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EXISTENCE RESULT FOR PERIODIC BOUNDARY VALUE PROBLEM OF SET DIFFERENTIAL EQUATIONS USING MONOTONE ITERATIVE TECHNIQUE

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1. **Introduction**

The study of set differential equations(SDE)[1] is useful as it encompasses the study of scalar differential equations and vector differential equations as special cases and further this study is done in a semilinear metric space. The monotone iterative technique (MIT) [2] is a flexible mechanism to obtain monotone sequence that converge to the extremal solutions of the considered problem.

The study of periodic boundary value problems(PBVP) is complicated and more so in the case of SDEs, where the constraints are many. Hence the construction of MIT for PBVP for set differential equations has not been done till now. In [3] MIT for PBVP was developed using monotone sequences, which are solutions of the initial value problem [IVPs] of linear differential equations. These solutions are unique and hence the monotone sequences obtained are unique and they converge to a unique function which is shown to be a solution of the considered PBVP. The special advantage obtained with this approach is that working with IVPs of linear differential equations is easy and the uniqueness of the solution of the PBVP is guaranteed with no extra assumptions or effort.

In this paper, using the approach utilized in [3] we develop the MIT for PBVP for SDEs.

2. **Preliminaries**

We begin with the definition of $K_c(\mathbb{R}^n)$, the semilinear metric space in which we work.We next define the Hausdorff metric and Hukuhara difference and proceed to define the Hukuhara derivative and Hukuhara integral. Further we also state some

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important properties that are useful tools in our paper. We also define a partial order in $K_c(\mathbb{R}^n)$, see [1]

Let $K_c(\mathbb{R}^n)$ denote the collection of all nonempty, compact and convex subsets of $\mathbb{R}^n$. We define the Hausdorff metric by

$$D[A,B] = \max\left[\sup_{x\in B} d(x,A), \sup_{y\in A} d(y,B)\right], \tag{2.1}$$

where $d(x,A) = \inf[d(x,y) : y \in A]$, A and B are bounded sets in $\mathbb{R}^n$. We note that $K_c(\mathbb{R}^n)$ with this metric is a complete metric space.

It is known that if the space $K_c(\mathbb{R}^n)$ is equipped with the natural algebraic operations of addition and non-negative scalar multiplication, then $K_c(\mathbb{R}^n)$ becomes a semilinear metric space which can be embedded as a complete cone into a corresponding Banach space. The Hausdorff metric (2.1) satisfies the following properties:

$$D[A+C, B+C] = D[A,B] \text{ and } D[A,B] = D[B,A], \tag{2.2}$$

$$D[\lambda A, \lambda B] = \lambda D[A,B], \tag{2.3}$$

$$D[A,B] \leq D[A,C] + D[C,B], \tag{2.4}$$

for all $A, B, C \in K_c(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}_+$.

Let $A, B \in K_c(\mathbb{R}^n)$. The set $C \in K_c(\mathbb{R}^n)$ satisfying $A = B + C$ is known as the Hukuhara difference of the sets A and B and is denoted by the symbol $A - B$. We say that the mapping $F : I \to K_c(\mathbb{R}^n)$ has a Hukuhara derivative $D_H F(t_0)$ at a point $t_0 \in I$, if

$$\lim_{h\to 0+} \frac{F(t_0+h) - F(t_0)}{h} \text{ and } \lim_{h\to 0+} \frac{F(t_0) - F(t_0-h)}{h}$$

exist in the topology of $K_c(\mathbb{R}^n)$ and are equal to $D_H F(t_0)$. Here I is any interval in $\mathbb{R}$.

With these preliminaries, we consider the set differential equation

$$D_H U = F(t,U), \quad U(t_0) = U_0 \in K_c(\mathbb{R}^n), \quad t_0 \geq 0, \tag{2.5}$$

where $F \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$. The mapping $U \in C^1[J, K_c(\mathbb{R}^n)]$, $J = [t_0, t_0+a]$ is said to be a solution of (2.5) on J if it satisfies (2.5) on J. Since $U(t)$ is continuously differentiable, we have

$$U(t) = U_0 + \int_{t_0}^{t} D_H U(s)ds, \quad t \in J. \tag{2.6}$$

Hence, we can associate with the IVP (2.5) the Hukuhara integral

$$U(t) = U_0 + \int_{t_0}^{t} F(s, U(s))ds, \quad t \in J. \tag{2.7}$$

where the integral is the Hukuhara integral which is defined as,

$$\int F(s)ds = \left\{ \int f(s)ds : f \text{ is any continuous selector of } F \right\}$$

Observe also that U(t) is a solution of (2.5) on J iff it satisfies (2.7) on J.

We now proceed to define a partial order in the metric space $(K_c(\mathbb{R}^n), D)$. We begin with the definition of a cone in this set up.

Let $K(K^o)$ be the subfamily of $K_c(\mathbb{R}^n)$ consisting of set $U \in K_c(\mathbb{R}^n)$ such that any $u \in U$ is a non-negative (positive) vector of n components satisfying $u_i \geq 0$ $(u_i > 0)$ for $i = 1, \ldots, n$. Then K is a cone in $K_c(\mathbb{R}^n)$ and K^0 is the nonempty interior of K.

Definition 2.1. For any U and $V \in K_c(\mathbb{R}^n)$, if there exists $Z \in K_c(\mathbb{R}^n)$ such that $Z \in K(K^0)$ and $U = V + Z$ then we say that $U \geq V$ $(U > V)$. Similarly we can define $U \leq V$ $(U < V)$.

We state the following results from [1] to develop the MIT for the considered problem

Theorem 2.2. *Assume that*

(a) *Let* $V, W \in C^1[R_+, K_c(\mathbb{R}^n)]$, $F \in C[R_+ \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$, $F(t, X)$ *is monotone nondecreasing in* X *for each* $t \in R_+$ *and* $D_H V \leq F(t, V)$, $D_H W \geq F(t, W)$, $t \in R_+$*;*

(b) *for any* $X, Y \in K_c(\mathbb{R}^n)$ *such that* $X \geq Y$, $t \in R_+$, $F(t, X) \leq F(t, Y) + L(X - Y)$ *for some* $L > 0$*. Then* $V(t_0) \leq W(t_0)$ *implies* $V(t) \leq W(t)$, $t \geq t_0$.

Corollary 2.3. *Let* $V, W \in C^1[R_+, K_c(\mathbb{R}^n)]$, $\sigma \in C[R_+, K_c(\mathbb{R}^n)]$*. Suppose that* $D_H V \leq \sigma$, $D_H W \geq \sigma$, *for* $t \geq t_0$*. Then* $V(t) \leq W(t)$, $t \geq t_0$, *provided* $V(t_0) \leq W(t_0)$.

Theorem 2.4. *If* $\{U_n(t)\}$ *is a sequence of equicontinuous and equibounded multimappings defined on an interval* J*, we can extract a subsequence that converges uniformly to a continuous multimapping* $U(t)$ *on* J.

3. Monotone iterative technique

In this section, we develop MIT to obtain a solution for the PBVP for SDE given by

$$D_H U = F(t, U) + G(t, U), \quad U(0) = U(T), \tag{3.1}$$

where $F, G \in C[J \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$ and $J = [0, T]$.

We need the following definition which gives several possible notions of lower and upper solutions relative to (3.1).

Definition 3.1. Let $V, W \in C[J, K_c(\mathbb{R}^n)]$. Then V, W are said to be

(a) natural lower and upper solutions of (3.1) if

$$\left.\begin{aligned} D_H V &\leq F(t,V) + G(t,V), \quad V(0) \leq V(T) \\ D_H W &\geq F(t,W) + G(t,W), \; W(0) \geq W(T), \quad t \in J; \end{aligned}\right\} \tag{3.2}$$

(b) coupled lower and upper solutions of Type I of (3.1) if

$$\left.\begin{aligned} D_H V &\leq F(t,V) + G(t,W), \quad V(0) \leq V(T), \\ D_H W &\geq F(t,W) + G(t,V), \quad W(0) \geq W(T), \quad t \in J; \end{aligned}\right\} \tag{3.3}$$

(c) coupled lower and upper solutions of Type II of (3.1) if

$$\left.\begin{aligned} D_H V &\leq F(t,W) + G(t,V), \quad V(0) \leq V(T), \\ D_H W &\geq F(t,V) + G(t,W), \quad W(0) \geq W(T), \quad t \in J; \end{aligned}\right\} \tag{3.4}$$

(d) coupled lower and upper solutions of Type III of (3.1) if

$$\left.\begin{aligned} D_H V &\leq F(t,W) + G(t,W), \quad V(0) \leq V(T), \\ D_H W &\geq F(t,V) + G(t,V), \quad W(0) \geq W(T), \quad t \in J. \end{aligned}\right\} \tag{3.5}$$

We observe that whenever we have $V(t) \leq W(t)$, $t \in J$, if $F(t,X)$ is nondecreasing in X for each $t \in J$ and $G(t,Y)$ is nonincreasing in Y for each $t \in J$, the lower and upper solutions defined by (3.2) and (3.5) reduce to (3.4) and consequently, it is sufficient to investigate the cases (3.3) and (3.4).

We now proceed to use the notions developed above and develop the monotone iterative technique for the periodic boundary value problem. In this paper we use sequence of iterates which are solutions of IVPs for linear set differential equations. Since the solution of the linear SDE is unique, the sequence of iterates is a unique sequence converging to an extremal solution of the PBVP. In this approach, we do not need to prove the existence of the solutions of the PBVP for SDE, as it follows from the construction of the monotone sequences.

Theorem 3.2. *Assume that*

(A_1) $V, W \in C^1[J, K_c(\mathbb{R}^n)]$ *are coupled lower and upper solutions of Type I relative to* (3.1) *with* $V(t) \leq W(t)$, $t \in J$;

(A_2) $F, G \in C[J \times K_c(\mathbb{R}^n), \mathbb{R}^n]$, $F(t,X)$ *is nondecreasing in* X *for each* $t \in J$ *and* $G(t,Y)$ *is nonincreasing in* Y *for each* $t \in J$;

(A_3) F *and* G *map bounded sets into bounded sets in* $K_c(\mathbb{R}^n)$.

Then there exist monotone sequences $\{V_n\}$, $\{W_n\}$ *in* $K_c(\mathbb{R}^n)$ *such that* $V_n \to \rho$, $W_n \to R$ *in* $K_c(\mathbb{R}^n)$ *where* (ρ, R) *are the coupled minimal and maximal solutions of* (3.1), *that is, they satisfy*

$$D_H \rho = F(t,\rho) + G(t,R), \quad \rho(0) = \rho(T),$$

$$D_H R = F(t,R) + G(t,\rho), \quad R(0) = R(T).$$

Proof. For each $n \geq 0$,

$$D_H V_{n+1} = F(t, V_n) + G(t, W_n), \quad V_{n+1}(0) = V_n(T), \tag{3.6}$$

$$D_H W_{n+1} = F(t, W_n) + G(t, V_n), \quad W_{n+1}(0) = W_n(T), \tag{3.7}$$

where $V(0) \leq U(0) \leq W(0)$. We set $V_0 = V$, $W_0 = W$.

Our aim is to prove

$$V_0 \leq V_1 \leq \cdots \leq V_n \leq W_n \leq \cdots \leq W_1 \leq W_0, \quad t \in J. \tag{3.8}$$

Since V_o is the coupled lower solution of Type I of (3.1), we have, using the fact $V_0 \leq W_0$ and the nondecreasing character of F,

$$D_H V_0 \leq F(t, V_0) + G(t, W_0).$$

Also from (3.6) we get for $n = 0$,

$$D_H V_1 = F(t, V_0) + G(t, W_0), \tag{3.9}$$

$$V_1(0) = V_0(T). \tag{3.10}$$

Clearly equations (3.9), (3.10) have a unique solution denoted by $V_1(t)$, $t \in J$ Consequently following the proof of Theorem 2.2, we arrive at $V_0 \leq V_1$ on J. A similar argument shows that $W_1 \leq W_0$ on J. For the purpose of showing $V_1 \leq W_1$, consider (3.9), (3.10) along with

$$D_H W_1 = F(t, W_0) + G(t, V_0), \tag{3.11}$$

$$W_1(0) = W_0(T). \tag{3.12}$$

Then the monotone nature of F and G yield

$$D_H V_1 \leq F(t, W_0) + G(t, W_0), \quad D_H W_1 = F(t, W_0) + G(t, W_0), \quad t \in J,$$

and also $W_1(0) \geq V_1(0)$. By Corollary 2.3, we get $V_1 \leq W_1$ on J. Thus,

$$V_0 \leq V_1 \leq W_1 \leq W_0, \text{ on } J. \tag{3.13}$$

Assume that for $j \geq 1$,

$$V_{j-1} \leq V_j \leq W_j \leq W_{j-1}, \text{ on } J. \tag{3.14}$$

Then we will show that

$$V_j \leq V_{j+1} \leq W_{j+1} \leq W_j, \text{ on } J. \tag{3.15}$$

To do this consider,

$$D_H V_j = F(t, V_{j-1}) + G(t, W_{j-1}), \tag{3.16}$$

$$V_j(0) = V_{j-1}(T), \tag{3.17}$$

$$D_H V_{j+1} = F(t, V_j) + G(t, W_j), \tag{3.18}$$

$$V_{j+1}(0) = V_j(T). \tag{3.19}$$

The relations (3.14), (3.17) and (3.19) yield $V_j(0) \le V_{j+1}(0)$. Further,

$$D_H W_{j+1} \ge F(t, V_j) + G(t, W_j) \ge F(t, V_{j-1}) + G(t, W_{j-1}) \quad t \in J.$$

Here we employed (3.14), and the monotone nature of F and G. Applying Corollary 2.3, we get $V_j \le V_{j+1}$ on J. Similarly we get $W_{j+1} \le W_J$ on J. Next we show that $V_{j+1} \le W_{j+1}$, $t \in J$. We have from (3.6), (3.7)

$$D_H V_{j+1} = F(t, V_j) + G(t, W_j), \quad V_{j+1}(0) = V_j(T), \tag{3.20}$$

$$D_H W_{j+1} = F(t, W_j) + G(t, V_j), \quad W_{j+1}(0) = W_j(T). \tag{3.21}$$

Using (3.14) and monotone character of F and G, we arrive at

$$D_H V_{j+1} \le F(t, W_j) + G(t, W_j), \tag{3.22}$$

$$D_H W_{j+1} \ge F(t, W_j) + G(t, W_j). \tag{3.23}$$

Also $V_{j+1}(0) = V_j(T) \le W_j(T) = W_{j+1}(0)$, and therefore Corollary 2.3 yields that $V_{j+1} \le W_{j+1}$, $t \in J$. Hence (3.15) follows and consequently, by induction (3.15) is valid for all n. Clearly the sequences $\{V_n\}$,$\{W_n\}$ are uniformly bounded on J. To show that these sequences are equicontinuous, consider for any $s \ge t$, where $t, s \in J$,

$$\begin{aligned} D[V_n(t), V_n(s)] &= D\Big[U_0 + \int_0^t (F(\xi, V_{n-1}(\xi)) + G(\xi, W_{n-1}(\xi)))d\xi, \\ &\qquad U_0 + \int_0^s (F(\xi, V_{n-1}(\xi)) + G(\xi, W_{n-1}(\xi)))d\xi\Big] \\ &= D\Bigg[\int_0^t (F(\xi, V_{n-1}(\xi)) + G(\xi, W_{n-1}(\xi)))d\xi, \\ &\qquad \int_0^s (F(\xi, V_{n-1}(\xi)) + G(\xi, W_{n-1}(\xi)))d\xi\Bigg] \\ &\le \int_s^t D[(F(\xi, V_{n-1}(\xi)) + G(\xi, W_{n-1}(\xi)))d\xi, \theta]d\xi \le M|t-s|. \end{aligned}$$

Here we utilized the properties of integral and the metric D, together with the fact $F + G$ are bounded since $\{V_n\}$, $\{W_n\}$ are uniformly bounded. Hence $\{V_n(t)\}$ is equicontinuous on J. The corresponding Ascoli's theorem, Theorem 2.4, gives a subsequence $\{V_{n_k}\}$ which converges uniformly to $\rho(t) \in K_c(\mathbb{R}^n)$, $t \in J$, and since $\{V_n(t)\}$ is nondecreasing sequence, the entire sequence $\{V_n(t)\}$ converges uniformly to $\rho(t)$ on J.

Similar arguments apply to the sequence $\{W_n(t)\}$ and we obtain $W_n(t) \to R(t)$ uniformly on J. It therefore follows, using the integral representation of (3.6), (3.7) that $\rho(t)$, $R(t)$ satisfy

$$D_H \rho(t) = F(t, \rho(t)) + G(t, R(t)), \quad \rho(0) = \rho(T);$$

$$D_H R(t) = F(t, R(t)) + G(t, \rho(t)), \quad R(0) = R(T).$$

and that

$$V_0 \le \rho \le R \le W_0, \quad t \in J.$$

Next we claim that (ρ, R) are coupled minimal and maximal solutions of (3.1), that is $U(t)$ is any solution of (3.1) such that

$$V_0 \le U \le W_0, \quad t \in J. \tag{3.24}$$

then

$$V_0 \le \rho \le U \le R \le W_0, \quad t \in J. \tag{3.25}$$

Suppose that for some n,

$$V_n \le U \le W_n \text{ on } J. \tag{3.26}$$

Then we have using the monotone nature of F, G and (3.1)

$$D_H U = F(t, U) + G(t, U) \ge F(t, V_n) + G(t, W_n), \quad U(0) = U(T).$$

$$D_H V_{n+1} = F(t, V_n) + G(t, W_n), \quad V_{n+1}(0) = V_n(T).$$

Corollary 2.3 yields, $V_{n+1} \le U$ on J. Similarly $W_{n+1} \ge U$ on J. Hence by induction (3.26) is true for all $n \ge 1$. Now taking the limit as $n \to \infty$, we get (3.25), proving the claim. The proof is therefore complete. □

Corollary 3.3. *If, in addition to the assumptions of Theorem* 3.2, *F and G satisfy, whenever $X \ge Y$, $X, Y \in K_c(\mathbb{R}^n)$,*

$$F(t, X) \le F(t, Y) + N_1(X - Y)$$

and

$$G(t, X) + N_2(X - Y) \ge G(t, Y)$$

where $N_1, N_2 > 0$. Then $\rho = R = U$ is the unique solution of (3.1) .

Proof. Since $\rho \le R$, we have $R = \rho + m$ or $m = R - \rho$. Now

$$\begin{aligned} D_H\rho + D_H m = D_H R &= F(t, R) + G(t, \rho) \\ &\le F(t, \rho) + N_1(m) + G(t, R) + N_2(m) \\ &= D_H\rho + (N_1 + N_2)m \end{aligned}$$

which means

$$D_H m \le (N_1 + N_2)m, \quad m(0) = 0$$

which by Theorem 2 leads to $R \le \rho$ on J, proving the uniqueness of $\rho = R = U$, completing the proof. □

Remark 3.4. (1) In Theorem 3.2, if $G(t, Y) = 0$, then we get a result when F is non decreasing.

(2) If $F(t, X) = 0$ in Theorem 3.2, then we obtain the results for G non increasing.

It can be observed that the iterative scheme used in Theorem 3.2 is in tune with the assumed hypothesis of the existence of lower and upper solutions of Type I. The idea of using iterative scheme parallel to the lower and upper solutions of Type II was introduced in [4] and continued in the works of [3,5,6] for developing monotone iterative technique for different problems. In Theorem 3.2 we have used standard iterates while in the following theorem we propose to use iterates corresponding to lower and upper solutions of Type II, with the hypothesis that lower and upper solutions of Type I exist. This forces us to consider the initial conditions of the iterates of IVPs in the following theorem in a special way, as used in [7].

Theorem 3.5. *Let the hypothesis of Theorem* 3.2 *hold and* $U(t)$ *be any solution of SDE* (3.1) *with* $V_0 \leq U \leq W_0$ *on* J. *Then the iterative scheme given by*

$$D_H V_{n+1} = F(t, W_n) + G(t, V_n), \tag{3.27}$$

$$V_{n+1}(0) = W_n(T). \tag{3.28}$$

and

$$D_H W_{n+1}(t) = F(t, V_n) + G(t, W_n), \tag{3.29}$$

$$W_{n+1}(0) = V_n(T). \tag{3.30}$$

yield alternating sequences $\{V_{2n}, W_{2n+1}\}$ *converging to* ρ *and* $\{W_{2n}, V_{2n+1}\}$ *converging to* R *uniformly on* J *such that the relation*

$$V_0 \leq W_1 \leq \cdots \leq V_{2n} \leq W_{2n+1} \leq U \leq V_{2n+1} \leq W_{2n} \leq \cdots \leq V_1 \leq W_0. \tag{3.31}$$

holds on J. *Further* ρ *and* R *are coupled minimal and maximal solutions of Type II for the SDE* (3.1) *satisfying* $\rho \leq U \leq R$ *on* J.

Proof. Clearly the IVPs (3.27), (3.28), (3.29) and (3.30) have unique solutions for each $n = 0, 1, 2, \ldots$ denoted by V_{n+1}(t) and W_{n+1}(t) respectively. Setting $n = 0$ in the iterative scheme we obtain that V_1 and W_1 are solutions of the IVPs for SDEs given by

$$D_H V_1 = F(t, W_0) + G(t, V_0), \tag{3.32}$$

$$V_1(0) = W_0(T), \tag{3.33}$$

and

$$D_H W_1(t) = F(t, V_0) + G(t, W_0), \tag{3.34}$$

$$W_1(0) = V_0(T). \tag{3.35}$$

Also since V_0 and W_0 are lower and upper solutions of Type I, we have

$$D_H V_0(t) \leq F(t, V_0) + G(t, W_0), \tag{3.36}$$

$$V_0(0) \leq V_0(T), \tag{3.37}$$

and

$$D_H W_0(t) \geq F(t, W_0) + G(t, V_0), \tag{3.38}$$

$$W_0(0) \geq W_0(T). \tag{3.39}$$

Further,

$$W_0(0) \geq W_0(T) = V_1(0). \tag{3.40}$$

0n applying Corollary 2.3 to the relations (3.32), (3.38), (3.40) we arrive at

$$V_1 \leq W_0 \text{ on } J. \tag{3.41}$$

The relations (3.35) and (3.37) yield that

$$V_0(0) \leq W_1(0). \tag{3.42}$$

Again using Corollary 2.3 on the relations (3.34), (3.36) and (3.42) we get

$$V_0 \leq W_1 \text{ on } J. \tag{3.43}$$

We claim that $W_1 \leq U \leq V_1$ on J and the proof is as follows. Since U is a solution of the SDE (3.1),

$$D_H U = F(t, U) + G(t, U), \tag{3.44}$$

$$U(0) = U(T), \tag{3.45}$$

and from hypothesis $V_0 \leq U \leq W_0$ on J. Since F and G are monotone,

$$D_H U \geq F(t, W_0) + G(t, V_0), \tag{3.46}$$

Also,

$$U(0) = U(T) \leq W_0(T) = V_1(0). \tag{3.47}$$

The relations (3.32), (3.46) and (3.47), on applying Corollary 2.3, give $U \leq V_1$ on J. Similarly, we can show that $W_1 \leq U$ on J. Thus, $V_0 \leq W_1 \leq U \leq V_1 \leq W_0$ on J. We now proceed to prove that $V_0 \leq W_1 \leq V_2 \leq W_3 \leq U$ and $U \leq V_3 \leq W_2 \leq V_1 \leq W_0$ on J. To do this, set $n = 1$ in (3.27), then

$$D_H V_2 = F(t, W_1) + G(t, V_1), \quad V_2(0) = W_1(T). \tag{3.48}$$

Using the monotone nature of F and G in (3.48) gives

$$D_H V_2 \geq F(t, V_0) + G(t, W_0), \tag{3.49}$$

and

$$V_2(0) = W_1(T) \geq V_0(T) = W_1(0). \tag{3.50}$$

Now the relations (3.34), (3.49) and (3.50) together with Corollary 2.3 yield $W_1 \leq V_2$ on J. Working in a similar fashion we arrive at

$$D_H W_2 \leq F(t, V_0) + G(t, W_0), \tag{3.51}$$

and

$$W_2(0) = V_1(0). \tag{3.52}$$

Applying Corollary 2.3 to the relations (3.32), (3.51) and (3.52) leads to $W_2 \leq V_1$ on J. To prove $V_2 \leq W_3$, set $n = 1$ in (3.27), (3.28) and $n = 2$ in (3.29), (3.30) then

$$\begin{aligned} iD_H V_2 &= F(t, W_1) + G(t, V_1), \\ V_2(0) &= W_1(T). \end{aligned}$$

and

$$\begin{aligned} D_H W_3 &= F(t, V_2) + G(t, W_2), \\ W_2(0) &= V_2(T). \end{aligned}$$

Since $W_1 \leq V_2$ and $W_2 \leq V_1$ on J, using the monotone nature of F and G gives

$$D_H W_1 \geq F(t, W_1) + G(t, V_1),$$

and using the fact that

$$W_3(0) = V_2(T) \geq W_1(T) = V_2(0)$$

we conclude, by Corollary 2.3, that $V_2 \leq W_3$ on Jr. Working as earlier, it can be easily shown that $W_3 \leq U \leq V_3$ on J. Now assume that the relation (3.31) holds for some integer $n = k$ such that

$$W_{2k-1} \leq V_{2k} \leq W_{2k+1} \leq U \leq V_{2k+1} \leq W_{2k} \leq V_{2k-1}. \tag{3.53}$$

To apply mathematical induction we need to prove that

$$W_{2k+1} \leq V_{2k+2} \leq W_{2k+3} \leq U \leq V_{2k+3} \leq W_{2k+2} \leq V_{2k+1} \text{ on } J. \tag{3.54}$$

For this, set $n = 2k + 1$ in (3.27), (3.28) and $n = 2k$ in (3.29), (3.30). Then,

$$D_H V_{2k+2} = F(t, W_{2k+1}) + G(t, V_{2k+1}), \quad V_{2k+2}(0) = W_{2k+1}(T) \tag{3.55}$$

and

$$D_H W_{2k+1} = F(t, V_{2k}) + G(t, W_{2k}), \;\; W_{2k+1}(0) = V_{2k}(T). \tag{3.56}$$

By the monotone nature of F and G and since $V_{2k} \leq W_{2k+1}$, $V_{2k+1} \leq W_{2k}$ we get,

$$D_H V_{2k+2} \geq F(t, V_{2k}) + G(t, W_{2k}), \tag{3.57}$$

and

$$V_{2k+2}(0) = W_{2k+1}(T) \geq V_{2k}(T) = W_{2k+1}(0). \tag{3.58}$$

From the relations (3.56), (3.57) and (3.58), on applying Corollary 2.3, we obtain $V_{2k} \leq W_{2k+1}$ on J. Similarly, $V_{2k+2} \leq W_{2k+3}$, $W_{2k+2} \leq V_{2k+1}$ and $V_{2k+3} \leq W_{2k+2}$ all hold on J.

To show that $V_{2k+2} \leq U$. The fact that U is a solution of (3.1) and from (3.53), using the monotone character of F and G, we arrive at,

$$D_H U = F(t, W_{2k+1}) + G(t, V_{2k+1})$$
$$U(0) = U(T) \geq W_{2k+1}(T) = V_{2k+2}(0).$$

Applying Corollary 2.3, to the above relations along with relation (3.55) gives $V_{2k+2} \leq U$ on J. Working as in the above case, we can show that $U \leq W_{2k+2}$, $W_{2k+3} \leq U$, $U \leq V_{2k+3}$ and $V_{2k+2} \leq U$ on J. Thus we are in a position to apply mathematical induction and claim that the relation (3.31) holds. Working as in Theorem 3.2, we can show that the sequences $\{V_{2n}\}$, $\{V_{2n+1}\}$, $\{W_{2n}\}$, $\{W_{2n+1}\}$ are equicontinuous and uniformly bounded. Thus from Theorem 2.4, which is the Arzela-Ascoli theorem for sequences of set functions, and the monotone nature of the sequences, we conclude that they are uniformly convergent and that $V_{2n} \to \rho, W_{2n+1} \to \rho$ and $W_{2n} \to R$ and $V_{2n+1} \to R$ as $n \to \infty$.

The proof is complete if we show that ρ and R are coupled minimal and maximal solutions of the SDE (3.1). This follows by considering the corresponding Hukuhara integral and using the properties of uniform continuity of F and G and uniform convergence of the sequences $\{V_{2n}\}$, $\{W_{2n+1}\}$, and $\{V_{2n+1}\}$, $\{W_{2n}\}$. As the details are routine, we omit them and the proof of the theorem is complete. □

Remark 3.6. A close analysis of the above two theorems suggests that if we propose to develop the MIT for the PBVP of set differential equations using coupled lower and upper solutions of Type II, then we can do so by considering two different types of iterates of the linear IVPs used in Theorem 3.2 and Theorem 3.5.

It can be observed that if we use the iterates in Theorem 3.2, in association with coupled lower and upper solutions of Type II then the initial values must be of the type $V_{n+1}(0) = V_n(T)$ and $W_{n+1}(0) = W_n(T)$. But if use the iterates in Theorem 3.5, along with the coupled lower and upper solutions of Type II, we need to take the initial conditions as $V_{n+1}(0) = W_n(T)$ and $W_{n+1}(0) = V_n(T)$. With this observation we state the following two results and omit the proof as they are very similar to the earlier theorems.

Theorem 3.7. *Assume that hypotheses (A2) and (A3) of Theorem 3.2 hold and V_0, W_0 are coupled lower and upper solutions of Type II with $V_0(t) \leq W(t)$. Further, for every $n \geq 1$, let the iterates be given by*

$$D_H V_{n+1} = F(t, V_n) + G(t, W_n),$$
$$V_{n+1}(0) = V_n(T),$$
$$D_H W_{n+1} = F(t, W_n) + G(t, V_n),$$
$$W_{n+1}(0) = W_n(T).$$

Then there exist monotone sequences $\{V_n\}$ *and* $\{W_n\}$ *such that* $V_0 \leq V_1 \leq \cdots \leq V_n \leq W_n \leq \cdots \leq W_1 \leq W_0$ *on* J*, which converge uniformly to* ρ *and* R *respectively, where* ρ *and* R *are coupled minimal and maximal solutions of SDE* (3.1)*.*

Theorem 3.8. *Assume that the assumptions (A2) and (A3) of Theorem* 3.2 *hold. Further, let* V_0, W_0 *be coupled lower and upper solutions of Type II for the SDE* (3.1) *such that* $V_0(t) \leq W_0(t)$ *for* $t \in J$*. Then for any solution* U *of SDE* (3.1) *with* $V_0(t) \leq U(t) \leq W_0(t)$*,* $t \in J$*, there exist alternating sequences* $\{V_{2n}\}, \{W_{2n+1}\}$*,* $\{V_{2n+1}\}, \{W_{2n}\}$ *satisfying* $V_0 \leq W_1 \leq \cdots \leq W_{2n+1} \leq U \leq V_{2n+1} \leq \cdots \leq V_1 \leq W_0$ *on* J*, for every* $n \geq 1$*, where* $V_{2n} \to \rho$*,* $W_{2n+1} \to \rho$ *and* $W_{2n} \to R$*, and* $W_{2n+1} \to R$*. The iterative schemes are given by the relations* (3.7) *and* (3.8)*.*

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