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POSITIVE AND NONDECREASING SOLUTIONS TO A SINGULAR BOUNDARY VALUE PROBLEM FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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This paper is dedicated to Professor Jeff Webb on the occasion of his retirement

ABSTRACT. In this paper we establish the existence and uniqueness of a positive and nondecreasing solution to a singular boundary value problem of a class of nonlinear fractional differential equations. Our analysis relies on a fixed point theorem in partially ordered sets.

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1. INTRODUCTION

Many papers and books on fractional differential equations have appeared recently. Most of them are devoted to the solvability of the linear fractional equation in terms of a special function (see, for example [3, 12]) and to problems of analyticity in the complex domain [11]. Moreover, Delbosco and Rodino [7] considered the existence of a solution for the nonlinear fractional differential equation $D^{\alpha}_{0^+}u = f(t, u)$, where $0 < \alpha < 1$ and $f\colon [0, a] \times \mathbb{R} \to \mathbb{R}$, $0 < a \le +\infty$ is a given continuous function in $(0, a) \times \mathbb{R}$. They obtained existence results by using the Schauder fixed point theorem and the Banach contraction principle. Recently, Zhang [19] considered the existence of a positive solution for the equation $D^{\alpha}_{0^+}u = f(t, u)$, where $0 < \alpha < 1$ and $f\colon [0, 1] \times [0, \infty) \to [0, \infty)$ is a given continuous function by using the sub and super-solution method.

In this paper, we discuss the existence and uniqueness of a positive and nondecreasing solution to the boundary value problem

$$\begin{aligned} &D^{\alpha}_{0^+}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ &u(0) = u'(1) = u''(0) = 0, \end{aligned} \tag{1.1}$$

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where $2 < \alpha \le 3$, D^{α}_{0+} is the Caputo's differentiation and $f\colon (0,1] \times [0,\infty) \to [0,\infty)$ with $\lim_{t\to 0^+} f(t,\cdot) = \infty$ (that is f is singular at $t = 0$). Note that this problem was considered in [17], where the authors proved the existence of one positive solution for (1.1) by using Krasnoselskii's fixed point theorem and a nonlinear alternative of Leray-Schauder type in a cone and assuming certain hypotheses on the function f. In [17] the nondecreasing character and the uniqueness of the solution is not treated. In this paper we will prove the existence and uniqueness of a positive and nondecreasing solution for the Problem (1.1) by using a fixed point theorem in partially ordered sets. In [9], the authors study Problem (1.1) using a different fixed point theorem that the one used in this paper. Existence of fixed points in partially ordered sets has been considered recently in [6, 8, 13, 14, 15, 16]. This work is inspired in the papers [1, 9, 17]. For existence theorems for fractional differential equation and applications, we refer to the survey [10]. Concerning the definitions and basic properties we refer the reader to [18]. Recently, some existence results for fractional boundary value problems have appeared in the literature (see, for example [2, 4, 5]).

2. PRELIMINARIES AND PREVIOUS RESULTS

For the convenience of the reader, we present here some notations and lemmas that will be used in the proofs of our main results.

Definition 2.1. The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $f\colon (0,\infty) \to \mathbb{R}$ is given by

$$D^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n-1 < \alpha \le n$, provide that the right-hand side is pointwise defined on $(0,\infty)$.

The following lemmas appear in [17].

Lemma 2.2. *Given $f\colon [0,1] \to \mathbb{R}$ continuous and $2 < \alpha \le 3$, the unique solution of*

$$\begin{aligned} &D^{\alpha}_{0+} u(t) + f(t) = 0, \qquad 0 < t < 1,\\ &u(0) = u'(1) = u''(0) = 0, \end{aligned}$$

is given by

$$u(t) = \int_0^1 G(t,s) f(s) ds,$$

where

$$G(t,s) = \begin{cases} \dfrac{(\alpha-1)t(1-s)^{\alpha-2} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1,\\[2mm] \dfrac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & 0 \le t \le s \le 1. \end{cases}$$

Remark 2.3. Note that $G(t,s) > 0$ for $t \in (0,1)$, $G(0,s) = 0$ and $G(t,1) = 0$ (see [17]).

Lemma 2.4. *Let $0 < \sigma < 1$, $2 < \alpha \leq 3$ and $F\colon (0,1] \to \mathbb{R}$ is a continuous function with $\lim_{t\to 0^+} F(t) = \infty$. Suppose that $t^\sigma F(t)$ is a continuous function on $[0,1]$. Then the function defined by*

$$H(t) = \int_0^1 G(t,s)F(s)ds$$

is continuous on $[0,1]$, where $G(t,s)$ is the Green function defined in Lemma 2.2.

Now, we present the fixed point theorem which we will use later. This result appears in [1]. Let $\mathcal{S}$ denote the class of those functions $\beta\colon [0,\infty) \to [0,1)$ satisfying

$$\beta(t_n) \to 1 \quad \text{implies} \quad t_n \to 0 \ .$$

Theorem 2.5 (Theorem 2.1 of [1])**.** *Let $(X,\leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Let $T\colon X \to X$ be a nondecreasing mapping such that there exists an element $x_0 \in X$ with $x_0 \leq T(x_0)$. Suppose that there exists $\beta \in \mathcal{S}$ such that*

$$d(Tx,Ty) \leq \beta\big(d(x,y)\big)\cdot d(x,y) \quad \text{for each } x,y \in X \text{ with } x \geq y. \tag{2.1}$$

Assume that either T is continuous or X is such that

$$\text{if } \big(x_n\big) \text{ is a nondecreasing sequence with } x_n \to x \text{ in } X, \text{ then } x_n \leq x, \forall n \in \mathbb{N}. \tag{2.2}$$

Besides, if

$$\text{for } x,y \in X \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y, \tag{2.3}$$

then T has an unique fixed point.

In our considerations, we will work in the Banach space

$$\mathcal{C}[0,1] = \{x\colon [0,1] \to \mathbb{R}, \text{continuous}\} \ ,$$

with the standard norm $\|x\| = \max_{0\leq t\leq 1}|x(t)|$. Note that this space can be equipped with a partial order given by

$$x,y \in \mathcal{C}[0,1], x \leq y \Leftrightarrow x(t) \leq y(t) \text{ for } t \in [0,1].$$

In [14] it is proved that $(\mathcal{C}[0,1],\leq)$ with the distance induced by the norm

$$d(x,y) = \max_{0\leq t\leq 1}\{|x(t)-y(t)|\},$$

satisfies condition (2.2) of Theorem 2.5. Moreover, for $x,y \in \mathcal{C}[0,1]$, as the function $\max\{x,y\}$ is continuous in $[0,1]$, $(\mathcal{C}[0,1],\leq)$ satisfies condition (2.3).

3. MAIN RESULT

Let $\mathcal{A}$ denote the class of those functions $\phi\colon [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(i) ϕ is nondecreasing.
(ii) For any $x > 0$, $\phi(x) < x$.
(iii) $\dfrac{\phi(x)}{x} \in \mathcal{S}$, where $\mathcal{S}$ is the class of functions appearing in Section 2.

For example, the functions $\phi(t) = \mu\, t$ with $0 \le \mu < 1$, $\phi(t) = \frac{t}{1+t}$ and $\phi(t) = \ln(1+t)$ belong to $\mathcal{A}$. In what follows, we formulate our main result.

Theorem 3.1. *Let $0 < \sigma < 1$, $2 < \alpha \le 3$, $f\colon (0,1] \times [0,\infty) \to [0,\infty)$ is continuous, $\lim_{t\to 0^+} f(t,\cdot) = \infty$ and $t^\sigma f(t,y)$ is a continuous function on $[0,1]\times[0,\infty)$. Assume that there exists $0 < \lambda \le \frac{\Gamma(\alpha-\sigma)}{\Gamma(1-\sigma)}$ and $\phi \in \mathcal{A}$ such that,*

$$0 \le t^\sigma (f(t,y) - f(t,x)) \le \lambda \cdot \phi(y-x) \tag{3.1}$$

for $x, y \in [0,\infty)$ with $y \ge x$ and $t \in [0,1]$. Then Problem (1.1) has a unique nonnegative solution.

Proof. Consider the cone

$$P = \{u \in \mathcal{C}[0,1] : u(t) \ge 0\}.$$

Note that, as P is a closed set of $\mathcal{C}[0,1]$, P is a complete metric space with the above mentioned distance in $\mathcal{C}[0,1]$. Now, for $u \in P$ we define the operator T by

$$(Tu)(t) = \int_0^1 G(t,s) f(s,u(s))ds.$$

By Lemma 2.4, $Tu \in \mathcal{C}[0,1]$. Moreover, taking into account Remark 2.3 and, as $t^\sigma f(t,y) \ge 0$ for $(t,y) \in [0,1]\times[0,\infty)$ by hypothesis, we get

$$(Tu)(t) = \int_0^1 G(t,s)s^{-\sigma}s^\sigma f(s,u(s))ds \ge 0.$$

Hence, $T(P) \subset P$. In what follows, we check that the hypotheses in Theorem 2.5 are satisfied. Firstly, the operator T is nondecreasing. In fact, taking into account assumption (3.1), for $u \ge v$ we obtain

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s) f(s,u(s))ds \\ &= \int_0^1 G(t,s)s^{-\sigma}s^\sigma f(s,u(s))ds \\ &\ge \int_0^1 G(t,s)s^{-\sigma}s^\sigma f(s,v(s))ds = (Tv)(t).\end{aligned}$$

Besides, for $u \geq v$ and $u \neq v$

$$\begin{aligned}
d(Tu,Tv) &= \max_{t\in[0,1]} |(Tu)(t)-(Tv)(t)| \\
&= \max_{t\in[0,1]}((Tu)(t)-(Tv)(t)) = \max_{t\in[0,1]}\left[\int_0^1 G(t,s)(f(s,u(s))-f(s,v(s)))ds\right] \\
&= \max_{t\in[0,1]}\left[\int_0^1 G(t,s)s^{-\sigma}s^{\sigma}(f(s,u(s))-f(s,v(s)))ds\right] \\
&\leq \max_{t\in[0,1]}\left[\int_0^1 G(t,s)s^{-\sigma}\lambda\cdot\phi\big(u(s)-v(s)\big)ds\right].
\end{aligned}$$

The nondecreasing character of ϕ gives us

$$\begin{aligned}
d(Tu,Tv) &\leq \lambda\cdot\phi\big(\|u-v\|\big)\max_{t\in[0,1]}\left[\int_0^1 G(t,s)s^{-\sigma}ds\right] \\
&= \lambda\cdot\phi\big(d(u,v)\big)\cdot \\
&\quad \max_{t\in[0,1]}\left[\int_0^t \frac{(\alpha-1)t(1-s)^{\alpha-2}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}ds+\int_t^1\frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}s^{-\sigma}ds\right] \\
&\leq \lambda\cdot\phi\big(d(u,v)\big)\cdot \\
&\quad \max_{t\in[0,1]}\left[\int_0^t \frac{(\alpha-1)t(1-s)^{\alpha-2}}{\Gamma(\alpha)}s^{-\sigma}ds+\int_t^1\frac{t(1-s)^{\alpha-2}\cdot s^{-\sigma}}{\Gamma(\alpha-1)}ds\right] \\
&\leq \lambda\cdot\phi\big(d(u,v)\big)\cdot \\
&\quad \max_{t\in[0,1]}\left[\int_0^t \frac{(\alpha-1)(1-s)^{\alpha-2}}{\Gamma(\alpha)}s^{-\sigma}ds+\int_t^1\frac{(1-s)^{\alpha-2}\cdot s^{-\sigma}}{\Gamma(\alpha-1)}ds\right] \\
&= \frac{\lambda\cdot\phi\big(d(u,v)\big)}{\Gamma(\alpha-1)}\cdot\int_0^1(1-s)^{\alpha-2}\cdot s^{-\sigma}ds \\
&= \frac{\lambda\cdot\phi\big(d(u,v)\big)}{\Gamma(\alpha-1)}\cdot\frac{\Gamma(1-\sigma)\cdot\Gamma(\alpha-1)}{\Gamma(\alpha-\sigma)} \\
&= \lambda\cdot\phi\big(d(u,v)\big)\cdot\frac{\Gamma(1-\sigma)}{\Gamma(\alpha-\sigma)} \\
&\leq \frac{\Gamma(\alpha-\sigma)}{\Gamma(1-\sigma)}\cdot\phi\big(d(u,v)\big)\cdot\frac{\Gamma(1-\sigma)}{\Gamma(\alpha-\sigma)} \\
&= \phi\big(d(u,v)\big) \\
&= \frac{\phi\big(d(u,v)\big)}{d(u,v)}\cdot d(u,v).
\end{aligned}$$

Thus, for $u \geq v$ and $u \neq v$

$$d(Tu,Tv) \leq \beta\big(d(u,v)\big)\cdot d(u,v),$$

where $\beta(x) = \frac{\phi(x)}{x}$. Obviously, the last inequality is satisfied for $u = v$. Thus, condition (2.1) in Theorem 2.5 holds with $\beta(x) = \frac{\phi(x)}{x}$. Finally, take into account that the zero function satisfies $0 \leq T0$, and, obviously, as $(P, \leq)$ satisfies condition (2.3), Theorem 2.5 says us that, Problem (1.1) has an unique nonnegative solution. □

Remark 3.2. We point out that in [17] the authors need to show that P is completely continuous in order to apply Krasnoselskii's fixed point theorem in a cone. However, our reasoning does not need that. On the other hand, we are able to guarantee the uniqueness of one positive solution to Problem (1.1)

In the sequel, we present an example which illustrates Theorem 3.1.

Example 3.3. Consider the fractional differential equation

$$\begin{aligned} &D_{0^+}^{\frac{5}{2}} u(t) + \frac{(t-\frac{1}{2})^2 u(t)}{\sqrt{t}\big(1+u(t)\big)} = 0, \quad 0<t<1\ , \\ &u(0) = u'(1) = u''(0) = 0. \end{aligned} \tag{3.2}$$

In this case, $f(t,u) = \dfrac{(t-\frac{1}{2})^2 u}{\sqrt{t}(1+u)}$ for $(t,u) \in (0,1]\times[0,\infty)$. Obviously, f is continuous in $(0,1]\times[0,\infty)$ and $\lim_{t\to 0^+} f(t,\cdot) = \infty$. Moreover, for $u \geq v$ and $t \in [0,1]$ we have

$$0 \leq \sqrt{t}\big(f(t,u) - f(t,v)\big) = \left(t-\frac{1}{2}\right)^2 \left(\frac{u}{1+u} - \frac{v}{1+v}\right),$$

because $g(x) = \frac{x}{1+x}$ is nondecreasing on $[0,\infty)$ $\left(g'(x) = \dfrac{1}{(1+x)^2}\right)$. Besides, for $u \geq v$ and $t \in [0,1]$, we obtain

$$\begin{aligned} \sqrt{t}\big(f(t,u) - f(t,v)\big) &= \left(t-\frac{1}{2}\right)^2 \left(\frac{u}{1+u} - \frac{v}{1+v}\right) \\ &= \left(t-\frac{1}{2}\right)^2 \left[\frac{u-v}{(1+u)(1+v)}\right] \\ &\leq \left(t-\frac{1}{2}\right)^2 \left[\frac{u-v}{1+(u-v)}\right] \\ &\leq \frac{1}{4}\left[\frac{u-v}{1+(u-v)}\right], \end{aligned}$$

and, it is easily proved that $\phi(x) = \frac{x}{1+x}$ belongs to the class $\mathcal{A}$. As $\dfrac{\Gamma(\alpha-\sigma)}{\Gamma(1-\sigma)} = \dfrac{\Gamma(\frac{5}{2}-\frac{1}{2})}{\Gamma(1-\frac{1}{2})} = \dfrac{\Gamma(2)}{\Gamma(\frac{1}{2})} = \dfrac{1}{\sqrt{\pi}} \geq \dfrac{1}{4}$, Theorem 3.1 says us that Problem (3.2) has an unique nonnegative solution.

Remark 3.4. In Remark 3.5 of [9] it is proved that the Green's function $G(t,s)$ is strictly increasing in the first variable in the interval $(0,1)$ and we omit the proof.

Remark 3.4 gives us the following theorem which is a better result than Theorem 3.3 of [17] because the solution of our Problem (1.1) is positive in $(0,1)$ and strictly increasing. The proof of Theorem 3.5 is similar to the proof of Theorem 3.6 of [9] and we omit it.

Theorem 3.5. *Under assumptions of Theorem* 3.1*, our Problem* (1.1) *has an unique nonnegative and strictly increasing solution.*

Remark 3.6. Theorem 3.5 gives us an unique positive and strictly increasing solution to Problem (3.2) of Example 1. On the other hand, this fact cannot be deduced by the results of [17].

Remark 3.7. Denote by $\mathcal{M}$ the class of functions $\phi\colon [0,\infty) \to [0,\infty)$ continuous and such that if $\varphi(x) = x - \phi(x)$ then

(i) $\varphi\colon [0,\infty) \to [0,\infty)$ and it is nondecreasing.
(ii) $\varphi(0) = 0$.
(iii) φ is positive in $(0,\infty)$.

In [9] we prove Theorem 3.1 using functions belonging to the class $\mathcal{M}$. The following example proves that the class $\mathcal{A}$ contains functions which are not in $\mathcal{M}$. This example appears in [1]. Let $\phi_0\colon [0,\infty) \to [0,\infty)$ be defined as

$$\phi_0(t) = \begin{cases} 0\,, & 0 \le t \le 2 \\ 2\,t - 4\,, & 2 < t \le 3 \\ \dfrac{2}{3}\,t\,, & 3 < t. \end{cases}$$

It is easily proved that $\phi_0 \in \mathcal{A}$. On the other hand, it is easily seen that $\varphi(t) = t - \phi_0(t)$ is not increasing and, consequently, $\phi_0 \notin \mathcal{M}$. The following example proves that there exist functions belonging to the class $\mathcal{M}$ which are not in $\mathcal{A}$. In fact, we consider $\phi\colon [0,\infty) \to [0,\infty)$ given by

$$\phi(x) = x - \arctan x.$$

It is easily seen that $\phi \in \mathcal{M}$. On the other hand, $\phi \notin \mathcal{A}$ because $\beta(x) = \dfrac{\phi(x)}{x} = 1 - \dfrac{\arctan x}{x}$ and $\beta(t_n) \to 1$ when $t_n \to \infty$. Therefore, the classes $\mathcal{A}$ and $\mathcal{M}$ are not comparable. This means that the results of this paper cover cases which cannot be treated by the results of [9] and viceversa.

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