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INFINITELY MANY HOMOCLINIC SOLUTIONS FOR SECOND ORDER DIFFERENCE EQUATIONS WITH p-LAPLACIAN

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ABSTRACT. By using critical point theory, the authors study the existence of infinitely many homoclinic solutions to the difference equation

$$-\Delta\big(a(k)\phi_p(\Delta u(k-1))\big) + b(k)\phi_p(u(k)) = \lambda f(k, u(k))), \quad k \in \mathbb{Z},$$

where $p > 1$ is a real number, $\phi_p(t) = |t|^{p-2}t$ for $t \in \mathbb{R}$, $\lambda > 0$ is a parameter, $a, b : \mathbb{Z} \to (0, \infty)$, and $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is a continuous function in the second variable. Some known work in the literature is extended.

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1. INTRODUCTION

The theory of nonlinear discrete dynamical systems has been widely used to examine discrete models appearing in many fields such as statistics, computing, electrical circuit analysis, dynamical systems, economics, biology, etc [1, 6]. In recent years, there has been an increasing interest in the literature on the use of variational methods to study the existence and multiplicity of periodic and homoclinic solutions for discrete systems. We refer the reader to [2, 4, 7, 8, 9, 10] and the reference therein for some recent work on this topic. In this paper, we are concerned with the existence of solutions of the second order difference equation with a p-Laplacian

$$\begin{cases} -\Delta\big(a(k)\phi_p(\Delta u(k-1))\big) + b(k)\phi_p(u(k)) = \lambda f(k, u(k)), \quad k \in \mathbb{Z}, \\ u(k) \to 0 \quad \text{as } |k| \to \infty, \end{cases} \tag{1.1}$$

where $p > 1$ is a real number, $\phi_p(t) = |t|^{p-2}t$ for $t \in \mathbb{R}$, $\lambda > 0$ is a parameter, Δ is the forward difference operator defined by $\Delta u(k) = u(k+1) - u(k)$ for $k \in \mathbb{Z}$, $a, b : \mathbb{Z} \to (0, \infty)$, and $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is a continuous function in the second variable. As in the literature, a solution of problem (1.1) is referred to as a homoclinic solution of the equation

$$-\Delta\big(a(k)\phi_p(\Delta u(k-1))\big) + b(k)\phi_p(u(k)) = \lambda f(k, u(k))), \quad k \in \mathbb{Z}.$$

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In a recent paper [4], Iannizzotto and Tersian studied the existence of two solutions of (1.1) with $a(k) \equiv 1$ on $\mathbb{Z}$. Let

$$F(k,t) = \int_0^t f(k,s)ds \quad \text{for all } (k,t) \in \mathbb{Z} \times \mathbb{R}, \tag{1.2}$$

and assume that the following conditions hold:

(H1) $b(k) \geq b_0 > 0$ for all $k \in \mathbb{Z}$, $b(k) \to \infty$ as $|k| \to \infty$;
(H2) $\lim_{t\to 0} \frac{|f(k,t)|}{|t|^{p-1}} = 0$ uniformly for all $k \in \mathbb{Z}$;
(H3) $\sup_{|t|\leq T} |F(\cdot,t)| \in \ell^1$ for all $T > 0$;
(H4) $\limsup_{|t|\to\infty} \frac{F(k,t)}{|t|^p} \leq 0$ uniformly for all $k \in \mathbb{Z}$;
(H5) $F(k_0,t_0) > 0$ for some $k_0 \in \mathbb{Z}$ and $t_0 \in \mathbb{R}$.

They proved the following result (see [4, Theorem 1]).

Proposition 1.1. *Assume that (H1)–(H5) hold and $a(k) \equiv 1$ on $\mathbb{Z}$. Then, for all $\lambda > 0$ large enough, problem* (1.1) *has at least two nonzero solutions. Moreover, whenever $u : \mathbb{Z} \to \mathbb{R}$ is a nontrivial solution of problem* (1.1)*, there exist integers k_+ and k_- such that both sequences $\{u(k)\}_{k\leq k_-}$ and $\{u(k)\}_{k\geq k_+}$ are strictly monotone.*

In this paper, we will extend Proposition 1.1. In particular, we find sufficient conditions under which problem (1.1) has infinitely many solutions for λ large enough. Our proof is mainly based on a critical point theorem obtained by Kajikiya in [5]; see Lemma 3.4 below.

The following assumptions will be used in our theorems.

(C1) There exists a constant $M > 0$ such that

$$a(k) \leq Mb(k) \quad \text{for all } k \in \mathbb{Z};$$

(C2) there exist a constant $\rho > 0$ and two positive functions $w_1, w_2 \in \ell^1$ such that

$$w_1(k)|t|^p \leq F(k,t) \leq w_2(k)|t|^p \tag{1.3}$$

for all $k \in \mathbb{Z}$ and $|t| \leq \rho$;
(C3) $f(k,-t) = -f(k,t)$ for all $k \in \mathbb{Z}$ and $t \in \mathbb{R}$.

Remark 1.2. Concerning condition (C2), we want to emphasize that we only need that (1.3) holds for small $|t| \in \mathbb{R}$, and there is no restriction on the behavior of $F(k,t)$ when $|t|$ is large.

The rest of this paper is organized as follows. Section 2 contains some preliminary lemmas and Section 3 contains the main results in this paper and their proofs.

2. PRELIMINARIES

In this section, we will establish the corresponding variational framework for problem (1.1) and present some lemmas to be used later.

For all $1 \le p < \infty$, let ℓ^p be the set of all functions $u : \mathbb{Z} \to \mathbb{R}$ such that

$$\|u\|_p = \left(\sum_{k\in\mathbb{Z}} |u(k)|^p \right)^{1/p} < \infty,$$

and let ℓ^∞ be the set of all functions $u : \mathbb{Z} \to \mathbb{R}$ such that

$$\|u\|_\infty = \sup_{k\in\mathbb{Z}} |u(k)| < \infty.$$

The following lemma can be found in [3, pp. 3 and 429] and [4, Proposition 2].

Lemma 2.1. *For all $1 < p < \infty$, $(\ell^p, \|\cdot\|_p)$ is a reflexive and separable Banach space whose dual is $(\ell^q, \|\cdot\|_q)$, where $1/p + 1/q = 1$. Moreover, $(\ell^\infty, \|\cdot\|_\infty)$ is a Banach space, and for all $1 \le p < \infty$, the embedding $\ell^p \hookrightarrow \ell^\infty$ is continuous as*

$$\|u\|_\infty \le \|u\|_p \quad \textit{for all } u \in \ell^p.$$

Let

$$X = \left\{ u : \mathbb{Z} \to \mathbb{R} \ : \ \sum_{k\in\mathbb{Z}} \left[a(k)|\Delta u(k-1)|^p + b(k)|u(k)|^p \right] < \infty \right\} \tag{2.1}$$

and

$$\|u\| = \left(\sum_{k\in\mathbb{Z}} \left[a(k)|\Delta u(k-1)|^p + b(k)|u(k)|^p \right] \right)^{1/p}. \tag{2.2}$$

Obviously, under condition (H1), we have

$$\|u\|_\infty \le \|u\|_p \le b_0^{-1/p} \|u\|. \tag{2.3}$$

Lemma 2.2. *For all $1 < p < \infty$, $(X, \|\cdot\|)$ is a reflexive and separable Banach space, and the embedding $X \hookrightarrow \ell^p$ is compact.*

Proof. By Lemma 2.1, $(\ell^p, \|\cdot\|_p)$ is a reflexive and separable Banach space. Then, the Cartesian product space $\ell^p_2 = \ell^p \times \ell^p$ is also a reflexive and separable Banach space with respect to the norm

$$\|v\|_{\ell^p_2} = \left(\sum_{i=1}^{2} \|v_i\|_p^p \right)^{1/p}, \quad v = (v_1, v_2) \in \ell^p_2. \tag{2.4}$$

Consider the space

$$Y = \left\{ \left((b(k))^{1/p} u(k), (a(k))^{1/p} \Delta u(k-1) \right) \ : \ u \in X, \ k \in \mathbb{Z} \right\}.$$

Then, Y is a closed subset of ℓ^p_2. Hence, Y is also a reflexive and separable Banach space with respect to the norm (2.4).

Let the operator $T : X \to Y$ be defined by

$$Tu = \big((b(k))^{1/p}u(k), (a(k))^{1/p}\Delta u(k-1)\big) \quad \text{for any } u \in X.$$

Obviously,

$$\|u\| = \|Tu\|_{\ell_2^p}.$$

Thus, $T : X \to Y$ is an isometric isomorphic mapping and X is isometric isomorphic to Y. Therefore, X is a reflexive and separable Banach space.

Finally, by using a proof similar to the ones used to prove [4, Proposition 3] or [7, Lemma 2.1], it can be shown that the embedding $X \hookrightarrow \ell^p$ is compact. The details are omitted. This completes the proof of the lemma. □

For any $u \in X$ and $\lambda > 0$, let

$$\Phi(u) = \frac{1}{p}\sum_{k\in\mathbb{Z}} \big[a(k)|\Delta u(k-1)|^p + b(k)|u(k)|^p\big],$$

$$\Psi(u) = \sum_{k\in\mathbb{Z}} F(k, u(k)),$$

and

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u). \tag{2.5}$$

Lemma 2.3. *For the functionals Φ, Ψ, and I_λ, we have the following:*

(a) *Assume that (H1) holds. Then $\Phi \in C^1(X, \mathbb{R})$ with*

$$\langle \Phi'(u), v\rangle = \sum_{k\in\mathbb{Z}} [a(k)\phi_p(\Delta u(k-1))\Delta v(k-1) + b(k)\phi_p(u(k))v(k)]$$

for all $u, v \in X$.

(b) *Assume that (C2) holds. Then $\Psi \in C^1(\ell^p, \mathbb{R})$ with*

$$\langle \Psi'(u), v\rangle = \sum_{k\in\mathbb{Z}} f(k, u(k))v(k) \quad \textit{for all } u, v \in \ell^p.$$

(c) *Assume that (H1) and (C2) hold. Then, for all $\lambda > 0$, every critical point $u \in X$ of I_λ is a solution of problem* (1.1)*.*

Remark 2.4. Part (a) of Lemma 2.3 with $a(k) \equiv 1$ on $\mathbb{Z}$ has been proved in [4, Proposition 5]; part (b) of the lemma has been shown in [4, Proposition 6] under (H2); and part (c) of the lemma with $a(k) \equiv 1$ on $\mathbb{Z}$ has been verified in [4, Proposition 7] under (H1) and (H2). The present version of the lemma can be proved by essentially the same arguments. We omit the details.

3. MAIN RESULTS

Theorem 3.1 below extends Proposition 1.1. In particular, it shows that, under some suitable assumptions on f, problem (1.1) has infinitely many solutions.

Theorem 3.1. *Assume that (H1), (H3), (H4), and (C1)–(C3) hold. Then, there exists a constant* $\underline{\lambda} > 0$ *such that for all* $\lambda > \underline{\lambda}$, *problem* (1.1) *has a sequence* $\{u_n(k)\}$ *of nontrivial solutions satisfying*

$$u_n \to 0 \text{ in } X \quad \text{and} \quad I_\lambda(u_n) \le 0, \tag{3.1}$$

where X *and* I_λ *are defined by* (2.1) *and* (2.5), *respectively.*

The following corollary is a direct consequence of Theorem 3.1.

Corollary 3.2. *Assume that (H1) and (C1) hold,* $w \in \ell^1$, *and* $1 < q < p$. *Let*

$$g(t) = \begin{cases} \phi_p(t) & \text{if } |t| \le 1, \\ \phi_q(t) & \text{if } |t| > 1. \end{cases}$$

Then, there exists a constant $\underline{\lambda} > 0$ *such that for all* $\lambda > \underline{\lambda}$, *the problem*

$$\begin{cases} -\Delta\big(a(k)\phi_p(\Delta u(k-1))\big) + b(k)\phi_p(u(k)) = \lambda w(k) g(u(k)), \quad k \in \mathbb{Z}, \\ u(k) \to 0 \quad \text{as } |k| \to \infty, \end{cases} \tag{3.2}$$

has a sequence $\{u_n(k)\}$ *of nontrivial solutions satisfying* (3.1).

In the rest of this section, we prove our results. To this end, we first recall the notion of genus.

Definition 3.3. Let X be a Banach space and A be a subset of X. We say that A is symmetric if $u \in A$ implies $-u \in A$. For a closed symmetric set A with $0 \notin A$, the genus $\gamma(A)$ of A is defined as the smallest integer k such that there exists an odd continuous mapping from A to $\mathbb{R}^k \setminus \{0\}$. If there does not exist such k, we define $\gamma(A) = \infty$. Moreover, we set $\gamma(\emptyset) = 0$. Let Γ_k denote the family of closed symmetric subsets of X such that if $A \in \Gamma_k$, then $0 \notin A$ and $\gamma(A) \ge k$.

Our analysis mainly relies on the following lemma, which follows from [5, Theorem 1].

Lemma 3.4. *Let* X *be an infinite dimensional Banach space and* $I \in C^1(X, \mathbb{R})$ *satisfy the following two conditions:*

(A1) $I(u)$ *is even, bounded from below,* $I(0) = 0$, *and* $I(u)$ *satisfies the Palais-Smale (PS) condition, i.e., every sequence* $\{u_n\} \subset X$, *such that* $I(u_n)$ *is bounded and* $I'(u_n) \to 0$ *as* $n \to \infty$, *has a convergent subsequence. Here, the sequence* $\{u_n\}$ *is called a PS sequence.*

(A2) *For each* $n \in \mathbb{N}$, *there exists an* $A_n \in \Gamma_n$ *such that* $\sup_{u \in A_n} I(u) < 0$.

Then, $I(u)$ *has a sequence of critical points* u_n *such that*

$$I(u_n) \le 0, \quad u_n \ne 0, \quad \textit{and} \quad \lim_{n\to\infty} u_n = 0.$$

Lemma 3.5. *Assume that (H1), (H3), (H4), (C2), and (C3) hold. Then, the functional* I_λ *defined by* (2.5) *satisfies condition (A1) of Lemma* 3.4 *with* $I = I_\lambda$, *i.e.,* $I_\lambda(u)$ *is even, bounded from below,* $I_\lambda(0) = 0$, *and* $I_\lambda(u)$ *satisfies the PS condition.*

Proof. Obviously, $I_\lambda(0) = 0$ and $I_\lambda(u)$ is even by (C3). Under (H1)–(H4), the rest part of the lemma has been proved in [4, Proposition 9] when $a(k) \equiv 1$ on $\mathbb{Z}$. The proof there can be slightly modified to prove the present version of the lemma. We omit the details. □

Lemma 3.6. *Assume that (H1), (C1), and (C2) hold. Then, for each* $n \in \mathbb{N}$, *there exist* $H_n \in \Gamma_n$ *and* $\underline{\lambda} > 0$ *satisfying*

$$\sup_{u \in H_n} I_\lambda(u) < 0 \quad \textit{for all } \lambda > \underline{\lambda}.$$

Proof. For any fixed $n \in \mathbb{N}$, define

$$v_i(k) = \delta_{ik} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \ne k, \end{cases} \qquad \text{for } i = 1, \dots, n \text{ and } k \in \mathbb{Z},$$

and

$$F_n = \operatorname{span}\{v_i(k) \ : \ i = 1, \dots, n\}.$$

Then, $\dim F_n = n$. Let

$$S = \left\{u \in X \ : \ \|u\| = \rho b_0^{1/p}\right\}, \quad H_n = S \cap F_n, \quad \text{and} \quad H_\infty = \lim_{n\to\infty} H_n$$

where ρ is given in (C2). Then, by the property of genus (see, for example, [5, Lemma 2.6]), $\gamma(H_n) = n$. Let

$$\kappa = \inf_{n\in\mathbb{N}} \inf_{u\in H_n} \sum_{k=1}^{n} w_1(k)|u(k)|^p. \tag{3.3}$$

Since $H_n \subset H_{n+1}$ for any $n \in \mathbb{N}$, we have

$$\kappa = \inf_{u \in H_\infty} \sum_{k=1}^{\infty} w_1(k)|u(k)|^p.$$

We claim that $\kappa > 0$. In fact, assume, to the contrary, that $\kappa = 0$. Then for any $l \in \mathbb{N}$, there exists $u_l \in H_\infty$ such that

$$\sum_{k=1}^{\infty} w_1(k)|u_l(k)|^p < \frac{1}{l}. \tag{3.4}$$

Thus,

$$\lim_{l\to\infty} u_l(k) = 0 \quad \text{for } k \in \mathbb{Z}.$$

Then, from the fact that $\sum_{k=1}^{\infty} b(k)|u_l(k)|^p \leq \|u_l\|^p = \rho^p b_0 < \infty$, we see that there exists $L \in \mathbb{N}$ such that

$$\sum_{k=1}^{\infty} b(k)|u_l(k)|^p < \frac{\rho^p b_0}{(2^p M+1)} \quad \text{for } l \geq L. \tag{3.5}$$

By Minkowski's inequality and (C1), we have

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} a(k)|\Delta u_l(k-1)|^p &= \sum_{k\in\mathbb{Z}} a(k)|u_l(k) - u_l(k-1)|^p \\
&\leq \left[\left(\sum_{k\in\mathbb{Z}} a(k)|u_l(k)|^p\right)^{1/p} + \left(\sum_{k\in\mathbb{Z}} a(k)|u_l(k-1)|^p\right)^{1/p}\right]^p \\
&= 2^p \sum_{k=1}^{\infty} a(k)|u_l(k)|^p \\
&\leq 2^p M \sum_{k=1}^{\infty} b(k)|u_l(k)|^p.
\end{aligned}$$

Thus, from (2.2), we see that

$$\begin{aligned}
\rho^p b_0 = \|u_l\|^p &= \sum_{k\in\mathbb{Z}} \left[a(k)|\Delta u_l(k-1)|^p + b(k)|u_l(k)|^p\right] \\
&\leq (2^p M+1) \sum_{k=1}^{\infty} b(k)|u_l(k)|^p \quad \text{for } l \in \mathbb{N}.
\end{aligned}$$

This contradicts (3.5). Hence, the claim is true.

Now, for any $u \in H_n$, from (2.3), we have $\|u\|_\infty \leq b_0^{-1/p}\|u\| = \rho$. This, together with (C2) and (3.3), implies that

$$\begin{aligned}
\sup_{u\in H_n} I_\lambda(u) &= \sup_{u\in H_n} \left\{\frac{1}{p}\sum_{k\in\mathbb{Z}} \left[a(k)|\Delta u(k-1)|^p + b(k)|u(k)|^p\right] - \lambda \sum_{k\in\mathbb{Z}} F(k, u(k))\right\} \\
&\leq \sup_{u\in H_n} \left\{\frac{1}{p}\|u\|^p - \lambda \sum_{k\in\mathbb{Z}} w_1(k)|u(k)|^p\right\} \\
&= \sup_{u\in H_n} \left\{\frac{1}{p}\rho^p b_0 - \lambda \sum_{k=1}^{n} w_1(k)|u(k)|^p\right\} \\
&\leq \sup_{u\in H_n} \left\{\frac{1}{p}\rho^p b_0 - \lambda\kappa\right\}
\end{aligned} \tag{3.6}$$

Let $\underline{\lambda} = \rho^p b_0/(\kappa p)$. Then, (3.6) implies that

$$\sup_{u\in H_n} I_\lambda(u) < 0 \quad \text{for all } \lambda > \underline{\lambda}.$$

This completes the proof of the lemma. $\square$

Proof of Theorem 3.1. For $\underline{\lambda}$ given in Lemma 3.6, by Lemmas 3.5 and 3.6, conditions (A1) and (A2) of Lemma 3.4 with $I = I_\lambda$ are satisfied if $\lambda > \underline{\lambda}$. Hence, Lemma 3.4 and Lemma 2.3 (c) imply that for all $\lambda > \underline{\lambda}$, problem (1.1) has a sequence $\{u_n(k)\}$ of nontrivial solutions satisfying the required properties. This completes the proof the theorem. □

Proof of Corollary 3.2. With $f(k,t) = w(k)g(t)$ and $F(k,t)$ defined by (1.2), it is easy to see that (H3), (H4), (C2), and (C3) hold. Moreover, in (C2), we can let $w_1 = w_2 = w$ and $\rho = 1$, The conclusion then follows from Theorem 3.1. This completes the proof of the corollary. □

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