Communications in Applied Analysis **19** (2015), 43–60

MULTIPLE FIXED POINT THEOREMS WITH OMITTED RAY CONDITIONS

RICHARD AVERY[1,a], JOHNNY HENDERSON[2], AND XUEYAN LIU[3]

[1]College of Arts and Sciences, Dakota State University
Madison, South Dakota 57042, USA
E-mail: rich.avery@dsu.edu

[2]Department of Mathematics, Baylor University
Waco, TX 76798-7328 USA
E-mail: Johnny_Henderson@baylor.edu

[3]Department of Mathematics, University of Tennessee at Chattanooga,
Chattanooga, TN 37403, USA
E-mail: Xueyan-Liu@utc.edu

In honor of Professor Allan Peterson's 45 years of contributions to mathematics and service to the University of Nebraska.

ABSTRACT. This paper presents several multiple fixed point theorems that utilize a functional version of Altman's condition as an alternative to the standard inward-outward techniques used in the Krasnoselskii and Leggett-Williams type fixed point theorems. An example is included to illustrate the new technique in showing the existence of solutions to boundary value problems.

AMS (MOS) Subject Classification. 47H10.

1. INTRODUCTION

Altman [1] showed that, if there exists an $x_0 \in G$ such that an operator T satisfies the inequality,

$$\|Tx - x\|^2 \geq \|Tx - x_0\|^2 - \|x - x_0\|^2,$$

for all $x \in \partial G$, then the operator T has a fixed point in $\overline{G}$. In the Omitted Ray Fixed Point Theorem [3], Avery, Henderson and Liu used Altman like functional conditions of the form

$$\gamma(Tx - x_0) < \gamma(x - x_0) + \gamma(Tx - x)$$

and

$$\psi(Tx - x_0) > \psi(x - x_0) + \psi(Tx - x)$$

[a]Corresponding author.

Received November 2, 2014

in conjunction with Leggett-Williams fixed point arguments [7] to create index one and index zero Lemmas. Those Lemmas were then used to replace the inward and outward type conditions in the Krasnoselskii [6] and Leggett-Williams [7] fixed point theorems and their generalizations. In this paper we will utilize those Lemmas to create multiple fixed point theorems utilizing these Altman like functional conditions. We conclude with an application that illustrates a new technique to verify the existence of solutions to a right focal boundary value problem.

2. PRELIMINARIES

In this section we will state the definitions that are used in the remainder of the paper.

Definition 2.1. Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a *cone* if for all $x \in P$ and $\lambda \geq 0$, $\lambda x \in P$, and if $x, -x \in P$ then $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2. An operator is called *completely continuous* if it is continuous and maps bounded sets into precompact sets.

Definition 2.3. A map α is said to be a *nonnegative continuous concave functional* on a cone P of a real Banach space E if $\alpha : P \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0,1]$. Similarly we say the map β is a *nonnegative continuous convex functional* on a cone P of a real Banach space E if $\beta : P \to [0, \infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $t \in [0,1]$. We say the map γ is a *continuous sub-homogeneous functional* on a real Banach space E if $\gamma : E \to \mathbb{R}$ is continuous and

$$\gamma(tx) \leq t\gamma(x) \text{ for all } x \in E, \ \ t \in [0,1] \text{ and } \gamma(0) = 0.$$

Similarly we say the map ρ is a *continuous super-homogeneous functional* on a real Banach space E if $\rho : E \to \mathbb{R}$ is continuous and

$$\rho(tx) \geq t\rho(x) \text{ for all } x \in E, \ \ t \in [0,1] \text{ and } \rho(0) = 0.$$

Let ψ and δ be nonnegative continuous functionals on P; then, for positive real numbers a and b, we define the following sets:

$$P(\psi, b) = \{x \in P : \psi(x) < b\}$$

and

$$P(\delta, \psi, b, a) = P(\delta, b) - \overline{P(\psi, a)} = \{x \in P : a < \psi(x) \text{ and } \delta(x) < b\}.$$

Definition 2.4. Let D be a subset of a real Banach space E. If $r : E \to D$ is continuous with $r(x) = x$ for all $x \in D$, then D is a *retract* of E, and the map r is a *retraction.* The *convex hull* of a subset D of a real Banach space X is given by

$$conv(D) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in D,\ \lambda_i \in [0,1],\ \sum_{i=1}^{n} \lambda_i = 1,\ \text{and } n \in \mathbb{N} \right\}.$$

The following theorem is due to Dugundji and its proof can be found in [4, *p* 44].

Theorem 2.1. *For Banach spaces X and Y, let $D \subset X$ be closed and let $F : D \to Y$ be continuous. Then F has a continuous extension $\tilde{F} : X \to Y$ such that $\tilde{F}(X) \subset \overline{conv(F(D))}$.*

Corollary 1. *Every closed convex set of a Banach space is a retract of the Banach space.*

The following theorem, which establishes the existence and uniqueness of the fixed point index, is from [5, *pp* 82–86]; an elementary proof can be found in [4, *pp* 58 & 238]. The proof of our main result in the next section will invoke the properties of the fixed point index.

Theorem 2.2. *Let X be a retract of a real Banach space E. Then, for every bounded relatively open subset U of X and every completely continuous operator $A : \overline{U} \to X$ which has no fixed points on ∂U (relative to X), there exists an integer $i(A, U, X)$ satisfying the following conditions:*

(G1) *Normality: $i(A, U, X) = 1$ if $Ax \equiv y_0 \in U$ for any $x \in \overline{U}$;*

(G2) *Additivity: $i(A, U, X) = i(A, U_1, X) + i(A, U_2, X)$ whenever U_1 and U_2 are disjoint open subsets of U such that A has no fixed points on $\overline{U} - (U_1 \cup U_2)$;*

(G3) *Homotopy Invariance: $i(H(t, \cdot), U, X)$ is independent of $t \in [0,1]$ whenever $H : [0,1] \times \overline{U} \to X$ is completely continuous and $H(t, x) \neq x$ for any $(t, x) \in [0,1] \times \partial U$;*

(G4) *Solution: If $i(A, U, X) \neq 0$, then A has at least one fixed point in U.*

Moreover, $i(A, U, X)$ is uniquely defined.

3. MAIN RESULTS

Below is the index-one Lemma [2] used in the functional generalizations of the Leggett-Williams Fixed Point Theorems.

Lemma 3.1. *Suppose P is a cone in a real Banach space E, α is a nonnegative continuous concave functional on P, β is a nonnegative continuous convex functional on P, and $T : P \to P$ is a completely continuous operator. If there exist nonnegative numbers a and b such that*

$(R1)$ $\{x \in P\ : a \leq \alpha(x)$ *and* $\beta(x) < b\} \neq \emptyset$;
$(R2)$ *if* $x \in P$ *with* $\beta(x) = b$ *and* $\alpha(x) \geq a$, *then* $\beta(Tx) < b$;
$(R3)$ *if* $x \in P$ *with* $\beta(x) = b$ *and* $\alpha(Tx) < a$, *then* $\beta(Tx) < b$;

and if $\overline{P(\beta, b)}$ *is bounded, then* $i(T, P(\beta, b), P) = 1$.

The following Lemma [3] utilizes a functional approach which is similar to Altman's condition to modify key components in Leggett-Williams [7] type arguments. This approach allows us to replace

$$\beta(Tx) < b$$

with

$$\gamma(Tx - x_0) < \gamma(x - x_0) + \gamma(Tx - x) \text{ and } \delta(Tx - x_0) < \delta(x - x_0) + \delta(Tx - x)$$

in conditions $(A2)$ and $(A3)$ of our result below. This is all done utilizing the core structure and spirit of the original work of Leggett-Williams [7].

Lemma 3.2. *Suppose P is a cone in a real Banach space E, α is a nonnegative continuous concave functional on P, β is a nonnegative continuous convex functional on P, γ and δ are continuous sub-homogeneous functionals on E, and $T : P \to P$ is a completely continuous operator. If there exist $x_0 \in P$ and nonnegative numbers a and b such that*

$(A1)$ $x_0 \in \{x \in P\ : a \leq \alpha(x)$ *and* $\beta(x) < b\}$;
$(A2)$ *if* $x \in P$ *with* $\beta(x) = b$ *and* $\alpha(x) \geq a$, *then* $\gamma(Tx - x_0) < \gamma(x - x_0) + \gamma(Tx - x)$;
$(A3)$ *if* $x \in P$ *with* $\beta(x) = b$ *and* $\alpha(Tx) < a$, *then* $\delta(Tx - x_0) < \delta(x - x_0) + \delta(Tx - x)$;

and if $\overline{P(\beta, b)}$ *is bounded, then* $i(T, P(\beta, b), P) = 1$.

In some applications it is difficult to work with $\alpha(Tx) < a$ from condition $(A3)$ in Lemma 3.2. That inequality can be replaced by $\alpha(x) < a$ which results in the following Lemma.

Lemma 3.3. *Suppose P is a cone in a real Banach space E, α is a nonnegative continuous concave functional on P, β is a nonnegative continuous convex functional on P, γ and δ are continuous sub-homogeneous functionals on E, and $T : P \to P$ is a completely continuous operator. If there exist $x_0 \in P$ and nonnegative numbers a and b such that*

$(A1)$ $x_0 \in \{x \in P\ : a \leq \alpha(x)$ *and* $\beta(x) < b\}$;
$(A2)$ *if* $x \in P$ *with* $\beta(x) = b$ *and* $\alpha(x) \geq a$, *then* $\gamma(Tx - x_0) < \gamma(x - x_0) + \gamma(Tx - x)$;
$(A3')$ *if* $x \in P$ *with* $\beta(x) = b$ *and* $\alpha(x) < a$, *then* $\delta(Tx - x_0) < \delta(x - x_0) + \delta(Tx - x)$;

and if $\overline{P(\beta, b)}$ *is bounded, then* $i(T, P(\beta, b), P) = 1$.

Similarly, it may be difficult to work with $\alpha(x) \geq a$ from condition $(A2)$ in Lemma 3.2. That inequality can be replaced by $\alpha(Tx) \geq a$ which results in the following Lemma.

Lemma 3.4. *Suppose P is a cone in a real Banach space E, α is a nonnegative continuous concave functional on P, β is a nonnegative continuous convex functional on P, γ and δ are continuous sub-homogeneous functionals on E, and $T : P \to P$ is a completely continuous operator. If there exist $x_0 \in P$ and nonnegative numbers a and b such that*

$(A1)$ $x_0 \in \{x \in P : a \leq \alpha(x)$ *and* $\beta(x) < b\}$;
$(A2')$ *if* $x \in P$ *with* $\beta(x) = b$ *and* $\alpha(Tx) \geq a$, *then* $\gamma(Tx - x_0) < \gamma(x - x_0) + \gamma(Tx - x)$;
$(A3)$ *if* $x \in P$ *with* $\beta(x) = b$ *and* $\alpha(Tx) < a$, *then* $\delta(Tx - x_0) < \delta(x - x_0) + \delta(Tx - x)$;

and if $\overline{P(\beta, b)}$ is bounded, then $i(T, P(\beta, b), P) = 1$.

Note that when conditions $(A2')$ and $(A3)$ or $(A2)$ and $(A3')$ are applied in the above Lemmas that there are Altman like conditions that must be satisfied for every point on the boundary of the functional wedge $P(\beta, b)$. Using the technique introduced by Leggett-Williams [7] we applied the mixed conditions $(A2)$ and $(A3)$ in Lemma 3.2, however there are no known results comparable to this Lemma which utilize the mixed conditions $(A2')$ and $(A3')$. Below is the index-zero Lemma used in the functional generalizations of the Leggett-Williams Fixed Point Theorems [2].

Lemma 3.5. *Suppose P is a cone in a real Banach space E, κ is a nonnegative continuous concave functional on P, θ is a nonnegative continuous convex functional on P, and $T : P \to P$ is a completely continuous operator. If there exist nonnegative numbers c and d such that*

$(R4)$ $\{x \in P : c < \kappa(x)$ *and* $\theta(x) \leq d\} \neq \emptyset$ *and* $P(\kappa, c) \neq \emptyset$;
$(R5)$ *if* $x \in P$ *with* $\kappa(x) = c$ *and* $\theta(x) \leq d$, *then* $\kappa(Tx) > c$;
$(R6)$ *if* $x \in P$ *with* $\kappa(x) = c$ *and* $\theta(Tx) > d$, *then* $\kappa(Tx) > c$;

and if $\overline{P(\kappa, c)}$ is bounded, then $i(T, P(\kappa, c), P) = 0$.

Following a similar approach that was utilized in the index-one Lemmas above, and using an argument related to Altman's condition, we will replace

$$\kappa(Tx) > c$$

with

$$\rho(Tx - x_1) > \rho(x - x_1) + \rho(Tx - x) \quad \text{and} \quad \psi(Tx - x_1) > \psi(x - x_1) + \psi(Tx - x)$$

in conditions $(A5)$ and $(A6)$ of our result below. This is again done utilizing the core structure and spirit of the original work of Leggett-Williams [7].

Lemma 3.6. *Suppose P is a cone in a real Banach space E, κ is a nonnegative continuous concave functional on P, θ is a nonnegative continuous convex functional on P, ρ and ψ are continuous super-homogeneous functionals on E, and $T : P \to P$ is a completely continuous operator. If there exist $x_1 \in P$ and nonnegative numbers c and d such that*

(A4) $x_1 \in \{x \in P \,:\, c < \kappa(x) \ \text{ and } \ \theta(x) \leq d\}$ *and* $P(\kappa, c) \neq \emptyset$;
(A5) *if* $x \in P$ *with* $\kappa(x) = c$ *and* $\theta(x) \leq d$, *then* $\rho(Tx - x_1) > \rho(x - x_1) + \rho(Tx - x)$;
(A6) *if* $x \in P$ *with* $\kappa(x) = c$ *and* $\theta(Tx) > d$, *then* $\psi(Tx - x_1) > \psi(x - x_1) + \psi(Tx - x)$;

and if $\overline{P(\kappa, c)}$ *is bounded, then* $i(T, P(\kappa, c), P) = 0$.

In some applications it is difficult to work with $\theta(Tx) > d$ from condition $(A6)$ in Lemma 3.6, and it can be replaced by $\theta(x) > d$ which results in the following Lemma.

Lemma 3.7. *Suppose P is a cone in a real Banach space E, κ is a nonnegative continuous concave functional on P, θ is a nonnegative continuous convex functional on P, ρ and ψ are continuous super-homogeneous functionals on E, and $T : P \to P$ is a completely continuous operator. If there exist $x_1 \in P$ and nonnegative numbers c and d such that*

(A4) $x_1 \in \{x \in P \,:\, c < \kappa(x) \ \text{ and } \ \theta(x) \leq d\}$ *and* $P(\kappa, c) \neq \emptyset$;
(A5) *if* $x \in P$ *with* $\kappa(x) = c$ *and* $\theta(x) \leq d$, *then* $\rho(Tx - x_1) > \rho(x - x_1) + \rho(Tx - x)$;
(A6′) *if* $x \in P$ *with* $\kappa(x) = c$ *and* $\theta(x) > d$, *then* $\psi(Tx - x_1) > \psi(x - x_1) + \psi(Tx - x)$;

and if $\overline{P(\kappa, c)}$ *is bounded, then* $i(T, P(\kappa, c), P) = 0$.

Also, in some applications it is difficult to work with $\theta(x) \leq d$ from condition $(A5)$ in Lemma 3.6, and it can be replaced by $\theta(Tx) \leq d$ which results in the following Lemma.

Lemma 3.8. *Suppose P is a cone in a real Banach space E, κ is a nonnegative continuous concave functional on P, θ is a nonnegative continuous convex functional on P, ρ and ψ are continuous super-homogeneous functionals on E, and $T : P \to P$ is a completely continuous operator. If there exist $x_1 \in P$ and nonnegative numbers c and d such that*

(A4) $x_1 \in \{x \in P \,:\, c < \kappa(x) \ \text{ and } \ \theta(x) \leq d\}$ *and* $P(\kappa, c) \neq \emptyset$;
(A5′) *if* $x \in P$ *with* $\kappa(x) = c$ *and* $\theta(Tx) \leq d$, *then* $\rho(Tx - x_1) > \rho(x - x_1) + \rho(Tx - x)$;
(A6) *if* $x \in P$ *with* $\kappa(x) = c$ *and* $\theta(Tx) > d$, *then* $\psi(Tx - x_1) > \psi(x - x_1) + \psi(Tx - x)$;

and if $\overline{P(\kappa, c)}$ *is bounded then* $i(T, P(\kappa, c), P) = 0$.

The following double fixed point theorem is the result of an index one wedge contained in an index zero wedge and the proof can be found in [3]. When $T(0) = 0$ this only guarantees a single positive fixed point.

Theorem 3.1. *Suppose P is a cone in a real Banach space E, α and κ are nonnegative continuous concave functionals on P, β and θ are nonnegative continuous convex functionals on P, γ and δ are continuous sub-homogeneous functionals on E, ρ and ψ are continuous super-homogeneous functionals on E, and $T : P \to P$ is a completely continuous operator. Furthermore, suppose that there exist $x_0, x_1 \in P$ and nonnegative numbers a, b, c and d such that*

(A1) $x_0 \in \{x \in P \ : \ a \leq \alpha(x) \ \text{ and } \ \beta(x) < b\}$;
(A2) *if* $x \in P$ *with* $\beta(x) = b$ *and* $\alpha(x) \geq a$, *then* $\gamma(Tx - x_0) < \gamma(x - x_0) + \gamma(Tx - x)$;
(A3) *if* $x \in P$ *with* $\beta(x) = b$ *and* $\alpha(Tx) < a$, *then* $\delta(Tx - x_0) < \delta(x - x_0) + \delta(Tx - x)$;
(A4) $x_1 \in \{x \in P \ : \ c < \kappa(x) \ \text{ and } \ \theta(x) \leq d\}$ *and* $P(\kappa, c) \neq \emptyset$;
(A5) *if* $x \in P$ *with* $\kappa(x) = c$ *and* $\theta(x) \leq d$, *then* $\rho(Tx - x_1) > \rho(x - x_1) + \rho(Tx - x)$;
(A6) *if* $x \in P$ *with* $\kappa(x) = c$ *and* $\theta(Tx) > d$, *then* $\psi(Tx - x_1) > \psi(x - x_1) + \psi(Tx - x)$.

If $\overline{P(\beta, b)} \subsetneq P(\kappa, c)$ *and* $\overline{P(\kappa, c)}$ *is bounded, then T has at least two fixed points x^* and x^{**} with*

$$x^* \in P(\beta, b) \ \text{ and } \ x^{**} \in P(\kappa, \beta, c, b).$$

The following double fixed point theorem is the result of an index zero wedge contained in an index one wedge which is contained in an index zero wedge.

Theorem 3.2. *Suppose P is a cone in a real Banach space E, α, κ_i and κ_o are nonnegative continuous concave functionals on P, β, θ_i and θ_o are nonnegative continuous convex functionals on P, γ and δ are continuous sub-homogeneous functionals on E, ρ_i, ρ_o, ψ_i and ψ_o are continuous super-homogeneous functionals on E, and $T : P \to P$ is a completely continuous operator. Furthermore, suppose that there exist $x_0, x_{1_i}, x_{1_o} \in P$ and nonnegative numbers a, b, c_i, c_o, d_i and d_o such that*

(A1) $x_0 \in \{x \in P \ : \ a \leq \alpha(x) \ \text{ and } \ \beta(x) < b\}$;
(A2) *if* $x \in P$ *with* $\beta(x) = b$ *and* $\alpha(x) \geq a$, *then* $\gamma(Tx - x_0) < \gamma(x - x_0) + \gamma(Tx - x)$;
(A3) *if* $x \in P$ *with* $\beta(x) = b$ *and* $\alpha(Tx) < a$, *then* $\delta(Tx - x_0) < \delta(x - x_0) + \delta(Tx - x)$;
($A4_i$) $x_{1_i} \in \{x \in P \ : \ c_i < \kappa_i(x) \ \text{ and } \ \theta_i(x) \leq d_i\}$ *and* $P(\kappa_i, c_i) \neq \emptyset$;
($A5_i$) *if* $x \in P$ *with* $\kappa_i(x) = c_i$ *and* $\theta_i(x) \leq d_i$, *then* $\rho_i(Tx - x_{1_i}) > \rho_i(x - x_{1_i}) + \rho_i(Tx - x)$;
($A6_i$) *if* $x \in P$ *with* $\kappa_i(x) = c_i$ *and* $\theta_i(Tx) > d_i$, *then* $\psi_i(Tx - x_{1_i}) > \psi_i(x - x_{1_i}) + \psi_i(Tx - x)$;
($A4_o$) $x_{1_o} \in \{x \in P \ : \ c_o < \kappa_o(x) \ \text{ and } \ \theta_o(x) \leq d_o\}$ *and* $P(\kappa_o, c_o) \neq \emptyset$;
($A5_o$) *if* $x \in P$ *with* $\kappa_o(x) = c_o$ *and* $\theta_o(x) \leq d_o$, *then* $\rho_o(Tx - x_{1_o}) > \rho_o(x - x_{1_o}) + \rho_o(Tx - x)$;
($A6_o$) *if* $x \in P$ *with* $\kappa_o(x) = c_o$ *and* $\theta_o(Tx) > d_o$, *then* $\psi_o(Tx - x_{1_o}) > \psi_o(x - x_{1_o}) + \psi_o(Tx - x)$.

If $\overline{P(\kappa_i, c_i)} \subsetneq P(\beta, b)$, $\overline{P(\beta, b)} \subsetneq P(\kappa_o, c_o)$ *and* $\overline{P(\kappa_o, c_o)}$ *is bounded, then* T *has at least two fixed points* x^* *and* x^{**} *with*

$$x^* \in P(\beta, \kappa_i, b, c_i) \ \text{ and } \ x^{**} \in P(\kappa_o, \beta, c_o, b).$$

Proof. We will prove the that $i(T, P(\beta, \kappa_i, b, c_i), P) \neq 0$ and $i(T, P(\kappa_o, \beta, c_o, b), P) \neq 0$ which will guarantee the existence of at least two fixed points x^* and x^{**} with

$$x^* \in P(\beta, \kappa_i, b, c_i) \ \text{ and } \ x^{**} \in P(\kappa_o, \beta, c_o, b).$$

In Lemma 3.2 we verified that T has no fixed points on $\partial P(\beta, b)$, and in Lemma 3.6 we verified that T has no fixed points on $\partial P(\kappa_i, c_i)$ and $\partial P(\kappa_o, c_o)$. Hence T has no fixed points on $\overline{P(\kappa_o, c_o)} - (P(\beta, b) \cup P(\kappa_o, \beta, c_o, b))$, and T has no fixed points on $\overline{P(\beta, b)} - (P(\kappa_i, c_i) \cup P(\beta, \kappa_i, b, c_i))$.

Claim 1: $i(T, P(\beta, \kappa_i, b, c_i), P) \neq 0$

The sets $P(\kappa_i, c_i)$ and $P(\beta, \kappa_i, b, c_i)$ are nonempty, disjoint, open subsets of $P(\beta, b)$, since $\overline{P(\kappa_i, c_i)} \subsetneq P(\beta, b)$ implies that $P(\beta, \kappa_i, b, c_i) = P(\beta, b) - \overline{P(\kappa_i, c_i)} \neq \emptyset$. Therefore, by the additivity property $(G2)$ of the fixed point index

$$i(T, P(\beta, b), P) = i(T, P(\kappa_i, c_i), P) + i(T, P(\beta, \kappa_i, b, c_i), P).$$

From Lemma 3.2 we have that $i(T, P(\beta, b), P) = 1$, and from Lemma 3.6 we have that $i(T, P(\kappa_i, c_i), P) = 0$. Consequently, we have $i(T, P(\beta, \kappa_i, b, c_i), P) = 1$.

Claim 2: $i(T, P(\kappa_o, \beta, c_o, b), P) \neq 0$

The sets $P(\beta, b)$ and $P(\kappa_o, \beta, c_o, b)$ are nonempty, disjoint, open subsets of $P(\kappa_o, c_o)$, since $\overline{P(\beta, b)} \subsetneq P(\kappa_o, c_o)$ implies that $P(\kappa_o, \beta, c_o, b) = P(\kappa_o, c_o) - \overline{P(\beta, b)} \neq \emptyset$. Therefore, by the additivity property $(G2)$ of the fixed point index

$$i(T, P(\kappa_o, c_o), P) = i(T, P(\beta, b), P) + i(T, P(\kappa_o, \beta, c_o, b), P).$$

From Lemma 3.2 we have that $i(T, P(\beta, b), P) = 1$, and from Lemma 3.6 we have that $i(T, P(\kappa_o, c_o), P) = 0$. Consequently, we have $i(T, P(\kappa_o, \beta, c_o, b), P) = -1$.

Since

$$i(T, P(\beta, \kappa_i, b, c_i), P) \neq 0 \ \text{ and } \ i(T, P(\kappa_o, \beta, c_o, b), P) \neq 0$$

with

$$P(\beta, \kappa_i, b, c_i) \cap P(\kappa_o, \beta, c_o, b) = \emptyset,$$

by the solution property $(G4)$ of the fixed point index, the operator T has at least two fixed points x^* and x^{**} with

$$x^* \in P(\beta, \kappa_i, b, c_i) \ \text{ and } \ x^{**} \in P(\kappa_o, \beta, c_o, b).$$

□

The following triple fixed point theorem is the result of an index one wedge contained in an index zero wedge which is contained in an index one wedge.

Theorem 3.3. *Suppose P is a cone in a real Banach space E, α_i, α_o and κ are nonnegative continuous concave functionals on P, β_i, β_o and θ are nonnegative continuous convex functionals on P, γ_i, γ_o, δ_i and δ_o are continuous sub-homogeneous functionals on E, ρ and ψ are continuous super-homogeneous functionals on E, and $T : P \to P$ is a completely continuous operator. Furthermore, suppose that there exist $x_{0_i}, x_{0_o}, x_1 \in P$ and nonnegative numbers a_o, a_i, b_o, b_i, c and d such that*

$(A1_i)$ $x_{0_i} \in \{x \in P \,:\, a_i \leq \alpha_i(x)$ *and* $\beta_i(x) < b_i\}$;
$(A2_i)$ *if* $x \in P$ *with* $\beta_i(x) = b_i$ *and* $\alpha_i(x) \geq a_i$, *then* $\gamma_i(Tx - x_{0_i}) < \gamma_i(x - x_{0_i}) + \gamma_i(Tx - x)$;
$(A3_i)$ *if* $x \in P$ *with* $\beta_i(x) = b_i$ *and* $\alpha_i(Tx) < a_i$, *then* $\delta_i(Tx - x_{0_i}) < \delta_i(x - x_{0_i}) + \delta_i(Tx - x)$;
$(A1_o)$ $x_{0_o} \in \{x \in P \,:\, a_o \leq \alpha_o(x)$ *and* $\beta_o(x) < b_o\}$;
$(A2_o)$ *if* $x \in P$ *with* $\beta_o(x) = b_o$ *and* $\alpha_o(x) \geq a_o$, *then* $\gamma_o(Tx - x_{0_o}) < \gamma_o(x - x_{0_o}) + \gamma_o(Tx - x)$;
$(A3_o)$ *if* $x \in P$ *with* $\beta_o(x) = b_o$ *and* $\alpha_o(Tx) < a_o$, *then* $\delta_o(Tx - x_{0_o}) < \delta_o(x - x_{0_o}) + \delta_o(Tx - x)$;
$(A4)$ $x_1 \in \{x \in P \,:\, c < \kappa(x)$ *and* $\theta(x) \leq d\}$ *and* $P(\kappa, c) \neq \emptyset$;
$(A5)$ *if* $x \in P$ *with* $\kappa(x) = c$ *and* $\theta(x) \leq d$, *then* $\rho(Tx - x_1) > \rho(x - x_1) + \rho(Tx - x)$;
$(A6)$ *if* $x \in P$ *with* $\kappa(x) = c$ *and* $\theta(Tx) > d$, *then* $\psi(Tx - x_1) > \psi(x - x_1) + \psi(Tx - x)$.

If $\overline{P(\beta_i, b_i)} \subsetneq P(\kappa, c)$, $\overline{P(\kappa, c)} \subsetneq P(\beta_o, b_o)$ *and* $\overline{P(\beta_o, b_o)}$ *is bounded, then* T *has at least three fixed points* x^*, x^{**} *and* x^{***} *with*

$$x^* \in P(\beta_i, b_i), \; x^{**} \in P(\kappa, \beta_i, c, b_i) \; \text{ and } \; x^{***} \in P(\beta_o, \kappa, b_o, c).$$

Proof. We will prove the that $i(T, P(\beta_i, b_i), P) \neq 0$, $i(T, P(\kappa, \beta_i, c, b_i), P) \neq 0$ and $i(T, P(\beta_o, \kappa, b_o, c), P) \neq 0$ which will guarantee the existence of at least three fixed points x^*, x^{**} and x^{***} with

$$x^* \in P(\beta_i, b_i), \; x^{**} \in P(\kappa, \beta_i, c, b_i) \; \text{ and } \; x^{***} \in P(\beta_o, \kappa, b_o, c).$$

In Lemma 3.2 we verified that T has no fixed points on $\partial P(\beta_i, b_i)$ and $\partial P(\beta_o, b_o)$, and in Lemma 3.6 we verified that T has no fixed points on $\partial P(\kappa, c)$. Hence T has no fixed points on $\overline{P(\kappa, c)} - (P(\beta_i, b_i) \cup P(\kappa, \beta_i, c, b_i))$ and T has no fixed points on $\overline{P(\beta_o, b_o)} - (P(\kappa, c) \cup P(\beta_o, \kappa, b_o, c))$.

By Lemma 3.2 we have that $i(T, P(\beta_i, b_i), P) = 1 \neq 0$.

Claim 1: $i(T, P(\kappa, \beta_i, c, b_i), P) \neq 0$

The sets $P(\beta_i, b_i)$ and $P(\kappa, \beta_i, c, b_i)$ are nonempty, disjoint, open subsets of $P(\kappa, c)$, since $\overline{P(\beta_i, b_i)} \subsetneq P(\kappa, c)$ implies that $P(\kappa, \beta_i, c, b_i) = P(\kappa, c) - \overline{P(\beta_i, b_i)} \neq \emptyset$. Therefore, by the additivity property $(G2)$ of the fixed point index

$$i(T, P(\kappa, c), P) = i(T, P(\beta_i, b_i), P) + i(T, P(\kappa, \beta_i, c, b_i), P).$$

From Lemma 3.6 we have that $i(T, P(\kappa, c), P) = 0$, and from Lemma 3.2 we have that $i(T, P(\beta_i, b_i), P) = 1$. Consequently, we have $i(T, P(\kappa, \beta_i, c, b_i), P) = -1$.

Claim 2: $i(T, P(\beta_o, \kappa, b_o, c), P) \neq 0$

The sets $P(\kappa, c)$ and $P(\beta_o, \kappa, b_o, c)$ are nonempty, disjoint, open subsets of $P(\beta_o, b_o)$, since $\overline{P(\kappa, c)} \subsetneq P(\beta_o, b_o)$ implies that $P(\beta_o, \kappa, b_o, c) = P(\beta_o, b_o) - \overline{P(\kappa, c)} \neq \emptyset$. Therefore, by the additivity property $(G2)$ of the fixed point index

$$i(T, P(\beta_o, b_o), P) = i(T, P(\kappa, c), P) + i(T, P(\beta_o, \kappa, b_o, c), P).$$

From Lemma 3.2 we have that $i(T, P(\beta_o, b_o), P) = 1$, and from Lemma 3.6 we have that $i(T, P(\kappa, c), P) = 0$. Consequently, we have $i(T, P(\beta_o, \kappa, b_o, c), P) = 1$.

Since

$$i(T, P(\beta_i, b_i), P) \neq 0, \;\; i(T, P(\kappa, \beta_i, c, b_i), P) \neq 0 \;\text{ and }\; i(T, P(\beta_o, \kappa, b_o, c), P) \neq 0$$

with the corresponding sets being pairwise disjoint, by the solution property $(G4)$ of the fixed point index, the operator T has at least three fixed points x^*, x^{**} and x^{***} with

$$x^* \in P(\beta_i, b_i), \;\; x^{**} \in P(\kappa, \beta_i, c, b_i) \;\text{ and }\; x^{***} \in P(\beta_o, \kappa, b_o, c).$$

$\square$

4. APPLICATION

In this section we will extend the application from [3] to yield a double existence result to a right focal boundary value problem. To proceed, consider the second-order nonlinear right focal boundary value problem

$$x''(t) + f(x(t)) = 0, \quad t \in (0, 1), \tag{4.1}$$

$$x(0) = 0 = x'(1), \tag{4.2}$$

where $f : \mathbb{R} \to [0, \infty)$ is continuous. If x is a fixed point of the operator T defined by

$$Tx(t) := \int_0^1 G(t, s) f(x(s)) ds,$$

where

$$G(t, s) = \min\{t, s\}, \quad (t, s) \in [0, 1] \times [0, 1]$$

is the Green's function for the operator L defined by

$$Lx(t) := -x'',$$

with right-focal boundary conditions

$$x(0) = 0 = x'(1),$$

then it is well known that x is a solution of the boundary value problem (4.1), (4.2). Throughout this section of the paper we will use the facts that $G(t,s)$ is nonnegative, and for each fixed $s \in [0,1]$, $G(t,s)$ is nondecreasing in t. Moreover, for each fixed $s \in [0,1]$, G satisfies the concavity property given by

$$\min_{s\in[0,1]} \frac{G(y,s)}{G(w,s)} \geq \frac{y}{w}, \quad \forall\ 0 \leq y \leq w \leq 1. \tag{4.3}$$

Define the cone $P \subset E = C[0,1]$ by

$$P := \{x \in E\ :\ x \text{ is nonnegative, nondecreasing, concave, and } x(0) = 0\}.$$

Thus if $x \in P$ and $\nu \in (0,1)$, then by the concavity of x we have $x(\nu) \geq \nu x(1)$ since

$$\frac{x(\nu) - x(0)}{\nu - 0} \geq \frac{x(1) - x(0)}{1 - 0}.$$

By mixing and matching the index zero lemmas and index one lemmas associated to the works of Krasnoselskii, Leggett-Williams, Altman and their functional generalizations, we can create a great deal of variability in our fixed point theorems. The fixed point theorem below is one such example of a double fixed point theorem.

Theorem 4.1. *Suppose P is a cone in a real Banach space E, α, κ_i and κ_o are nonnegative continuous concave functionals on P, β and θ_i are nonnegative continuous convex functionals on P, γ and δ are continuous sub-homogeneous functionals on E, ρ_o is a continuous super-homogeneous functionals on E, and $T : P \to P$ is a completely continuous operator. Furthermore, suppose that there exist nonnegative numbers a, b, c_i, c_o, and d_i, with $x_0, x_{1_i}, x_{1_o} \in P$ such that*

- *(A1)* $x_0 \in \{x \in P\ :\ a \leq \alpha(x)$ *and* $\beta(x) < b\}$;
- *(A2)* *if* $x \in P$ *with* $\beta(x) = b$ *and* $\alpha(x) \geq a$, *then* $\gamma(Tx - x_0) < \gamma(x - x_0) + \gamma(Tx - x)$;
- *(A3)* *if* $x \in P$ *with* $\beta(x) = b$ *and* $\alpha(Tx) < a$, *then* $\delta(Tx - x_0) < \delta(x - x_0) + \delta(Tx - x)$;
- *(R4)* $x_{1_i} \in \{x \in P\ :\ c_i < \kappa_i(x)$ *and* $\theta_i(x) \leq d_i\}$ *and* $P(\kappa_i, c_i) \neq \emptyset$;
- *(R5)* *if* $x \in P$ *with* $\kappa_i(x) = c_i$ *and* $\theta_i(x) \leq d_i$, *then* $\kappa_i(Tx) > c_i$;
- *(R6)* *if* $x \in P$ *with* $\kappa_i(x) = c_i$ *and* $\theta_i(Tx) > d_i$, *then* $\kappa_i(Tx) > c_i$;
- *(A4$_o$)* $c_o < \kappa_o(x_{1_o})$ *and* $P(\kappa_o, c_o) \neq \emptyset$;
- *(A5$_o$)* *if* $x \in P$ *with* $\kappa_o(x) = c_o$, *then* $\rho_o(Tx - x_{1_o}) > \rho_o(x - x_{1_o}) + \rho_o(Tx - x)$.

If $\overline{P(\kappa_i, c_i)} \subsetneq P(\beta, b)$, $\overline{P(\beta, b)} \subsetneq P(\kappa_o, c_o)$ *and* $\overline{P(\kappa_o, c_o)}$ *is bounded, then T has at least two fixed points x^* and x^{**} with*

$$x^* \in P(\beta, \kappa_i, b, c_i) \ \text{ and } \ x^{**} \in P(\kappa_o, \beta, c_o, b).$$

In the following theorem, we demonstrate how to apply Theorem 4.1, to prove the existence of at least one positive solution to (4.1), (4.2).

Theorem 4.2. *If b and c_i are positive real numbers with $c_i < \frac{b}{2}$ and if $f : [0, \infty) \to [0, \infty)$ is a continuous function such that*

(a) $7b < f(x) \le 8b$ *for* $0 \le x \le \frac{b}{2}$,
(b) $2b < f(x) < \frac{15b}{2}$ *for* $\frac{b}{2} \le x \le \frac{5b}{8}$,
(c) $2b < f(x) < \frac{7b}{3}$ *for* $\frac{5b}{8} \le x \le b$,
(d) $\frac{3b}{2} < f(x) \le \frac{17b}{8}$ *for* $b \le x \le 2b$, *and*
(e) $\frac{1024b}{7} < f(x)$ *for* $7b \le x \le 8b$,

then the focal problem (4.1), (4.2) *has at least two positive solutions x^* and x^{**} such that*

$$c_i < x^*\left(\frac{1}{4}\right), \quad x^*\left(\frac{1}{2}\right) < b, \quad b < x^{**}\left(\frac{1}{2}\right) \quad \text{and} \quad x^{**}\left(\frac{7}{8}\right) < 7b.$$

Proof. For $x \in P$, if $t \in (0, 1)$, then by the properties of the Green's function $(Tx)''(t) = -f(x(t))$ and $Tx(0) = 0 = (Tx)'(1)$.

Also, for any $x \in P$, we have from (4.3) that, for any $0 \le y \le w \le 1$,

$$\begin{aligned} w(Tx(y)) &= w\int_0^1 G(y, s)\, f(x(s))\, ds \\ &\ge y\int_0^1 G(w, s)\, f(x(s))\, ds = y(Tx(w)). \end{aligned}$$

Therefore we have that $T : P \to P$. By the Arzela-Ascoli Theorem it is a standard exercise to show that T is a completely continuous operator using the properties of G and f.

For $x \in P$, let

$$\alpha(x) = \kappa_i(x) = x\left(\frac{1}{4}\right),$$

$$\beta(x) = \theta_i(x) = x\left(\frac{1}{2}\right),$$

and

$$\kappa_o(x) = x\left(\frac{7}{8}\right).$$

For $z \in E$, let

$$\delta(z) = \gamma(z) = \left| z\left(\frac{1}{2}\right) \right|$$

and

$$\rho_o(z) = \min_{t \in \{\frac{7}{8}, 1\}} z(t).$$

Furthermore, let $a = \frac{5b}{8}$, $d_i = b$, and $c_o = 7b$.

Clearly $P(\kappa_o, c_o)$ is a bounded subset of the cone P, since if $x \in P(\kappa_o, c_o)$, then

$$c_o \ge x\left(\frac{7}{8}\right) \ge \left(\frac{7}{8}\right) x(1),$$

and so (since x is increasing)

$$\|x\| = x(1) \le \frac{8c_o}{7} = 8b.$$

Also, if $x \in \overline{P(\kappa_i, c_i)}$, then

$$\frac{c_i}{2} \ge \frac{\kappa_i(x)}{2} = \frac{x\left(\frac{1}{4}\right)}{2} \ge \frac{x\left(\frac{1}{2}\right)}{4}$$

by the concavity of x. Thus if $x \in \overline{P(\kappa_i, c_i)}$ then

$$\beta(x) = x\left(\frac{1}{2}\right) \le 2c_i < b,$$

and hence

$$\overline{P(\kappa_i, c_i)} \subset P(\beta, b).$$

Clearly,

$$\overline{P(\beta, b)} \subset P(\kappa_o, c_o),$$

since if $x \in \overline{P(\beta, b)}$, then $\beta(x) = x\left(\frac{1}{2}\right) \le b$, and by the concavity of x,

$$\left(\frac{7}{8}\right) x\left(\frac{1}{2}\right) \ge \left(\frac{1}{2}\right) x\left(\frac{7}{8}\right).$$

Thus $x\left(\frac{7}{8}\right) \le \left(\frac{7}{4}\right) x\left(\frac{1}{2}\right) < 7b = c_o$.

Let x_0 and x_{1_o} be defined by

$$x_0(s) = \left\{ \begin{array}{ll} \frac{5bs}{2} & 0 \le s \le \frac{1}{4} \\ & \\ \frac{5b}{8} & \frac{1}{4} \le s \le 1 \end{array} \right\} \text{ and } x_{1_o}(s) = \frac{64c_o s}{49}, \quad s \in [0,1].$$

Consequently we have that

$$x_0 \in \left\{ x \in P \ : a = \frac{5b}{8} \le \alpha(x) \text{ and } \beta(x) < b \right\},$$

thus verifying $(A1)$ of Theorem 4.1. By letting $x_{1_i} = x_0$, we have that

$$x_{1_i} \in \{x \in P \ : c_i < \kappa_i(x) \text{ and } \theta_i(x) \le b = d_i\},$$

thus verifying $(R4)$ of Theorem 4.1. We also have

$$x_{1_o}\left(\frac{7}{8}\right) = \left(\frac{64c_o}{49}\right)\left(\frac{7}{8}\right) = \frac{8c_o}{7} > c_o,$$

and $0 \in P(\kappa_o, c_o)$, thus verifying $(A4_o)$ of Theorem 4.1. Also since $x_0 \in P(\beta, b) - \overline{P(\kappa_i, c_i)}$, we have that

$$\overline{P(\kappa_i, c_i)} \subsetneq P(\beta, b),$$

and since $4bt \in P(\kappa_o, c_o) - \overline{P(\beta, b)}$, we have that

$$\overline{P(\beta, b)} \subsetneq P(\kappa_o, c_o).$$

Claim 1: $\gamma(Tx - x_0) < \gamma(x - x_0) + \gamma(Tx - x)$ for all $x \in P$ with $\beta(x) = b$ and $\alpha(x) \geq a$.

Let $x \in P$ with $\beta(x) = b$ and $\alpha(x) \geq a = \frac{5b}{8}$. Thus by the nondecreasing and concavity properties of x, for $s \in \left[\frac{1}{4}, \frac{1}{2}\right]$ we have

$$\frac{5b}{8} \leq x\left(\frac{1}{4}\right) \leq x(s) \leq x\left(\frac{1}{2}\right) = b,$$

for $s \in \left[\frac{1}{2}, 1\right]$, we have

$$b \leq x(s) \leq \frac{7b}{4}$$

and for $s \in \left[0, \frac{1}{4}\right]$,

$$0 \leq x(s) \leq b.$$

Hence,

$$\begin{aligned}
Tx\left(\frac{1}{2}\right) &= \int_0^1 G\left(\frac{1}{2}, s\right) \ f(x(s))\ ds \\
&= \int_0^{\frac{1}{4}} s\ f(x(s))\ ds + \int_{\frac{1}{4}}^{\frac{1}{2}} s\ f(x(s))\ ds + \left(\frac{1}{2}\right)\int_{\frac{1}{2}}^1 \ f(x(s))\ ds \\
&< \int_0^{\frac{1}{4}} s\ \ (8b)\ \ ds + \int_{\frac{1}{4}}^{\frac{1}{2}} s\ \left(\frac{7b}{3}\right)\ ds + \left(\frac{1}{2}\right)\int_{\frac{1}{2}}^1 \ \left(\frac{17b}{8}\right)\ ds \\
&= \frac{8b}{32} + \frac{7b}{32} + \frac{17b}{32} = b,
\end{aligned}$$

and

$$\begin{aligned}
Tx\left(\frac{1}{2}\right) &= \int_0^1 G\left(\frac{1}{2}, s\right) \ f(x(s))\ ds \\
&= \int_0^{\frac{1}{4}} s\ f(x(s))\ ds + \int_{\frac{1}{4}}^{\frac{1}{2}} s\ f(x(s))\ ds + \left(\frac{1}{2}\right)\int_{\frac{1}{2}}^1 \ f(x(s))\ ds \\
&> \int_0^{\frac{1}{4}} s\ \ (2b)\ \ ds + \int_{\frac{1}{4}}^{\frac{1}{2}} s\ \ (2b)\ \ ds + \left(\frac{1}{2}\right)\int_{\frac{1}{2}}^1 \ \left(\frac{3b}{2}\right)\ ds \\
&= \frac{2b}{32} + \frac{6b}{32} + \frac{3b}{8} = \frac{5b}{8},
\end{aligned}$$

which verifies that

$$\frac{5b}{8} < Tx\left(\frac{1}{2}\right) < b.$$

Therefore,

$$\begin{aligned}
\gamma(Tx - x_0) &= \left|Tx\left(\frac{1}{2}\right) - x_0\left(\frac{1}{2}\right)\right| \\
&= Tx\left(\frac{1}{2}\right) - \frac{5b}{8} \\
&< b - \frac{5b}{8} \\
&= \left|x\left(\frac{1}{2}\right) - x_0\left(\frac{1}{2}\right)\right| \\
&\leq \gamma(x - x_0) + \gamma(Tx - x)
\end{aligned}$$

which verifies condition $(A2)$ of Theorem 4.1.

Claim 2: $\delta(Tx - x_0) < \delta(x - x_0) + \delta(Tx - x)$ for all $x \in P$ with $\beta(x) = b$ and $\alpha(Tx) < a$.

Let $x \in P$ with $\beta(x) = b$. Thus

$$\begin{aligned}
Tx\left(\frac{1}{4}\right) &= \int_0^1 G\left(\frac{1}{4}, s\right)\ f(x(s))\ ds \\
&= \int_0^{\frac{1}{4}} s\ f(x(s))\ ds + \left(\frac{1}{4}\right)\int_{\frac{1}{4}}^{\frac{1}{2}}\ f(x(s))\ ds + \left(\frac{1}{4}\right)\int_{\frac{1}{2}}^{1}\ f(x(s))\ ds \\
&> \int_0^{\frac{1}{4}} s\ \ (2b)\ \ ds + \left(\frac{1}{4}\right)\int_{\frac{1}{4}}^{\frac{1}{2}}\ (2b)\ \ ds + \left(\frac{1}{4}\right)\int_{\frac{1}{2}}^{1}\ \left(\frac{3b}{2}\right)\ ds \\
&= \frac{2b}{32} + \frac{2b}{16} + \frac{3b}{16} = \frac{3b}{8} > \frac{b}{4},
\end{aligned}$$

and hence we have that

$$\frac{b}{4} < Tx\left(\frac{1}{4}\right) < \frac{5b}{8}.$$

We also have that,

$$\begin{aligned}
Tx\left(\frac{1}{2}\right) - Tx\left(\frac{1}{4}\right) &= \int_0^1 G\left(\frac{1}{2}, s\right)\ f(x(s))\ ds - \int_0^1 G\left(\frac{1}{4}, s\right)\ f(x(s))\ ds \\
&= \int_{\frac{1}{4}}^{\frac{1}{2}} \left(s - \frac{1}{4}\right)\ f(x(s))\ ds + \left(\frac{1}{4}\right)\int_{\frac{1}{2}}^{1}\ f(x(s))\ ds \\
&< \int_{\frac{1}{4}}^{\frac{1}{2}} \left(s - \frac{1}{4}\right)\ \left(\frac{15b}{2}\right)\ ds + \left(\frac{1}{4}\right)\int_{\frac{1}{2}}^{1}\ \left(\frac{17b}{8}\right)\ ds \\
&= \frac{15b}{64} + \frac{17b}{64} = \frac{b}{2} < \frac{3b}{8},
\end{aligned}$$

where we used the fact that $x\left(\frac{1}{2}\right) \geq \frac{b}{2}$ by the concavity of x. Hence we have verified that

$$\frac{b}{4} < Tx\left(\frac{1}{2}\right) < b,$$

and thus

$$\frac{-3b}{8} < Tx\left(\frac{1}{2}\right) - \frac{5b}{8} < \frac{3b}{8}.$$

Therefore,

$$\begin{aligned}
\delta(Tx - x_0) &= \left|Tx\left(\frac{1}{2}\right) - x_0\left(\frac{1}{2}\right)\right| \\
&= \left|Tx\left(\frac{1}{2}\right) - \frac{5b}{8}\right| \\
&< \frac{3b}{8} \\
&= \left|x\left(\frac{1}{2}\right) - x_0\left(\frac{1}{2}\right)\right| \\
&\leq \delta(x - x_0) + \delta(Tx - x)
\end{aligned}$$

which verifies condition $(A3)$ of Theorem 4.1.

Claim 3: $\kappa_i(Tx) > c_i$ for all $x \in P$ with $\kappa_i(x) = c_i < \frac{b}{2}$.

Let $x \in P$ with $\kappa_i(x) = c_i$. Hence, by the concavity of x we have $\beta(x) \leq b$, thus

$$\begin{aligned}
\kappa_i(Tx) &= Tx\left(\frac{1}{4}\right) \\
&= \int_0^1 G\left(\frac{1}{4}, s\right)\ f(x(s))\ ds \\
&= \int_0^{\frac{1}{4}} s\ f\left(x(s)\right)\ ds + \left(\frac{1}{4}\right)\int_{\frac{1}{4}}^1\ f\left(x(s)\right)\ ds \\
&> \int_0^{\frac{1}{4}} s\ (7b)\ ds + \left(\frac{1}{4}\right)\int_{\frac{1}{4}}^1\ \left(\frac{3b}{2}\right)\ ds \\
&= \frac{7b}{32} + \frac{9b}{32} = \frac{b}{2} > c_i,
\end{aligned}$$

and we have verified both conditions $(R5)$ and $(R6)$ of Theorem 4.1.

Claim 4: $\rho_o(Tx - x_{1_o}) = Tx\left(\frac{7}{8}\right) - x_{1_o}\left(\frac{7}{8}\right)$ for all $x \in P$ with $\kappa_o(x) = c_0$.

Let $x \in P$ with $\kappa_o(x) = c_o$. For $t \in \left[\frac{7}{8}, 1\right]$, we have

$$(Tx)'(t) = \int_t^1 f(x(s))\ ds > \int_t^1 \left(\frac{1024c_o}{49}\right)\ ds = \left(\frac{1024c_o}{49}\right)(1-t),$$

and hence

$$\begin{aligned} Tx\,(1) - Tx\left(\frac{7}{8}\right) &= \int_{\frac{7}{8}}^{1} (Tx)'(s)\, ds \\ &> \int_{\frac{7}{8}}^{1} \left(\frac{1024c_o}{49}\right)(1-s)\, ds \\ &= \left(\frac{1024c_o}{49}\right)\left(\frac{1}{128}\right) = \frac{8c_o}{49} = x_{1_o}(1) - x_{1_o}\left(\frac{7}{8}\right). \end{aligned}$$

Therefore,

$$\rho(Tx - x_{1_o}) = Tx\left(\frac{7}{8}\right) - x_{1_o}\left(\frac{7}{8}\right),$$

since

$$Tx\left(\frac{7}{8}\right) - x_{1_o}\left(\frac{7}{8}\right) < Tx\,(1) - x_{1_o}\,(1)\,.$$

Claim 5: $\rho_o(Tx - x_{1_o}) > \rho_o(x - x_{1_o}) + \rho_o(Tx - x)$ for $x \in P$ with $\kappa_o(x) = c_o$.

Let $x \in P$ with $\kappa_o(x) = c_o$. By the previous claim we have

$$\rho_o(Tx - x_{1_o}) = Tx\left(\frac{7}{8}\right) - x_{1_o}\left(\frac{7}{8}\right),$$

and since

$$x\left(\frac{7}{8}\right) \geq \left(\frac{7}{8}\right) x(1),$$

we have that $x(1) \leq \left(\frac{8}{7}\right) x\left(\frac{7}{8}\right) = \left(\frac{8}{7}\right) c_o$. Thus, since $x_{1_o}(t) = \frac{64c_o t}{49}$, we have

$$\begin{aligned} x\left(\frac{7}{8}\right) - x_{1_o}\left(\frac{7}{8}\right) &= c_o - \left(\frac{8}{7}\right) c_o \\ &> \left(\frac{8}{7}\right) c_o - \left(\frac{64}{49}\right) c_o \\ &\geq x\,(1) - x_{1_o}\,(1)\,. \end{aligned}$$

Therefore,

$$\begin{aligned} \rho_o(Tx - x_{1_o}) &= Tx\left(\frac{7}{8}\right) - x_{1_o}\left(\frac{7}{8}\right) \\ &= Tx\left(\frac{7}{8}\right) - x\left(\frac{7}{8}\right) + x\left(\frac{7}{8}\right) - x_{1_o}\left(\frac{7}{8}\right) \\ &> Tx\left(\frac{7}{8}\right) - x\left(\frac{7}{8}\right) + x\,(1) - x_{1_o}\,(1) \\ &\geq \rho_o(Tx - x) + \rho_o(x - x_{1_o}). \end{aligned}$$

Therefore, the conditions of Theorem 4.1 are satisfied and the operator T has at least two fixed points x^* and x^{**} with

$$x^* \in P\,(\beta, \kappa_i, b, c_i) \quad \text{and} \quad x^{**} \in P\,(\kappa_o, \beta, c_o, b)\,.$$

□

Example: Let

$$b = 128 \text{ and } f(x) = \begin{cases} 900, & 0 \le x \le 64, \\ -40x + 3460, & 64 < x \le 80, \\ 260, & 80 < x \le 256, \\ 29x - 7164, & x > 256. \end{cases}$$

Then the boundary value problem

$$x''(t) + f(x(t)) = 0, \quad t \in (0,1),$$

with right focal boundary conditions

$$x(0) = 0 = x'(1),$$

has at least two positive solutions x^* and x^{**} which can be verified by the above theorem, with

$$64 \le x^*\left(\frac{1}{4}\right), \quad x^*\left(\frac{1}{2}\right) < 128, \quad 128 < x^{**}\left(\frac{1}{2}\right) \quad \text{and} \quad x^{**}\left(\frac{7}{8}\right) < 896.$$

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