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EXISTENCE OF POSITIVE SOLUTIONS OF A TERMINAL VALUE PROBLEM FOR SECOND ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we use the well–known Guo–Krasnoselskii fixed point theorem to establish conditions which guarantee the existence of at least one positive solution for a terminal value problem concerning a second order differential equation.

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1. INTRODUCTION

In this paper we discuss the second order nonlinear differential equation

$$x''(t) + f(t, x(t)) = 0, \quad t \in [0, \infty), \tag{1.1}$$

along with the terminal condition

$$\lim_{t\to\infty} x(t) = \xi, \tag{1.2}$$

where $f : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\xi \in (0, \infty)$. More precisely, we are looking for conditions yielding existence of positive solutions of (1.1), defined on the whole interval $[0, +\infty)$, which satisfy the terminal condition (1.2).

As we know this problem was initiated by Hille [7] in 1948 and consequently was the subject of several papers [2, 4, 6, 11, 12, 13]. In these papers the existence of at least one or exactly one solution is proved mainly by using the Schauder's fixed point theorem or the contraction principle respectively. Recently, an increasing interest has also been observed concerning the existence of positive solutions on the half–line for second order differential equations. Fixed point theorems on Banach spaces ordered by appropriate cones are usually the tools to derive such results (see, among others, [3, 9, 10, 14, 15] and the references therein).

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Our purpose in this paper is to establish simple conditions under which the above terminal value problem (1.1)–(1.2) has at least one positive solution. The results we present here are obtained by using the well–known Guo–Krasnoselskii fixed point theorem [5, 8].

2. PRELIMINARIES

Definition 2.1. A function $x \in C([0,\infty),\mathbb{R})$ is a solution of the problem (1.1)–(1.2) if and only if x satisfies the differential equation (1.1) and the terminal condition (1.2).

At this point we establish the following assumption.

(H) It holds that

$$|f(t,s)| \leq a(t)L(s) + b(t),$$

where $L, a, b : [0,\infty) \to [0,\infty)$ are continuous functions and L is increasing. Moreover, assume that

$$A = \int_0^\infty \int_s^\infty a(r)drds < \infty \quad \text{and} \quad B = \int_0^\infty \int_s^\infty b(r)drds < \infty.$$

Definition 2.2. Let B be a real Banach space. A cone in B is a nonempty closed set $K \subseteq B$, such that

$$\kappa u + \lambda v \in K \text{ for all } u, v \in K \text{ and all } \kappa, \lambda \geq 0,$$

and

$$u, -u \in K \text{ implies } u = 0.$$

Theorem 2.3 ([5, 8])**.** *Let B be a Banach space and let K be a cone in B. Assume that Ω_1, Ω_2 are open bounded subsets of B, with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ and let $T : K \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right) \to K$ be a completely continuous operator such that either*

$$\|Tu\| \leq \|u\|, \quad u \in K \cap \partial\Omega_1, \quad \text{and} \quad \|Tu\| \geq \|u\|, \quad u \in K \cap \partial\Omega_2$$

or

$$\|Tu\| \geq \|u\|, \quad u \in K \cap \partial\Omega_1, \quad \text{and} \quad \|Tu\| \leq \|u\|, \quad u \in K \cap \partial\Omega_2.$$

Then T has a fixed point in $K \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$.

3. MAIN RESULTS

Let $E = \{x \in C([0,\infty),\mathbb{R}) : x \text{ is bounded}\}$, $c \geq 0$ and consider the sets

$$K = \{x \in E : x(t) \geq 0\}$$

and

$$K_c = \{x \in K : \|x\| < c\}.$$

It is not difficult to verify that E endowed with the usual sup-norm, defined as

$$\|x\| := \sup\{|x(t)| : \ t \in [0, \infty)\},$$

is a Banach space and K is a cone in E. Also, for $c > 0$, define the operator $T : K_c \to C([0, \infty), \mathbb{R}))$ by the formula

$$(Tx)(t) = \xi - \int_t^\infty \int_s^\infty f(r, x(r))drds. \tag{3.1}$$

Under the assumption (H), operator T is well defined. Indeed, for any $x \in \overline{K}_c$ and every $t \geq 0$, we obviously have

$$0 \leq x(t) \leq c.$$

Hence taking into account assumption (H), we have

$$\begin{aligned} |Tx(t)| &= |\xi - \int_t^\infty \int_s^\infty f(r, x(r))drds| \\ &\leq \xi + \int_t^\infty \int_s^\infty |f(r, x(r))|drds \\ &\leq \xi + \int_t^\infty \int_s^\infty (a(r)L(x(r)) + b(r))drds \\ &\leq \xi + AL(c) + B \\ &< \infty, \end{aligned}$$

for all $t \in [0, \infty)$.

Since a completely continuous operator is a continuous function, which maps bounded sets into relatively compact sets, we need the following compactness criterion for subsets U of E, which is a consequence of the well–known Arzela–Ascoli theorem (see Avramescu [1]). In order to formulate this criterion, we note that the a set U of real functions defined on $[0, \infty)$ is called equiconvergent at ∞ if all functions in U have finite limits at ∞ and, in addition, for each $\epsilon > 0$, there exists $T \equiv T(\epsilon) > 0$ such that, for all functions $u \in U$, we have $|u(t) - \lim_{s\to\infty} u(s)| < \epsilon$ for all $t \geq T$.

Lemma 3.1. *Let U be an equicontinuous and uniformly bounded subset of the Banach space E. If U is equiconvergent at ∞, it is also relatively compact.*

Lemma 3.2. *Let $M > 0$ and suppose that assumption (H) is satisfied. Then a function $x \in \overline{K}_M$ is a solution of the problem (1.1)–(1.2) if and only if x is a fixed point of the operator $T : \overline{K}_M \to C([0, \infty), \mathbb{R}))$ defined by equation (3.1).*

Proof. Let $x \in \overline{K}_M$ be a fixed point of the operator T, i.e. $x(t) = Tx(t)$, $t \in [0, \infty)$. Then, by the definition of T, we have

$$x'(t) = \int_t^\infty f(r, x(r))dr, \quad t \geq 0,$$

and consequently

$$x''(t) = -f(t, x(t)), \quad t \geq 0.$$

Moreover

$$\begin{aligned}\lim_{t\to\infty} x(t) &= \lim_{t\to\infty} Tx(t)\\ &= \lim_{t\to\infty} \left(\xi - \int_t^\infty \int_s^\infty f(r, x(r))drds\right)\\ &= \xi.\end{aligned}$$

So, we proved that every fixed point of T in $\overline{K}_M$ is a solution of the problem (1.1)–(1.2).

Assume that x is a solution of the problem (1.1)–(1.2) in $\overline{K}_M$. We will prove that $x = Tx$. Integrating (1.1) on $[s, t]$, $t > s \geq 0$, we have

$$x'(t) - x'(s) = -\int_s^t f(r, x(r))dr$$

and

$$\lim_{t\to\infty} x'(t) - x'(s) = -\int_s^\infty f(r, x(r))dr$$

or

$$x'(s) = \int_s^\infty f(r, x(r))dr,$$

since by condition (1.2) we have $\lim_{t\to\infty} x'(t) = 0$. Now, integrating the above formula on $[t, \sigma]$, $\sigma > t \geq 0$, we have

$$x(\sigma) - x(t) = \int_t^\sigma \int_s^\infty f(r, x(r))drds$$

and, for $\sigma \to \infty$,

$$\xi - x(t) = \int_t^\infty \int_s^\infty f(r, x(r))drds,$$

i.e.

$$x(t) = Tx(t),$$

and the proof is complete. □

Lemma 3.3. *Suppose that assumption (H) is satisfied and that there exists $M > 0$ such that $AL(M) + B \leq \xi$. Then*

$$T(\overline{K}_M) \subseteq K.$$

Proof. For every $x \in \overline{K}_M$ and $t \in [0, \infty)$, we have

$$\begin{aligned}Tx(t) &= \xi - \int_t^\infty \int_s^\infty f(r, x(r))drds\\ &\geq \xi - \int_t^\infty \int_s^\infty (a(r)L(x(r)) + b(r))drds\end{aligned}$$

$$\begin{aligned}
&\geq \xi - \int_t^\infty \int_s^\infty (a(r)L(M) + b(r))drds \\
&= \xi - L(M)\int_t^\infty \int_s^\infty a(r)drds - \int_t^\infty \int_s^\infty b(r)drds \\
&\geq \xi - (L(M)A + B) \\
&\geq 0.
\end{aligned}$$

□

Theorem 3.4. *Suppose that assumption* (H) *is satisfied and that there exist* $M_1, M_2 \in (0,\infty)$ *such that* $M_1 < \xi < M_2$ *and*

$$AL(M_1) + B \leq \xi - M_1, \qquad AL(M_2) + B \leq min\{\xi, M_2 - \xi\}.$$

Then there exists at least one positive solution y *of the boundary value problem* (1.1)–(1.2) *such that*

$$M_1 \leq \|y\| \leq M_2.$$

Proof. Our purpose is to apply Theorem 2.3. Since $A + L(M_2) + B \leq \xi$, using Lemma 3.3, we have that

$$T(\overline{K}_{M_2} \backslash K_{M_1}) \subseteq K.$$

Now, we will prove that operator

$$T : \overline{K}_{M_2} \backslash K_{M_1} \to K$$

is completely continuous. First of all, we will show that $T(\overline{K}_{M_2} \backslash K_{M_1})$ is relatively compact. For that purpose, let $x \in \overline{K}_{M_2} \backslash K_{M_1}$. Then, for every $t \in [0,\infty)$, we have

$$\begin{aligned}
|Tx(t) &= \left| \xi - \int_t^\infty \int_s^\infty f(r, x(r))drds \right| \\
&\leq |\xi| + \left| \int_t^\infty \int_s^\infty f(r, x(r))drds \right| \\
&\leq |\xi| + \int_0^\infty \int_s^\infty |f(r, x(r))|drds \\
&\leq |\xi| + \int_0^\infty \int_s^\infty (a(r)L(x(r)) + b(r))drds \\
&\leq |\xi| + \int_0^\infty \int_s^\infty (a(r)L(M_2) + b(r))drds \\
&\leq |\xi| + AL(M_2) + B \\
&< \infty.
\end{aligned}$$

So, the set $T(\overline{K}_{M_2} \backslash K_{M_1})$ is uniformly bounded. Moreover, this set is equiconvergent at ∞, since for every $t \in [0,\infty)$, we have

$$|Tx(t) - \xi| \leq \int_t^\infty \int_s^\infty (a(r)L(M_2) + b(r))drds.$$

Furthermore, for every $x \in \overline{K}_{M_2} \backslash K_{M_1}$ and $0 \le t_1 \le t_2$, we have

$$\begin{aligned} |Tx(t_2) - Tx(t_1)| &= \left| \int_{t_1}^{\infty} \int_{s}^{\infty} f(r, x(r)) dr ds - \int_{t_2}^{\infty} \int_{s}^{\infty} f(r, x(r)) dr ds \right| \\ &= \left| \int_{t_1}^{t_2} \int_{s}^{\infty} f(r, x(r)) dr ds \right| \\ &\le \int_{t_1}^{t_2} \int_{s}^{\infty} |f(r, x(r))| dr ds \\ &\le \int_{t_1}^{t_2} \int_{s}^{\infty} (a(r) L(x(r)) + b(r)) dr ds \\ &\le \int_{t_1}^{t_2} \int_{s}^{\infty} (a(r) L(M_2) + b(r)) dr ds. \end{aligned}$$

So, the set $T(\overline{K}_{M_2} \backslash K_{M_1})$ is equicontinuous. Therefore, by Lemma 3.1, this set is relatively compact. Moreover, the mapping T is continuous. Indeed, let $x \in \overline{K}_{M_2} \backslash K_{M_1}$ and $(x_n)_{n \in \mathbb{N}}$ an arbitrary sequence in $\overline{K}_{M_2} \backslash K_{M_1}$, with $\lim x_n = x$. Then, we have $\lim x_n(t) = x(t)$, $t \ge 0$. Thus, by applying the Lebesgue dominated convergence theorem, we have

$$\lim_{n} \int_{t}^{\infty} \int_{s}^{\infty} f(r, x_n(r)) dr ds = \int_{t}^{\infty} \int_{s}^{\infty} f(r, x(r)) dr ds.$$

So, for every $t \ge 0$, we have the pointwise convergence

$$\lim_{n} Tx_n(t) = Tx(t).$$

It remains to prove that

$$\lim Tx_n = Tx.$$

Consider any subsequence (u_m) of (Tx_n). Because $T(\overline{K}_{M_2} \backslash K_{M_1})$ is relatively compact, there exists a subsequence (u_λ) of (u_m) and a function $y \in E$, so that $\lim u_\lambda = y$. Since the uniform convergence implies the pointwise one to the same limit function, we must have $y = Tx$, which means that $\lim Tx_n = Tx$.

Also, let $x \in K$ with $\|x\| = M_1$. Then $0 \le x(t) \le M_1$, $t \in [0, \infty)$, and since $AL(M_1) + B \le \xi - M_1$, we have

$$\begin{aligned} (Tx)(t) &= \xi - \int_{t}^{\infty} \int_{s}^{\infty} f(r, x(r)) dr ds \\ &\ge \xi - \int_{t}^{\infty} \int_{s}^{\infty} (a(r) L(x(r)) + b(r)) dr ds \\ &\ge \xi - \int_{t}^{\infty} \int_{s}^{\infty} (a(r) L(M_1) + b(r)) dr ds \\ &= \xi - (L(M_1) A + B) \\ &\ge M_1 = \|x\|. \end{aligned}$$

Also, for every $x \in K$ with $\|x\| = M_2$, and since $AL(M_2) + B \leq M_2 - \xi$, we have

$$\begin{aligned}(Tx)(t) &= \xi - \int_t^\infty \int_s^\infty f(r, x(r))drds \\ &\leq \xi + \int_t^\infty \int_s^\infty (a(r)L(x(r)) + b(r))drds \\ &\leq \xi + \int_t^\infty \int_s^\infty (a(r)L(M_2) + b(r))drds \\ &= \xi + L(M_2)A + B \\ &\leq M_2 = \|x\|.\end{aligned}$$

Consequently, by Theorem 2.3, the boundary value problem (1.1)–(1.2) has at least one positive solution y, such that

$$M_1 \leq \|y\| \leq M_2.$$

□

4. AN APPLICATION

Let $a, b : [0, \infty) \to [0, \infty)$ be continuous functions such that

$$\int_0^\infty \int_s^\infty a(r)drds = 1 \quad \text{and} \quad \int_0^\infty \int_s^\infty b(r)drds = 1.$$

Consider the boundary value problem consisting of the differential equation

$$x''(t) + f(t, x(t)) = 0, \quad t \in [0, \infty), \tag{4.1}$$

along with the terminal condition

$$\lim_{t\to\infty} x(t) = 2, \tag{4.2}$$

where $f : [0, +\infty) \to \mathbb{R}$ is any continuous function such that

$$|f(t, s)| \leq a(t) \left(\frac{2}{\pi}\text{Arctan}(s)\right) + b(t).$$

We will prove that the boundary value problem (4.1)–(4.2) has at least one positive solution y, with $\frac{1}{2} \leq \|y\| \leq 4$.

Indeed, here we have $\xi = 2$, $L(t) = \frac{2}{\pi}\text{Arctan}(t)$, $t \in [0, \infty)$, and $A = B = 1$. Also, it is easy to see that function L is increasing on $[0, \infty)$. So, we have

$$\xi - (L(M_1)A + B) \geq M_1 \Leftrightarrow \frac{2}{\pi}\text{Arctan}(M_1) + M_1 - 1 \leq 0,$$

$$\xi - (L(M_2)A + B) \geq 0 \Leftrightarrow \text{Arctan}(M_2) \leq \frac{\pi}{2}$$

and

$$\xi + L(M_2)A + B \leq M_2 \Leftrightarrow \text{Arctan}(M_2) \leq \frac{\pi}{2}(M_2 - 3).$$

The above equations are satisfied for $M_1 = \frac{1}{2}$ and $M_2 = 4$. This completes the proof.

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