in [0,T]} f(t, c_2)}{(1-\delta)c_2}$
*holds. Then* (1.1) *has at least two positive $T$-periodic solutions.*

**Example 3.10.** Consider the delay differential equation

$$
x'(t) = (\sin^4 t + \cos^4 t)x(t) - \frac{1}{\pi}[x(t) + x^2(t - \tau)], \tag{3.1}
$$

where $\tau > 0$ is a real constant. Here $a(t) = \sin^4 t + \cos^4 t$, $f(t, x) = \frac{1}{\pi}[x + x^2]$, $T = \frac{\pi}{2}$ and $\delta = e^{-\int_0^T a(s)\,ds} = 0.3$. Now choosing $c_1 = 0.3$ and $c_2 = 4$, we observe that the conditions of Theorem 3.8 are satisfied. Hence (3.1) has at least two positive $T$-periodic solutions.

In [29], Wang introduced the notation

$$
i_0 = \text{number of zeros in the set } \{\overline{f}_0, \overline{f}_\infty\}
$$

and

$$i_\infty = \text{number of infinities in the set } \{\overline{f}_0, \overline{f}_\infty\},$$

where

$$\overline{f}_0 = \lim_{x \to 0^+} \frac{f(x)}{x} \quad \text{and} \quad \overline{f}_\infty = \lim_{x \to \infty} \frac{f(x)}{x}$$

for the existence of positive $T$-periodic solutions of the differential equation

$$x'(t) = a(t)g(x(t))x(t) - \lambda b(t)f(x(t - \tau(t))), \tag{3.2}$$

where $\lambda > 0$ is a positive parameter, $a, b \in C(R, [0, \infty))$ are $T$-periodic functions, $\int_0^T a(t)\, dt > 0, \int_0^T b(t)\, dt > 0, \tau \in C(R, R)$ is $T$-periodic function, $f, g : [0, \infty) \to [0, \infty)$ are continuous, $0 < l \le g(x) < L < \infty$ for $x \ge 0, l, L$ are positive constants, $f(x) > 0$ for $x > 0$.

In the following, we apply Theorem 2.1 in Eq.(3.2) to obtain some new results differ those in [29]. The Banach space $X$ and a cone $K$ in $X$ are the same as above, while the operator $A$ is replaced by

$$(A_\lambda x)(t) = \lambda \int_t^{t+T} G_x(t, s)b(s)f(x(s - \tau(s)))\, ds,$$

where $G_x(t, s) = \frac{e^{-\int_t^s a(\theta)g(x(\theta))\, d\theta}}{1 - e^{-\int_0^T a(\theta)g(x(\theta))\, d\theta}}$ is the Green's kernel. The Green's kernel $G_x(t, s)$ satisfies the property

$$\frac{\delta^L}{1 - \delta^L} \le G_x(t, s) \le \frac{1}{1 - \delta^l}.$$

If we proceed as in the lines of Theorem 3.8, we obtain the following theorem:

**Theorem 3.11.** *Let $f$ be nondecreasing. Further assume that there are constants $0 < c_1 < c_2$ such that*
$(H_{14})$   $\frac{(1-\delta^L)c_2}{\delta^L f(c_2) \int_0^T b(s)\, ds} < \lambda < \frac{(1-\delta^l)c_1}{f(c_1) \int_0^T b(s)\, ds}$
*hold. Then (3.2) has at least two positive $T$-periodic solutions.*

**Theorem 3.12.** *Let $\overline{f}_0 < 1 - \delta^l$ and $\overline{f}_\infty < 1 - \delta^l$ hold. Further, assume that there exists a constant $c_2 > 0$ such that*
$(H_{15})$   $f(x(t - \tau(t))) > \frac{(1-\delta^L)}{\delta^{2L}}c_2$ *for*  $c_2 \le x \le \frac{(1-\delta^L)}{\delta^L(1-\delta^l)}c_2$
*holds. Then (3.2) has at least three positive $T$-periodic solutions for*

$$\frac{\delta^L}{\int_0^T b(t)\, dt} < \lambda < \frac{1}{\int_0^T b(t)\, dt}.$$

*Proof.* By $\overline{f}_\infty < 1 - \delta^l$, there exist $0 < \epsilon < 1 - \delta^l$ and $\xi > 0$ such that $f(x) \le \epsilon x$ for $x \ge \xi$. Let $\gamma = \max_{0 \le x \le \xi, 0 \le t \le T} f(x)$. Then $f(x) \le \epsilon x + \gamma$. Choose $c_4 > 0$ such that

$$c_4 > \max\{\frac{\gamma}{(1 - \delta^l) - \epsilon}, \frac{1 - \delta^L}{\delta^L(1 - \delta^l)}c_2\}.$$

Then, for $x \in \overline{K}_{c_4}$

$$
\begin{aligned}
\|A_\lambda x\| &= \sup_{0 \le t \le T} \lambda \int_t^{t+T} G(t,s) b(s) f(x(s - \tau(s))) \, ds \\
&\le \frac{1}{1 - \delta^l} \lambda \int_0^T b(s) f(x(s - \tau(s))) \, ds \\
&\le \frac{1}{1 - \delta^l} \lambda \int_0^T b(s) (\epsilon \|x\| + \gamma) \, ds \\
&\le \frac{1}{1 - \delta^l} (\epsilon c_4 + \gamma) < c_4,
\end{aligned}
$$

that is, $A : \overline{K}_{c_4} \to \overline{K}_{c_4}$.

Now, we define a nonnegative concave functional $\psi$ on $K$ as $\psi(x) = \min_{t \in [0,T]} x(t)$. Then $\psi(x) \le \|x\|$. Set $c_3 = \frac{1 - \delta^L}{\delta^L (1 - \delta^l)} c_2$ and $\phi_0(t) = \phi_0 = \frac{c_2 + c_3}{2}$. Then $c_2 < c_3$ and $\phi_0 \in \{x; x \in K(\psi, c_2, c_3), \psi(x) > c_2\}$. For $x \in K(\psi, c_2, c_3)$ it follows from $(H_{15})$ that

$$
\begin{aligned}
\psi(A_\lambda x) &= \min_{0 \le t \le T} \lambda \int_t^{t+T} G(t,s) b(s) f(x(s - \tau(s))) \, ds \\
&\ge \frac{\delta^L}{1 - \delta^L} \lambda \int_0^T b(s) f(x(s - \tau(s))) \, ds \\
&\ge \frac{\delta^L}{1 - \delta^L} \lambda \int_0^T b(s) \, ds \, \frac{(1 - \delta^L)}{\delta^{2L}} c_2 > c_2.
\end{aligned}
$$

Next by $\bar{f}_0 < 1 - \delta^l$, there exists a positive $\sigma < c_2$ such that

$$
f(x) < (1 - \delta^l) x \quad for \ \ 0 < x \le \sigma.
$$

Set $c_1 = \sigma$. Then $c_1 < c_2$. For $x \in \overline{K}_{c_1}$, we have

$$
\begin{aligned}
\|A_\lambda x\| &= \sup_{0 \le t \le T} \lambda \int_t^{t+T} G(t,s) b(s) f(x(s - \tau(s))) \, ds \\
&\le \frac{1}{1 - \delta^l} \lambda \int_0^T b(s) (1 - \delta^l) x \, ds \\
&\le \frac{1}{1 - \delta^l} \lambda \int_0^T b(s) (1 - \delta^l) c_1 \, ds < c_1.
\end{aligned}
$$

Finally, for $x \in K(\psi, c_2, c_4)$ with $\|A_\lambda x\| > c_3$, we have

$$
c_3 < \|A_\lambda x\| \le \frac{1}{1 - \delta^l} \lambda \int_0^T b(s) f(x(s - \tau(s))) \, ds,
$$

which in turn implies that

$$
\begin{aligned}
\psi(A_\lambda x) &\ge \frac{\delta^L}{1 - \delta^L} \lambda \int_0^T b(s) f(x(s - \tau(s))) \, ds \\
&> \frac{\delta^L}{1 - \delta^L} (1 - \delta^l) c_3 = c_2.
\end{aligned}
$$

Hence, by Theorem 2.2, (3.2) has at least three positive $T$-periodic solutions. $\qquad \square$

**Corollary 3.13.** *If $i_0 = 2$ and there exists a constant $c_2 > 0$ such that $(H_{15})$ holds, then (3.2) has at least three positive $T$-periodic solutions for*

$$\frac{\delta^L}{\int_0^T b(t)\, dt} < \lambda < \frac{1}{\int_0^T b(t)\, dt}.$$

**Remark 3.14.** Wang [29] obtained three different results for the existence of at least one positive periodic solution of (3.2) by using fixed point index theory [3]. In Corollary 3.13 we have shown that (3.2) has at least three positive $T$-periodic solutions when $i_0 = 2$. It would be interesting to obtain sufficient conditions for the existence of at least two or three positive periodic solutions of (3.2) when $i_0 \in \{0, 1\}$ and $i_\infty \in \{0, 1, 2\}$ by using leggett-Williams multiple fixed point theorems. Bai and Xu [1] obtained a sufficient condition (Theorem 3.12 in [1]) for the existence of three nonnegative $T$-periodic solutions of (3.2). Although the condition $i_0 = 2$ holds both in Theorem 3.2 in [1] and in our Corollary 3.13, our condition $(H_{15})$ and the condition $(H_5)$ in [1] are different. Accordingly, the ranges on the parameter $\lambda$ are also different.

Padhi et al. [23] have considered the functional differential equation

$$x'(t) = a(t)x(t) - \lambda b(t)f(x(t - \tau(t))), \tag{3.3}$$

which is a particular case of (3.2), $\lambda, a, b, T$ and $f$ are defined as in (3.2). The results of [23] can be extended to (3.2). In the following, we show that our Theorem 3.12 and Corollary 3.13 are different from some of the results given in [23]. Extending Theorem 3.4 and Corollary 3.5 of [23] to (3.2), we obtain the following results.

**Theorem 3.15.** *Let $\overline{f}_0 < T$ and $\overline{f}_\infty < T$ hold. If there exists a constant $c_2 > 0$ such that*

$(H_{16})$    $f(x(t - \tau(t))) > \frac{2T(1-\delta^L)}{\delta^L(1-\delta^l)}c_2$   *for*   $c_2 \leq x \leq \frac{(1-\delta^L)}{\delta^L(1-\delta^l)}c_2$    *and*    $0 \leq t \leq T,$

*then (3.2) has at least three positive $T$-periodic solutions for*

$$\frac{1 - \delta^l}{2T \int_0^T b(t)\, dt} < \lambda < \frac{1 - \delta^l}{T \int_0^T b(t)\, dt}.$$

**Corollary 3.16.** *Let $i_0 = 2$. If $(H_{16})$ holds, then (3.2) has at least three positive $T$-periodic solutions for*

$$\frac{1 - \delta^l}{2T \int_0^T b(t)\, dt} < \lambda < \frac{1 - \delta^l}{T \int_0^T b(t)\, dt}.$$

One may observe that the upper bounds on $\overline{f}_0$ and $\overline{f}_\infty$ in Theorem 3.15 and Theorem 3.12 are $T$ and $1 - \delta^l$, respectively. We may note that $T$ and $1 - \delta^l$ are not comparable. Similarly, one may compare our Corollary 3.13 with Corollary 3.16. Although the condition $i_0 = 2$ holds both in Corollary 3.13 and Corollary 3.16, the conditions $(H_{15})$ and $(H_{16})$ are different. Accordingly, the ranges on the parameter $\lambda$ are also different.

## 4. APPLICATIONS

Ye et al. [31] showed that the models (1.5)–(1.8) have at least one positive periodic solution, respectively. In the following section, we shall apply some of our results to obtain sufficient conditions for the existence of at least two positive periodic solutions of the models (1.5)–(1.8).

**Example 4.1.** The generalized logistic model of single species

$$x'(t) = x(t)[a(t) - b(t)x(t) - c(t)x(t - \tau(t))] \tag{4.1}$$

has at least two positive $T$-periodic solutions, where $a(t), b(t)$ and $c(t)$ are nonnegative continuous periodic functions.

*Proof.* Set $f(t, x) = x(t)[b(t)x(t) + c(t)x(t - \tau(t))]$. Since

$$\max_{t \in [0,T]} \frac{f(t, x)}{a(t)x} \leq \max_{t \in [0,T]} \{\frac{b(t)}{a(t)}\} \|x\| + \max_{t \in [0,T]} \{\frac{c(t)}{a(t)}\} \|x\| \to 0 \;\; as \;\; x \to 0,$$

we see that $(H_6)$ is satisfied. Further, since

$$\min_{t \in [0,T]} \frac{f(t, x)}{a(t)x} \geq \min_{t \in [0,T]} \delta\{\frac{b(t)}{a(t)}\} \|x\| \to \infty \;\; as \;\; x \to \infty,$$

then $(H_5)$ is satisfied. Thus by Theorem 3.3, (4.1) has at least two positive $T$-periodic solutions. $\qquad\square$

**Example 4.2.** The logistic equation of single species

$$x'(t) = x(t)[a(t) - \sum_{i=1}^{n} b_i(t)x(t - \tau_i(t))] \tag{4.2}$$

has at least two positive $T$-periodic solutions, where $a, b_i, \tau_i \in C(R, R^+)$ are $T$-periodic functions.

**Example 4.3.** The logistic equation with several delays (1.5) has at least two positive $T$-periodic solutions.

**Example 4.4.** The generalized Richards single species growth model (1.6) has at least two positive $T$-periodic solutions.

The proofs of the Examples 4.3–4.4 are similar to the proof of the Example 4.1.

Applying Corollary 3.5 to the generalized Michaelis-Menton type single species growth model (1.7), we obtain the following result:

**Example 4.5.** If

$$\min_{t \in [0,T]} \sum_{i=1}^{n} \frac{b_i(t)}{a(t)c_i(t)} > 1,$$

then (1.7) has at least two positive $T$-periodic solutions.

Now, we assume that the population is subject to harvesting. Under the catch-per-unit-effort hypothesis [3], consider the harvested population's growth model

$$x'(t) = x(t)[a(t) - \frac{b(t)x(t)}{1 + c(t)x(t)}] - qEx(t), \tag{4.3}$$

where $q$ and $E$ are positive constants denoting the catch ability coefficients and harvesting effort, respectively. Ye et al. [31] proved that if $0 < qE < \frac{1-\delta}{T}$ and $(\frac{b^m}{c} + qE) > \frac{1-\delta}{\delta^2 T}$, then (4.3) has at least one positive $T$-periodic solution, where $b^m = \min_{0 \le t \le T} b(t)$ and $0 < c(t) \le c$.

**Theorem 4.6.** *Suppose that* $0 < qE < \frac{1-\delta}{T}$ *and* $\frac{\delta \int_0^T b(t)\, dt}{c} + qE > \frac{\delta-1}{\delta^2 T}$. *Then* (4.3) *has at least two positive $T$-periodic solutions.*

*Proof.* Set $f(t,x) = \frac{b(t)x^2}{1+c(t)x} + qEx$. Then $qE < \frac{1-\delta}{T}$ implies the condition $(H_9)$. Choose $c_2 = \frac{\delta(1-qE\alpha\delta T)}{\alpha\delta^2 T \int_0^T b(t)\, dt - c(1-qE\alpha\delta T)}$. Then $\frac{c_2 \alpha \delta^2 T \int_0^T b(t)\, dt}{\delta + cc_2} + qE\alpha\delta T = 1$. Set $c_3 = \frac{c_2}{\delta} = \frac{(1-qE\alpha\delta T)}{\alpha\delta^2 T \int_0^T b(t)\, dt - c(1-qE\alpha\delta T)}$. Then $c_2 < c_3$. Now for $c_2 \le x \le \frac{c_3}{\delta}$, we have

$$
\begin{aligned}
f(t,x) \;&>\; \frac{c_2^2 \int_0^T b(t)\, dt}{1 + c\frac{c_2}{\delta}} + qEc_2 = \frac{c_2}{\alpha\delta T} \cdot [\frac{c_2 \alpha\delta^2 T \int_0^T b(t)\, dt}{\delta + cc_2} + qE\alpha\delta T] \\
&\ge\; \frac{c_2}{\alpha\delta T},
\end{aligned}
$$

that is, $(H_{10})$. Hence by Theorem 3.6, (4.3) has at least two positive $T$-periodic solutions. $\square$

**Remark 4.7.** One may see in the literature, that very few result exist on the existence of two periodic solutions of (1.1) with its application to the models (1.5)–(1.8). Hence a simple result on the existence of two periodic solutions of the above equations are of immense important.

## ACKNOWLEDGMENTS

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