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ON A BISTABLE QUASILINEAR PARABOLIC EQUATION: WELL-POSEDNESS AND STATIONARY SOLUTIONS

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Dedicated to Professor Jeff Webb on the occasion of his retirement.

ABSTRACT. In this paper we prove existence and uniqueness of variational inequality solutions for a bistable quasilinear parabolic equation arising in the theory of solid-solid phase transitions and discuss its stationary solutions, which can be discontinuous.

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1. INTRODUCTION

To generalise the Ginzburg-Landau phase transition theory to high gradients in the order parameter u, Rosenau [15, 16] proposed the following free energy functional:

$$E[u](t) = \int_\Omega [W(u) + \epsilon\Psi(|\nabla u|)]\, dx, \tag{1.1}$$

where the diffusion coefficient $\epsilon > 0$, the interface energy $\Psi(s)$ is a convex function of its variable that grows linearly in s; for example,

$$\Psi(s) = \sqrt{1 + s^2} - 1,$$

$W(u)$ is the bulk energy which we take to be the double-well function

$$W(u) = \frac{u^4}{4} - \frac{u^2}{2}.$$

Then formally, the L^2-gradient flow of (1.1) is given by

$$u_t = \epsilon\nabla \cdot (\psi(\nabla u)) + f(u), \tag{1.2}$$

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where $f(u) = -W'(u) := u - u^3$,

$$\psi(\nabla u) = \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}},$$

and $(x,t) \in \Omega \times (0,T) \equiv Q_T$ for some bounded domain $\Omega \subset \mathbb{R}^n$, $T > 0$. Of course (1.2) has to be supplemented with suitable initial and boundary conditions; here we consider the physically relevant Neumann boundary conditions, $\psi(\nabla u) \cdot \underline{n} = 0$ on $\partial\Omega$ which, since $\psi(0) = 0$, implies that $\nabla u \cdot \underline{n} = 0$ on $\partial\Omega$. For the motivation to consider the quasilinear diffusion operator in (1.2) see the work of Rosenau [15, 16]; applications of nonlinear diffusion equations with bistable reaction terms in ecology and meterials science are discussed in [9].

In this paper, using the methods of [7] we prove a well-posedness result for (1.2) and while this result holds for any dimension n, here we restrict ourselves to the one-dimensional case so that $\Omega \equiv (0, L)$, $L > 0$. As shown in [3], the bifurcation structure for the one-dimensional stationary problem associated with (1.2) depends on the parameter ϵ as well as the length L of the interval; these issues will be discussed in more detail in Section 4.

2. PRELIMINARIES

In this section we briefly recall some properties of the function space $BV(\Omega)$. A function of bounded variation is a $u \in L^1(\Omega)$ whose partial derivatives in the sense of distributions are measures with finite total variation

$$\int_\Omega |u_x|\, dx = \sup \left\{ \int_\Omega u\, v_x\, dx \;:\; v \in C_0^\infty(\Omega),\ |v(x)| \leq 1 \,\text{for}\, x \in \Omega \right\}.$$

The space $BV(\Omega)$ endowed with the norm

$$||u||_{BV(\Omega)} = ||u||_{L^1(\Omega)} + \int_\Omega |u_x|\, dx,$$

is a Banach space. The topology on BV which we will require is the BV-weak* topology defined by

$$u_j \xrightarrow{BV-w^*} u \Leftrightarrow u_j \to u \text{ in } L^1(\Omega) \text{ and } u_{jx} \rightharpoonup u_x \text{ in } M(\Omega),$$

where $M(\Omega)$ is the space of bounded measures on Ω and $u_{jx} \rightharpoonup u_x$ in $M(\Omega)$ means that

$$\int_\Omega u_{jx} \varphi\, dx \to \int_\Omega u_x \varphi\, dx,$$

for all $\varphi \in C_0(\Omega)$.

We also have the following compactness property: for every bounded sequence $\{u_j\} \subset BV(\Omega)$, there exists a subsequence $\{u_{j_k}\}$ and a function u in $BV(\Omega)$ such that $u_{j_k} \xrightarrow{BV-w^*} u$ as $k \to \infty$.

Following [8], we define $\int_\Omega \Psi(u_x)$ and supposing $\Psi(s) = \sqrt{1+s^2}$, we arrive at the following definition

$$\int_\Omega \sqrt{1+|u_x|^2}\,dx = \sup_{v\in C_0^\infty} \left\{ -\int_\Omega u\,v_x\,dx + \int_\Omega \sqrt{1-v^2}\,dx \;:\; |v(x)| \le 1\ \forall x \in \Omega \right\}.$$

Hence we obtain the following useful estimate:

$$\int_\Omega |u_x|\,dx - |\Omega| \le \int_\Omega \sqrt{1+|u_x|^2} - 1\,dx \le \int_\Omega |u_x|\,dx + |\Omega|, \tag{2.1}$$

for all $u \in BV(\Omega)$.

3. EXISTENCE AND UNIQUENESS OF WEAK SOLUTIONS OF THE PARABOLIC PROBLEM

The problem we are considering is

$$\begin{aligned} u_t &= (\psi(u_x))_x + f(u), \quad (x,t) \in Q_T \equiv \Omega \times (0,T), \qquad (3.1)\\ u_x(0,t) &= u_x(L,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T),\\ u(x,0) &= u_0(x), \quad x \in \Omega, \end{aligned}$$

where $(0,T)$ is any finite time interval over which we will prove existence of weak solutions. Note that without loss of generality we have put $\epsilon = 1$.

First, we need to define our notion of a weak solution. To begin with, let us suppose that u is smooth enough to permit us to perform the calculations which follow. For smooth test functions $v \in C^\infty(Q_T)$, we multiply our equation by $v-u$ and integrate by parts using Neumann boundary conditions to obtain

$$\int_{Q_T} (u_t - f(u))(v-u)\,dx\,dt + \int_{Q_T} \psi(u_x)(v_x - u_x)\,dx\,dt = 0.$$

Since $\Psi(s)$ is convex, we have that $\Psi(v_x) - \Psi(u_x) \ge \Psi'(u_x)(v_x - u_x)$ and hence

$$\int_{Q_T} (u_t - f(u))(v-u)\,dx\,dt + \int_{Q_T} (\Psi(v_x) - \Psi(u_x))\,dx\,dt \ge 0,$$

for smooth functions $v \in C^\infty(Q_T)$. This motivates the following definition of a weak solution to our problem.

Definition 3.1. Let $M(Q_T)$ denote the space of bounded measures on Q_T. A function $u \in L^\infty(Q_T) \cap L^\infty((0,T), BV(\Omega)) \cap \{u : u_x \in M(Q_T)\}$ is called a weak solution of problem (3.1) if $u_t \in L^2(Q_T)$ and u satisfies the variational inequality

$$\int_{Q_T} (u_t - f(u))(v-u)\,dx\,dt + \int_{Q_T} (\Psi(v_x) - \Psi(u_x))\,dx\,dt \ge 0, \tag{3.2}$$

for all $v \in L^\infty(Q_T) \cap \{v : v_x \in M(Q_T)\}$.

Thus v_x, the distributional derivative of the function v, will be a measure with finite total variation.

By the above discussion, classical solutions of (3.1) automatically satisfy variational inequality (3.2). To see that a smooth solution u of (3.2) also satisfies (3.1), choose as a test function $v = u + ch$ where $h \in C^\infty(Q_T)$, $c \in \mathbb{R}$, so that (3.2) becomes

$$\int_{Q_T} (u_t - f(u))(ch)\,dx\,dt + \int_{Q_T} \Psi(u_x + ch_x)\,dx\,dt \geq \int_{Q_T} \Psi(u_x)\,dx\,dt.$$

Hence from the Taylor series of $\Psi(u_x + ch_x)$ we have

$$c\int_{Q_T} (u_t - f(u))h\,dx\,dt + c\int_{Q_T} \Psi'(u_x)h_x\,dx\,dt + \frac{c^2}{2}\int_{Q_T} \Psi''(u_x)(h_x)^2 + \ldots \geq 0.$$

Considering firstly, $c > 0$, then $c < 0$ and letting $c \to 0$ from above and below yields

$$\int_{Q_T} (u_t - f(u))h\,dx\,dt + \int_{Q_T} \psi(u_x)h_x\,dx\,dt = 0, \;\; \forall\, h \in C^\infty(Q_T).$$

Integrating by parts and using the boundary conditions, we see that u classically satisfies (3.1).

Theorem 3.2. *The problem* (3.1) *admits a unique variational inequality solution for all $T > 0$ for every $u_0(x) \in BV(\Omega)$.*

Proof. For $\gamma > 0$, consider the following regularised problem:

$$\begin{aligned} u_t &= (\psi(u_x))_x + f(u), \quad (x,t) \in \Omega \times (0,T), \\ u_x(0,t) &= u_x(L,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) &= u_0^\gamma(x), \quad x \in \Omega, \end{aligned} \tag{3.3}$$

where $u_0^\gamma(x)$ satisfies

$$u_0^\gamma \in C^\infty(\bar{\Omega}), \qquad\qquad u_{0x}^\gamma = 0 \;\text{ on }\; \partial\Omega,$$
$$||u_0^\gamma - u_0||_{L^\infty(\Omega)} \to 0 \;\text{ as }\; \gamma \to 0, \qquad ||u_0^\gamma||_{L^\infty(\Omega)} \leq ||u_0||_{L^\infty(\Omega)} + 1 = m_0,$$

and

$$\int_\Omega |u_{0x}^\gamma|\,dx \leq C(\Omega)\int_\Omega |u_{0x}|\,dx. \tag{3.4}$$

The existence of such a sequence of regularising initial data $u_0^\gamma \in C^\infty(\Omega)$ follows from the fact the initial data $u_0 \in BV(\Omega)$ and because the space $C^\infty(\Omega)$ is dense in the space of functions of bounded variation. Let $u^\gamma(x,t)$ represent the unique classical solution to the regularised problem with the regular initial data $u_0^\gamma(x)$; these exist by standard parabolic theory, see for example Theorem 7.4, [12, Chapter V] which is proved for a more restricted type of quasilinear equation in divergence form but by the remark in [12, p.492], the proof survives without much change for problems of the type of (3.3).

We want to show that there exists a function $u \in BV(Q_T)$ such that $u^\gamma \to u$ in $L^1(Q_T)$ as $\gamma \to 0$, which will be a weak solution to our problem and that it does not depend on the choice of the sequence u^γ. As in [7], we will need to establish a series of convergence properties for, and a priori bounds on, the approximating solutions u^γ. Namely we show

Lemma 3.3.

A: *the sequence $\{u^\gamma\}$ is uniformly bounded in $L^\infty(Q_T)$ and the sequence $\{u_t^\gamma\}$ is uniformly bounded in $L^2(Q_T)$,*

B: *the sequence $\{u^\gamma\}$ is uniformly bounded in $L^\infty((0,T), BV(\Omega))$ and in $BV(Q_T)$,*

C: *the sequence $\{u^\gamma\}$ converges in the space $L^\infty((0,T), L^2(\Omega))$ and the sequence $\{u^\gamma(\cdot,t)\}$ converges in the space $L^2(\Omega)$ for all $t \in [0,T]$.*

Proof. [A]: In what follows, let Q_τ denote the space-time cylinder $\Omega \times (0,\tau)$ where τ is arbitrary in $[0,T]$. First of all, we have that

$$||u^\gamma||_{L^\infty(Q_T)} < m_0, \tag{3.5}$$

where $m_0 > 1$, by the parabolic maximum principle and properties of $f(\cdot)$.

We show next that the sequence $\{u_t^\gamma\}$ is uniformly bounded in $L^2(Q_T)$. Multiply the regularised problem by u_t^γ and integrate over Q_τ:

$$\begin{aligned}
\int_{Q_\tau} (u_t^\gamma)^2\,dx\,dt &= -\int_{Q_\tau} \psi(u_x^\gamma)u_{tx}^\gamma\,dx\,dt + \int_{Q_\tau} f(u^\gamma)u_t^\gamma\,dx\,dt \\
&= -\int_0^\tau \frac{d}{dt}\int_\Omega \Psi(u_x^\gamma)\,dx\,dt + \int_0^\tau \frac{d}{dt}\int_\Omega F(u^\gamma)\,dx\,dt \\
&= -\int_\Omega (\Psi(u_x^\gamma)|_{t=\tau} - \Psi(u_0^\gamma))\,dx + \int_\Omega (F(u^\gamma)|_{t=\tau} - F(u_0^\gamma))\,dx,
\end{aligned}$$

where $F(u) = \int_0^u f(s)\,ds$.
Hence

$$\begin{aligned}
||u_t^\gamma||^2_{L^2(Q_\tau)} + \int_\Omega \Psi(u_x^\gamma)|_{t=\tau}\,dx + \int_\Omega \left[\left.\frac{(u^\gamma)^4}{4}\right|_{t=\tau} + \frac{(u_0^\gamma)^2}{2}\right]dx& \\
\leq \int_\Omega \Psi(u_{0x}^\gamma)\,dx + \left(\frac{m_0^4}{4} + \frac{m_0^2}{2}\right)|\Omega|,&
\end{aligned} \tag{3.6}$$

from the bounds we have on u_0^γ and on u^γ. Hence using the bound on $\int_\Omega \Psi(u_x)\,dx$ in (2.1) and subsequently the bound in (3.4), it follows from (3.6) taking $\tau = T$, that

$$\begin{aligned}
||u_t^\gamma||^2_{L^2(Q_T)} &\leq \int_\Omega \Psi(u_{0x}^\gamma)\,dx + \left(\frac{m_0^4}{4} + \frac{m_0^2}{2}\right)|\Omega| \\
&\leq \int_\Omega |u_{0x}^\gamma|\,dx + \left(\frac{m_0^4}{4} + \frac{m_0^2}{2} + 1\right)|\Omega| \\
&\leq C(\Omega)\int_\Omega |u_{0x}|\,dx + C_1 < \infty,
\end{aligned}$$

since $u_0 \in BV(\Omega)$. Thus we have that the sequence $\{u_t^\gamma\}$ is uniformly bounded in $L^2(Q_T)$ and therefore also in $L^1(Q_T)$.

[B]: We also need to show that the sequence $\{u^\gamma\}$ is uniformly bounded in the space $L^\infty((0,T), BV(\Omega))$ and also that $\{u^\gamma\}$ is uniformly bounded in $BV(Q_T)$. To see the former, first note that (3.6) also implies that

$$\int_\Omega \Psi(u_x^\gamma)|_{t=\tau}\,dx \le C(\Omega)\int_\Omega |u_{0x}|\,dx + C_1,$$

but since τ was arbitrary in $[0,T]$ we have, using (2.1) once again, that for all $t \in [0,T]$

$$\begin{aligned} C(\Omega)\int_\Omega |u_{0x}|\,dx + C_1 &\ge \int_\Omega \Psi(u_x^\gamma)\,dx \ge \int_\Omega |u_x^\gamma|\,dx - |\Omega|, \\ \Rightarrow \int_\Omega |u_x^\gamma|\,dx &\le C(\Omega)\int_\Omega |u_{0x}|\,dx + C_1 + |\Omega| \le C_2. \end{aligned} \tag{3.7}$$

This, together with the fact that $u^\gamma(\cdot,t) \in L^1(\Omega)$ for all $t \in [0,T]$ implies that

$$||u^\gamma(\cdot,t)||_{BV(\Omega)} < C_3 \quad \forall\ t \in [0,T],$$

with C_3 independent of γ and of t and so $\sup_{0<t\le T} ||u^\gamma(\cdot,t)||_{BV(\Omega)} < C_3$. Hence we have that the sequence $\{u^\gamma\}$ is indeed uniformly bounded in $L^\infty((0,T), BV(\Omega))$.

Since $u^\gamma(\cdot,t) \in L^1(\Omega)\ \forall\ t \in [0,T]$, we infer that $u^\gamma \in L^1(Q_T)$ and since (3.7) implies that

$$\int_0^T \int_\Omega |u_x^\gamma|\,dx\,dt \le C_2 T,$$

we have that

$$||u^\gamma||_{BV(Q_T)} < C_4,$$

for C_4 independent of γ and so u^γ is also uniformly bounded in $BV(Q_T)$.

[C]: We now establish that the sequence $\{u^\gamma(\cdot,t)\}$ converges in the space $L^2(\Omega)$ as $\gamma \to 0$ for all $t \in [0,T]$ and that the sequence $\{u^\gamma\}$ converges in the space $L^\infty((0,T), L^2(\Omega))$ as $\gamma \to 0$. To this end, consider u^{γ_m} and u^{γ_n} both satisfying the regularised problem, multiply the difference of the two equations by the difference $u^{\gamma_m} - u^{\gamma_n}$, then integrate over Q_τ to obtain

$$\begin{aligned} \frac{1}{2}\int_{Q_\tau} \frac{\partial}{\partial t}(u^{\gamma_m} - u^{\gamma_n})^2\,dx\,dt = &-\int_{Q_\tau} (\psi(u_x^{\gamma_m}) - \psi(u_x^{\gamma_n}))(u_x^{\gamma_m} - u_x^{\gamma_n})\,dx\,dt \\ &+ \int_{Q_\tau} (f(u^{\gamma_m}) - f(u^{\gamma_n}))(u^{\gamma_m} - u^{\gamma_n})\,dx\,dt. \end{aligned} \tag{3.8}$$

Since the function $\psi(s)$ is monotonic, the first term on the right-hand side of (3.8) is non-positive and so (3.8) becomes

$$\begin{aligned}\int_0^\tau \frac{d}{dt}\left(\int_\Omega (u^{\gamma_m}-u^{\gamma_n})^2\,dx\right)dt &\le 2\int_{Q_\tau}(f(u^{\gamma_m})-f(u^{\gamma_n}))(u^{\gamma_m}-u^{\gamma_n})\,dx\,dt\\
&= 2\int_{Q_\tau}\left[(u^{\gamma_m}-u^{\gamma_n})-\{(u^{\gamma_m})^3-(u^{\gamma_n})^3\}\right](u^{\gamma_m}-u^{\gamma_n})\,dx\,dt\\
&= 2\int_{Q_\tau}\left[1-\{(u^{\gamma_m})^2+u^{\gamma_m}u^{\gamma_n}+(u^{\gamma_n})^2\}\right](u^{\gamma_m}-u^{\gamma_n})^2\,dx\,dt\\
&\le 2\int_{Q_\tau}|\{(u^{\gamma_m})^2+u^{\gamma_m}u^{\gamma_n}+(u^{\gamma_n})^2\}-1||u^{\gamma_m}-u^{\gamma_n}|^2\,dx\,dt\\
&\le 2|3m_0^2-1|\int_{Q_\tau}|u^{\gamma_m}-u^{\gamma_n}|^2\,dx\,dt\\
&= \int_0^\tau |6m_0^2-2|\left(\int_\Omega (u^{\gamma_m}-u^{\gamma_n})^2\,dx\right)dt. \qquad (3.9)\end{aligned}$$

Thus if we define $C(m_0)=|6m_0^2-2|$ then we have, since τ is arbitrary in $[0,T]$

$$\frac{d}{dt}\int_\Omega (u^{\gamma_m}-u^{\gamma_n})^2\,dx \le C(m_0)\int_\Omega (u^{\gamma_m}-u^{\gamma_n})^2\,dx.$$

Hence Gronwall's inequality implies that

$$\int_\Omega (u^{\gamma_m}-u^{\gamma_n})^2\,dx \le e^{C(m_0)\tau}\int_\Omega (u_0^{\gamma_m}-u_0^{\gamma_n})^2\,dx,$$

so that from (3.9)

$$\begin{aligned}\int_\Omega (u^{\gamma_m}-u^{\gamma_n})^2|_{t=\tau}\,dx &\le \int_0^\tau C(m_0)\left(\int_\Omega (u^{\gamma_m}-u^{\gamma_n})^2\,dx\right)dt+\int_\Omega (u_0^{\gamma_m}-u_0^{\gamma_n})^2\,dx\\
&\le (C(m_0)\tau\, e^{C(m_0)\tau}+1)\int_\Omega (u_0^{\gamma_m}-u_0^{\gamma_n})^2\,dx,\end{aligned}$$

but since τ was arbitrary in $[0,T]$ and u^{γ_m} and u^{γ_n} both satisfy the regularised problem, we have

$$||u^{\gamma_m}(\cdot,t)-u^{\gamma_n}(\cdot,t)||_{L^2(\Omega)}\to 0 \text{ as } \gamma_m,\gamma_n\to 0, \text{ for all } t\in[0,T].$$

So $u^{\gamma_n}(\cdot,t)$ is Cauchy in $L^2(\Omega)$ for all $t\in[0,T]$ hence the sequence $u^{\gamma_n}(\cdot,t)$ converges in $L^2(\Omega)$ for all $t\in[0,T]$ and from this it follows that u^γ converges in $L^\infty((0,T),L^2(\Omega))$. Thus Lemma 3.3 is established. □

We now pass to the limit as $\gamma\to 0$ making use of the above properties of the sequence u^γ. We have shown that there exists a unique $u\in L^\infty((0,T),L^2(\Omega))$ such that

$$||u^\gamma(\cdot,t)-u(\cdot,t)||_{L^2(\Omega)}\to 0 \text{ as } \gamma\to 0\ \forall t\in[0,T] \text{ and } ||u^\gamma-u||_{L^2(Q_T)}\to 0 \text{ as } \gamma\to 0,$$

but then this implies convergence in L^1 so that we also have,

$$||u^\gamma(\cdot,t)-u(\cdot,t)||_{L^1(\Omega)}\to 0 \text{ as } \gamma\to 0\ \ \forall t\in[0,T],$$

and

$$||u^\gamma - u||_{L^1(Q_T)} \to 0 \text{ as } \gamma \to 0 \tag{3.10}$$

using the Cauchy Schwarz inequality.

We have also shown uniform boundedness of u_t^γ in $L^2(Q_T)$, hence $||u_t^\gamma||_{L_2(Q_T)} \leq C$ and so by weak compactness in $L^2(Q_T)$, we can extract a subsequence that we still denote as $\{u_t^\gamma\}$ which is such that

$$u_t^\gamma \rightharpoonup u_t \text{ in } L^2(Q_T) \text{ with } u_t \in L^2(Q_t).$$

This implies that given $\varphi \in L^2(\Omega)$ we have

$$\int_0^t \langle u_t^\gamma(x,s), \varphi\rangle_{L^2(\Omega)} \, ds = \langle u^\gamma(x,t), \varphi\rangle_{L^2(\Omega)} - \langle u_0^\gamma(x), \varphi\rangle_{L^2(\Omega)},$$

and letting $\gamma \to 0$ gives

$$\int_0^t \langle u_t(x,s), \varphi\rangle_{L^2(\Omega)} \, ds = \langle u(x,t), \varphi\rangle_{L^2(\Omega)} - \langle u_0(x), \varphi\rangle_{L^2(\Omega)},$$

from which it follows that the limit function $u(x,t)$ satisfies the initial condition, $u(x,0) = u_0(x)$, and following the same reasoning as for (3.5), the limit function u is also uniformly bounded in $L^\infty(Q_T)$.

We now prove that the limit function u is in $BV(Q_T)$. We have shown that the sequence $\{u^\gamma\}$ is uniformly bounded in $BV(Q_T)$. Hence we can extract a subsequence denoted $\{u^{\gamma_i}\}$ that converges weakly to some BV function η. That is to say, $u^{\gamma_i}(x,t) \rightharpoonup \eta(x,t)$ in $BV(Q_T)$-weak* with $\eta \in BV(Q_T)$, but this means that $u^{\gamma_i} \to \eta$ in $L^1(Q_T)$, so from (3.10) by the uniqueness of the limit we must have

$$\eta = u \in BV(Q_T). \tag{3.11}$$

Hence by definition of BV functions on Q_T, we conclude from (3.11) that weak first derivative in space of u is a bounded measure on Q_T.

We can now show that the limit function u is such that $u(\cdot,t) \in BV(\Omega)$ for every $t \in [0,T]$. That the sequence $\{u^\gamma\}$ is uniformly bounded in $L^\infty((0,T), BV(\Omega))$ means that

$$||u^{\gamma_i}(\cdot,t)||_{BV(\Omega)} < C_5, \quad \text{for almost every } t \in [0,T].$$

Fix t_0 arbitrary in $[0,T]$. We can extract a subsequence $\{u^{\gamma_j}\}$ of $\{u^{\gamma_i}\}$ such that $u^{\gamma_j}(\cdot,t_0) \rightharpoonup U(\cdot,t_0)$ weak* in $BV(\Omega)$ with $U(\cdot,t_0) \in BV(\Omega)$. But this means that $u^{\gamma_j}(\cdot,t_0) \to U(\cdot,t_0)$ in $L^1(\Omega)$ and so we have once again from (3.10) that $u(\cdot,t) = U(\cdot,t) \in BV(\Omega)$ for all $t \in [0,T]$ since t_0 was arbitrary in $[0,T]$.

In [7] it is shown that for $u \in BV(\Omega)$ and Ψ convex, the functional $\int_\Omega \Psi(u_x)\, dx$ is lower semi-continuous with respect to L^1-convergence. Hence, since $u(\cdot,t) \in BV(\Omega)$

for almost all $t \in [0,T]$ and since $||u^\gamma(\cdot,t)-u(\cdot,t)||_{L^1(\Omega)} \to 0$ as $\gamma \to 0$ for all $t \in [0,T]$, we must have that

$$\int_\Omega \Psi(u_x)\,dx \le \liminf_{\gamma\to 0} \int_\Omega \Psi(u_x^\gamma)\,dx \ \text{ for all } t \in [0,T]. \tag{3.12}$$

We noted earlier that from (3.6), it follows that

$$\int_\Omega \Psi(u_x^\gamma)\,dx \le C(\Omega)\int_\Omega |u_{0x}| + C_1 + |\Omega| \quad \forall t \in [0,T]. \tag{3.13}$$

Hence taking the limit inferior of (3.13) as $\gamma \to 0$, we see that by (2.1)

$$\begin{aligned}\int_\Omega |u_x|\,dx - |\Omega| &\le \int_\Omega \Psi(u_x)\,dx \\ &\le \liminf_{\gamma\to 0}\int_\Omega \Psi(u_x^\gamma)\,dx \le C(\Omega)\int_\Omega |u_{0x}|\,dx + C_1 + |\Omega| \quad \forall t \in [0,T].\end{aligned}$$

Thus we are lead to conclude that $||u(\cdot,t)||_{BV(\Omega)} < \infty$ for almost all $t \in [0,T]$ and consequently

$$u \in L^\infty((0,T), BV(\Omega)).$$

For later, note that one may integrate (3.12) on $[0,T]$ to obtain

$$\liminf_{\gamma\to 0}\int_{Q_T} \Psi(u_x^\gamma)\,dx\,dt \ge \int_{Q_T} \Psi(u_x)\,dx\,dt.$$

An additional result that we will need when passing to the limit as $\gamma \to 0$ is that as $||u^\gamma - u||_{L^1(Q_T)} \to 0$ as $\gamma \to 0$, $||f(u) - f(u^\gamma)||_{L_1(Q_T)} \to 0$. This follows easily when one considers

$$\begin{aligned}&\int_0^T\int_\Omega |f(u) - f(u^\gamma)|\,dx\,dt = \int_0^T\int_\Omega |u - u^3 - (u^\gamma - (u^\gamma)^3)|\,dx\,dt \\ &\le \int_0^T\int_\Omega |u-u^\gamma|\,dx\,dt + \int_0^T\int_\Omega |u-u^\gamma||u^2 + uu^\gamma + (u^\gamma)^2|\,dx\,dt \\ &\le \int_0^T\int_\Omega |u-u^\gamma|\,dx\,dt + 3m_0^2\int_0^T\int_\Omega |u-u^\gamma|\,dx\,dt \ \to 0 \ \text{ as } \ \gamma \to 0.\end{aligned}$$

So far we have shown that the limit function u is such that

$$u \in L^\infty(Q_T) \cap L^\infty((0,T), BV(\Omega)) \cap \{u : u_x \in M(Q_T)\},$$

so that all that remains is to be proven is that the limit function u satisfies the variational inequality (3.2). Note that the variational inequality holds for the solutions u^γ of the regularised problems with test functions taken from the smooth sequence $\{v^n\}_{n\in\mathbb{N}} \subset C^\infty(Q_T)$ i.e.

$$\int_{Q_T} (u_t^\gamma - f(u^\gamma))(v^n - u^\gamma)\,dx\,dt + \int_{Q_T} \Psi(v_x^n) - \Psi(u_x^\gamma)\,dx\,dt \ge 0. \tag{3.14}$$

It is shown in [8, Thm 2.2] that the space $C^\infty(Q_T)$ is dense in $BV(Q_T)$ equipped with the topology defined by the distance

$$d(u,w) = ||u-w||_{L^1(Q_T)} + \left| \int_{Q_T} |u_x| - \int_{Q_T} |w_x| \right| + \left| \int_{Q_T} \Psi(u_x) - \int_{Q_T} \Psi(w_x) \right|,$$

which means in particular that one can approximate $BV(Q_T)$ functions by a sequence of $C^\infty(Q_T)$ functions, i.e. for $v \in BV(Q_T)$, there exists a sequence $\{v^n\} \in C^\infty(Q_T)$ such that

$$\int_{Q_T} \Psi(v_x^n)\, dx\, dt \to \int_{Q_T} \Psi(v_x) \ \text{ as } n \to \infty,$$

and

$$\int |v^n - v|\, dx\, dt \to 0 \text{ as } n \to \infty.$$

This combined with all the properties that have been established for solutions u^γ to the regularised problem, means that one may pass to the limit as $n \to \infty$ and subsequently as $\gamma \to 0$ in inequality (3.14) to obtain the result.

As usual, in order to prove uniqueness of a weak solution to our problem we suppose non-uniqueness and derive a contradiction. Hence suppose there are two weak solutions u_1 and u_2 satisfying problem (3.1) and therefore the variational inequality (3.2) with

$$u_1(x,0) = u_2(x,0) = u_0(x). \tag{3.15}$$

Take the variational inequality first with $u = u_1$, $v = u_2$ and then with $u = u_2$, $v = u_1$ so that

$$\int_{Q_\tau} \left(\frac{\partial u_1}{\partial t} - f(u_1) \right) (u_2 - u_1)\, dx\, dt + \int_{Q_\tau} \Psi((u_2)_x) - \Psi((u_1)_x))\, dx\, dt \geq 0,$$

and

$$\int_{Q_\tau} \left(\frac{\partial u_2}{\partial t} - f(u_2) \right) (u_1 - u_2)\, dx\, dt + \int_{Q_\tau} \Psi((u_1)_x) - \Psi((u_2)_x)\, dx\, dt \geq 0.$$

Adding these two inequalities gives

$$\int_{Q_\tau} \frac{\partial(u_1 - u_2)}{\partial t}(u_1 - u_2)\, dx\, dt \leq \int_{Q_\tau} (f(u_1) - f(u_2))(u_1 - u_2)\, dx\, dt.$$

As before

$$\begin{aligned} \int_\Omega (u_1 - u_2)^2\, dx|_{t=\tau} &\leq \int_0^\tau C(m_0) \left(\int_\Omega (u_1 - u_2)^2\, dx \right) dt + \int_\Omega (u_1(x,0) - u_2(x,0))^2\, dx \\ &\leq (C(m_0)\tau\, e^{C(m_0)T} + 1) \int_\Omega (u_1(x,0) - u_2(x,0))^2\, dx, \end{aligned}$$

using again the Gronwall inequality. Thus it follows from (3.15) that

$$||u_1(\cdot,\tau) - u_2(\cdot,\tau)||_{L^2(\Omega)} = 0,$$

and uniqueness follows from τ being arbitrary in $[0,T]$. $\square$

4. STATIONARY SOLUTIONS IN ONE DIMENSION

The one-dimensional Neumann stationary problem for (1.2)

$$\left(\frac{u'}{\sqrt{1+(u')^2}}\right)' + \lambda f(u) = 0, \qquad (4.1)$$
$$u'(0) = u'(L) = 0,$$

where we have set $\lambda = 1/\epsilon$, is a boundary value problem of prescribed mean curvature type and these have been studied extensively for various choices of nonlinearity $f(u)$ (see for example [3, 4, 10, 11, 13, 14]). Many of the references for (4.1) deal with the case of $f(u)$ being a homogeneous function such as u^p. In this case the form of bifurcation diagram in λ is independent of L. In [14] and in this work, the nonlinearity is non-homogeneous. For fixed L, using Liapunov-Schmidt reduction, we have:

Proposition 4.1. [3, Prop. 3.1] *The k-th bifurcation from the trivial solution of (4.1) is a supercritical pitchfork if $L > k\pi/\sqrt{2}$ and a subcritical pitchfork if the inequality is reversed.*

It is shown in [3] through an analysis of the time map associated with (4.1) that for any given value of L, there is a value $\lambda^*(L)$ beyond which there cannot exist classical, i.e. $C^2((0,L)) \cap C^1([0,L])$, solutions to (4.1) and solutions at $\lambda = \lambda^*(L)$ develop infinite gradient.

Solutions to (4.1) are defined in the BV sense as functions of bounded variation which satisfy the variational inequality

$$-\lambda \int_\Omega f(u)(v-u)\,dx + \int_\Omega \Psi(v_x) - \Psi(u_x)\,dx \geq 0 \;\; \forall v \in BV(\Omega),$$

which is obtained from (3.2) if one assumes that $u_t = 0$. If without loss of generality one considers monotone decreasing solutions to (4.1), a theorem proven in [3] is that discontinuous solutions constructed by patching together different level curves of the Hamiltonian

$$H(u, u') = 1 - \frac{1}{\sqrt{1+(u')^2}} - \lambda W(u),$$

which satisfy $u_x = 0$ at $x = 0$ and $x = L$ are solutions to (4.1) in the BV sense. This construction for $\lambda > \lambda^*(L)$ of solutions in the BV sense that are discontinuous at some x_0 in the interior of the interval delivers a continuum of equilibria for (4.1). One can easily generate initial conditions which, for L fixed and $\lambda > \lambda^*(L)$, converge to a discontinuous stationary solution of (1.2) by taking

$$u_0(x) = -\alpha \tanh\left(\beta\left(\frac{x}{L} - \delta\right)\right), \qquad (4.2)$$

which serves as an approximation to the discontinuous steady state with discontinuity at some $x_0 = \delta L$ for $\delta \in (0,1)$. Note that in (4.2), $u_0(0) = -u_0(L) = \alpha \in (0,1)$ and

β is large and such that $u_0'(x_0) = -\frac{\alpha\beta}{L}$. In Figure 1, we fix $L = 2.5$ (supercritical) and $\lambda = 5 > \lambda^*(L) \approx 4.019534$ and solve (1.2), (4.2) with $\alpha = 0.98$, $\beta = 500$ and $\delta = 0.24, 0.5$ and 0.76 respectively and the solutions indeed converge to a discontinuous steady state. We note with a view to Figures 1 and 3 that for a given L and a particular $\lambda > \lambda^*(L)$ there is a range $[x_1, x_2] \subset (0, L)$ (where $x_2 = L - x_1$) of possible positions of the interface x_0 and this range for such L and λ is determined from considerations of the time map for (4.1) (see [3] for details).

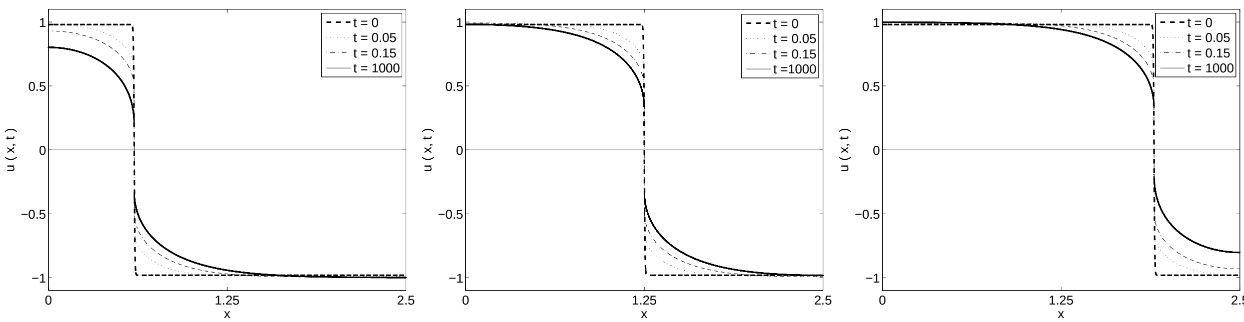

FIGURE 1. Convergence of three initial data to three of the infinitely many steady states of (1.2) for $L = 2.5$ and $\lambda = 5$.

For details of the numerics, please consult [5]. Note from this figure that these solutions have some stability properties (see [3] for a discussion of the right notion of stability for this case).

There are also similarly stable non-monotone solutions as in Figure 2 arising from non-monotone initial data

$$u_0(x) = \frac{4x(L-x)}{L^2} \sin\left(\frac{10\pi x^2}{L^2}\right),$$

if ϵ is taken to be sufficiently small in (1.2). This indicates that the structure of patterns that the bistable quasilinear equation gives rise to is much richer than in the semilinear case, in which only the constant solutions ± 1 attract all initial conditions with probability one.

Finally, given a continuum of stationary solutions, it is interesting to know which has the lowest energy. It turns out that it is the most asymmetric of the possible stationary solutions that minimize the energy over the continuum. Figure 3 depicts this situation for $\lambda = 5$ and $L = 2.5$ once again where the position of the interface $x_0 \in [x_1, x_2] \equiv [0.57021, 1.92979]$ since as mentioned above, non-classical monotone solutions cannot exist for values of x_0 outside of this region for this particular value of λ.

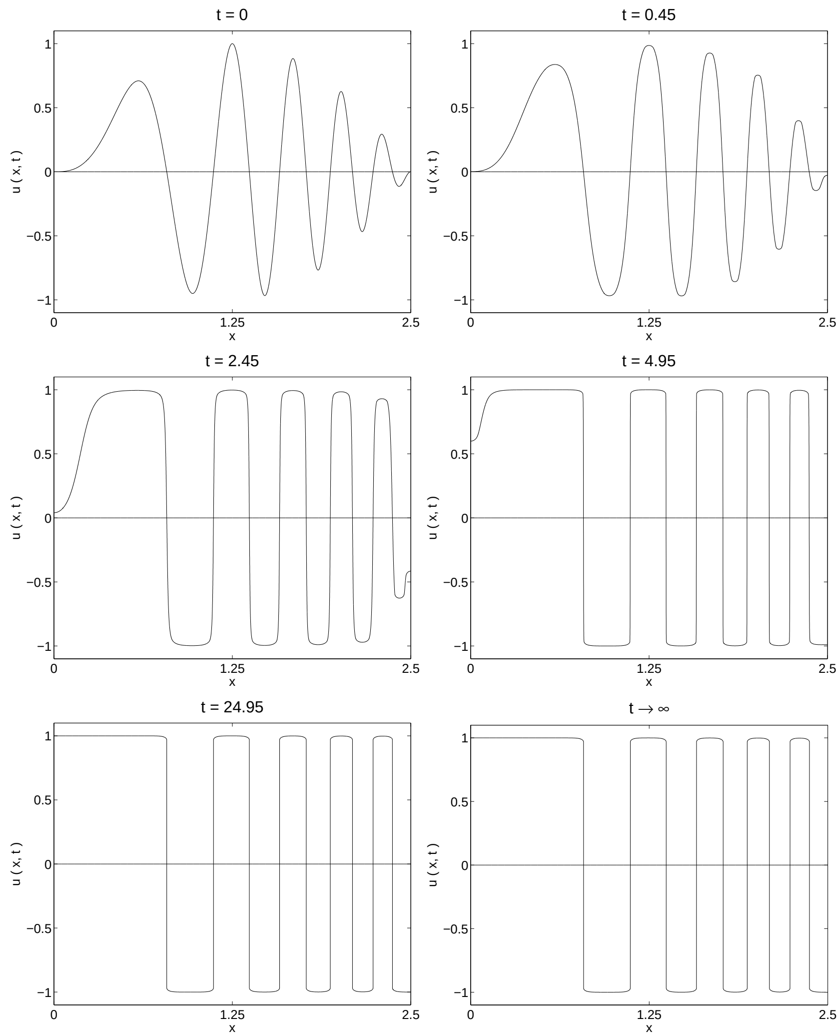

FIGURE 2. Convergence of a non-monotone initial condition to a non-monotone steady solution of (1.2) with $\epsilon = 0.001$ and $L = 2.5$

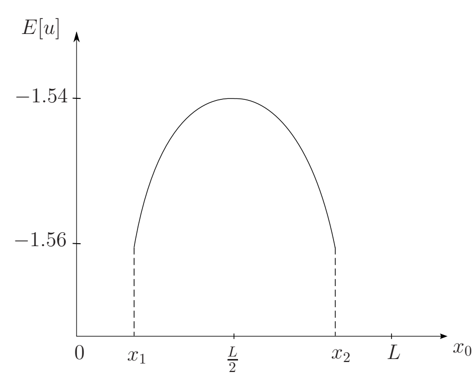

FIGURE 3. Plot of the energy $E[u]$ against the position x_0 of the interface for stationary solutions to (1.2) corresponding to $L = 2.5$ and $\lambda = 5$.

5. CONCLUDING REMARKS

We have presented a model for solid-solid phase transitions and have proved the existence of weak (variational inequality) solutions on all $[0, T]$, $T > 0$. We have also presented some results on discontinuous stationary solutions for the model, which have some stability properties in stark contrast to the semilinear situation. Much work remains to be done, in particular proving stabilisation of orbits. We expect that nonlinear semigroup techniques of [1] together with a Simon-Łojasiewicz inequality type result [6] will be required for that.

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