Communications in Applied Analysis **19** (2015), 329–342

MULTIPLE POSITIVE SOLUTIONS FOR TWO-POINT BOUNDARY VALUE PROBLEMS ASSOCIATED WITH TERNARY SYSTEM OF DIFFERENTIAL EQUATIONS

K. R. PRASAD[1], K. R. KUMAR[2], AND N. SREEDHAR[3]

[1]Department of Applied Mathematics, Andhra University
Visakhapatnam, 530 003, India
E-mail: rajendra92@rediffmail.com

[2]Department of Mathematics, VITAM College of Engineering
Visakhapatnam, 531 173, India
E-mail: rkkona72@rediffmail.com

[3]Department of Mathematics, GITAM University
Visakhapatnam, 530 045, India
E-mail: sreedharnamburi@rediffmail.com

ABSTRACT. This paper establishes the existence of multiple positive solutions for ternary system of higher order two-point boundary value problems by using five functionals fixed point theorem. We also establish the existence of at least $2k-1$ positive solutions to the boundary value problem for an arbitrary positive integer k.

AMS (MOS) Subject Classification. 34B15, 39A10, 34A40.

1. INTRODUCTION

There has been much attention focused in establishing the existence of positive solutions for higher order boundary value problems (BVPs) due to their applicability for all areas of science, engineering and technology. The concept that arises in industries like automobile, aerospace, chemical, pharmaceutical, petroleum, electronics and communications as well as emerging technologies such as nanotechnology and biotechnology can be modeled as BVPs. In these applied settings, the positive solutions are meaningful. In recent years, researchers have shown interest in establishing the existence of positive solutions for systems of BVPs. To mention a few along these lines are Cheng and Zhang [3], Henderson and Ntouyas [7]–[10], Liu, Kang and Wu [13], Xu and Yang [17]. Recently, Prasad, Murali and Rao [16] studied the existence of multiple positive solutions for the system of higher order two-point boundary value problems.

Received May 16, 2014

Till now in the literature, the study of positive solutions for system of two equations are available. We wish to extend these results to the systems of three equations,

$$\left.\begin{array}{l} y_1^{(m)}(t) + f_1(t, y_1(t), y_2(t), y_3(t)) = 0,\ t \in [a, b], \\ y_2^{(n)}(t) + f_2(t, y_1(t), y_2(t), y_3(t)) = 0,\ t \in [a, b], \\ y_3^{(l)}(t) + f_3(t, y_1(t), y_2(t), y_3(t)) = 0,\ t \in [a, b], \end{array}\right\} \tag{1.1}$$

satisfying two-point boundary conditions,

$$\left.\begin{array}{l} y_1^{(i)}(a) = 0,\ i = 0, 1, 2, \ldots, m-2, \\ y_1^{(p)}(b) = 0,\ (\ 1 \le p \le m-1,\ \text{but fixed}), \\ \\ y_2^{(j)}(a) = 0,\ j = 0, 1, 2, \ldots, n-2, \\ y_2^{(q)}(b) = 0,\ (\ 1 \le q \le n-1,\ \text{but fixed}), \\ \\ y_3^{(k)}(a) = 0,\ k = 0, 1, 2, \ldots, l-2, \\ y_3^{(r)}(b) = 0,\ (\ 1 \le r \le l-1,\ \text{but fixed}) \end{array}\right\} \tag{1.2}$$

where $m, n, l \ge 2$, $b > a \ge 0$ and $f_i : [a, b] \times \mathbb{R}^3 \to \mathbb{R}^+$ are continuous, for $i = 1, 2, 3$. By applying five functionals fixed point theorem for the BVP (1.1)–(1.2), we establish the existence of multiple positive solutions. This theorem generalizes the fixed point theorem of cone expansion and compression which is of norm type and it allows to choose functionals that satisfy certain conditions which are used in place of norm. In applications to BVPs the functionals will typically be the minimum or maximum of the function over a specific interval.

The rest of the paper is organized as follows. In section 2, we construct the Green's function for the homogeneous problem corresponding to BVP (1.1)–(1.2) and estimate bounds for the Green's function. In section 3, we establish criteria for the existence of at least three positive solutions for the BVP (1.1)–(1.2), by using five functionals fixed point theorem. We also establish the existence of at least $2k-1$ positive solutions to the BVP (1.1)–(1.2) for an arbitrary positive integer k. As an application, we give an example to obtain at least three positive solutions to the BVP.

2. GREEN'S FUNCTION AND BOUNDS

In this section, we construct the Green's function for the homogeneous BVP corresponding to (1.1)–(1.2) using Cauchy function concept and we estimate bounds for the Green's function.

Let $G_{m_1}(t, s)$ be the Green's function for the homogeneous BVP,

$$-y^{(m_1)}(t) = 0,\ t \in [a, b], \tag{2.1}$$

$$\left.\begin{array}{l} y^{(i_1)}(a) = 0,\ i_1 = 0, 1, 2, \ldots, m_1 - 2, \\ y^{(j_1)}(b) = 0,\ (1 \le j_1 \le m_1 - 1,\ \text{but fixed}) \end{array}\right\} \tag{2.2}$$

and after the computation it is given by

$$G_{m_1}(t,s) = \begin{cases} H_1(t,s), & a \le t \le s \le b, \\ H_2(t,s), & a \le s \le t \le b, \end{cases}$$

where

$$H_1(t,s) = \frac{(t-a)^{m_1-1}(b-s)^{m_1-j_1-1}}{(m_1-1)!(b-a)^{m_1-j_1-1}},$$

$$H_2(t,s) = \frac{(t-a)^{m_1-1}(b-s)^{m_1-j_1-1}}{(m_1-1)!(b-a)^{m_1-j_1-1}} - \frac{(t-s)^{m_1-1}}{(m_1-1)!}.$$

It is clear that the Green's function $G_{m_1}(t,s) > 0$, for all $(t,s) \in (a,b) \times (a,b)$ by simple algebraic calculations.

Lemma 2.1. *For $(t,s) \in [a,b] \times [a,b]$, we have*

$$G_{m_1}(t,s) \le G_{m_1}(b,s). \tag{2.3}$$

Proof. Let $a \le t \le s \le b$. Then, we have

$$\frac{\partial G_{m_1}(t,s)}{\partial t} = \frac{(t-a)^{m_1-2}(b-s)^{m_1-j_1-1}}{(m_1-2)!(b-a)^{m_1-j_1-1}} \ge 0.$$

Therefore, the Green's function $G_{m_1}(t,s)$ is increasing in t. Hence the inequality (2.3).

Let $a \le s \le t \le b$. Then, we have

$$\begin{aligned} \frac{\partial G_{m_1}(t,s)}{\partial t} &= \frac{(t-a)^{m_1-2}(b-s)^{m_1-j_1-1}}{(m_1-2)!(b-a)^{m_1-j_1-1}} - \frac{(t-s)^{m_1-2}}{(m_1-2)!} \\ &= \frac{1}{(m_1-2)!(b-a)^{m_1-j_1-1}}\Big[(t-a)^{m_1-2}(b-s)^{m_1-j_1-1} \\ &\qquad\qquad - (t-s)^{m_1-2}(b-a)^{m_1-j_1-1}\Big] \\ &= \frac{(t-a)^{m_1-j_1-1}(b-s)^{m_1-j_1-1}}{(m_1-2)!(b-a)^{m_1-j_1-1}}\Bigg[(t-a)^{j_1-1} \\ &\qquad\qquad - \frac{(t-s)^{m_1-2}(b-a)^{m_1-j_1-1}}{(t-a)^{m_1-j_1-1}(b-s)^{m_1-j_1-1}}\Bigg] \\ &\ge \frac{(t-a)^{m_1-j_1-1}(b-s)^{m_1-j_1-1}}{(m_1-2)!(b-a)^{m_1-j_1-1}}\Big[(t-a)^{j_1-1} - (t-s)^{j_1-1}\Big] \ge 0. \end{aligned}$$

Therefore, the Green's function $G_{m_1}(t,s)$ is increasing in t. Hence the inequality (2.3). □

Lemma 2.2. *Let $I = \left[\frac{3a+b}{4}, \frac{a+3b}{4}\right]$. For $(t,s) \in I \times [a,b]$, we have*

$$G_{m_1}(t,s) \ge \frac{1}{4^{m_1-1}} G_{m_1}(b,s). \tag{2.4}$$

Proof. Let $a \le t \le s \le b$ and $t \in I$. Then, we have

$$\frac{G_{m_1}(t,s)}{G_{m_1}(b,s)} \ge \left(\frac{t-a}{b-a}\right)^{m_1-1} \ge \frac{1}{4^{m_1-1}}.$$

Let $a \le s \le t \le b$ and $t \in I$. Then, we have

$$\begin{aligned}
\frac{G_{m_1}(t,s)}{G_{m_1}(b,s)} &= \frac{(t-a)^{m_1-1}(b-s)^{m_1-j_1-1}-(t-s)^{m_1-1}(b-a)^{m_1-j_1-1}}{(b-a)^{m_1-1}(b-s)^{m_1-j_1-1}-(b-s)^{m_1-1}(b-a)^{m_1-j_1-1}}\\
&= \frac{(t-a)^{m_1-j_1-1}(b-s)^{m_1-j_1-1}\left[(t-a)^{j_1}-\frac{(t-s)^{m_1-1}(b-a)^{m_1-j_1-1}}{(t-a)^{m_1-j_1-1}(b-s)^{m_1-j_1-1}}\right]}{(b-a)^{m_1-j_1-1}(b-s)^{m_1-j_1-1}[(b-a)^{j_1}-(b-s)^{j_1}]}\\
&\ge \frac{(t-a)^{m_1-j_1-1}[(t-a)^{j_1}-(t-s)^{j_1}]}{(b-a)^{m_1-j_1-1}[(b-a)^{j_1}-(b-s)^{j_1}]}\\
&\ge \left(\frac{t-a}{b-a}\right)^{m_1-1}\\
&\ge \frac{1}{4^{m_1-1}}.
\end{aligned}$$

Hence the result. □

3. MULTIPLE POSITIVE SOLUTIONS

In this section, we establish the existence of at least three positive solutions for the BVP (1.1)–(1.2), by using five functionals fixed point theorem. And then, we establish the existence of at least $2k-1$ positive solutions for an arbitrary positive integer k.

Let B be a real Banach space with cone P. A map $\alpha : P \to [0,\infty)$ is said to be nonnegative continuous concave functional on P if α is continuous and

$$\alpha(\lambda x + (1-\lambda)y) \ge \lambda\alpha(x) + (1-\lambda)\alpha(y),$$

for all $x, y \in P$ and $\lambda \in [0,1]$. Similarly, we say that a map $\beta : P \to [0,\infty)$ is said to be nonnegative continuous convex functional on P if β is continuous and

$$\beta(\lambda x + (1-\lambda)y) \le \lambda\beta(x) + (1-\lambda)\beta(y),$$

for all $x, y \in P$ and $\lambda \in [0,1]$.

Let γ, β, θ be nonnegative continuous convex functionals on P and α, ψ be nonnegative continuous concave functionals on P, then for nonnegative numbers h', a', b', d' and c', we define the following convex sets.

$$\begin{aligned}
P(\gamma, c') &= \{y \in P : \gamma(y) < c'\},\\
P(\gamma, \alpha, a', c') &= \{y \in P : a' \le \alpha(y); \gamma(y) \le c'\},\\
Q(\gamma, \beta, d', c') &= \{y \in P : \beta(y) \le d'; \gamma(y) \le c'\},\\
P(\gamma, \theta, \alpha, a', b', c') &= \{y \in P : a' \le \alpha(y); \theta(y) \le b'; \gamma(y) \le c'\},\\
Q(\gamma, \beta, \psi, h', d', c') &= \{y \in P : h' \le \psi(y); \beta(y) \le d'; \gamma(y) \le c'\}.
\end{aligned}$$

In obtaining multiple positive solutions of the BVP (1.1)–(1.2), the following so called five functionals fixed point theorem will be fundamental.

Theorem 3.1 ([2]). *Let P be a cone in the real Banach space B. Suppose α and ψ are nonnegative continuous concave functionals on P, and γ, β, θ are nonnegative continuous convex functionals on P, such that for some positive numbers c' and e', $\alpha(y) \leq \beta(y)$ and $\|y\| \leq e'\gamma(y)$ i for all y in $\overline{P(\gamma, c')}$. Suppose further that $T : \overline{P(\gamma, c')} \to \overline{P(\gamma, c')}$ is completely continuous and there exist constants h', d', a' and $b' \geq 0$ with $0 < d' < a'$ such that each of the following is satisfied.*

(B1) $\{y \in P(\gamma, \theta, \alpha, a', b', c') : \alpha(y) > a'\} \neq \emptyset$ *and* $\alpha(Ty) > a'$ *for* $y \in P(\gamma, \theta, \alpha, a', b', c')$,
(B2) $\{y \in Q(\gamma, \beta, \psi, h', d', c') : \beta(y) < d'\} \neq \emptyset$ *and* $\beta(Ty) < d'$ *for* $y \in Q(\gamma, \beta, \psi, h', d', c')$,
(B3) $\alpha(Ty) > a'$ *provided* $y \in P(\gamma, \alpha, a', c')$ *with* $\theta(Ty) > b'$,
(B4) $\beta(Ty) < d'$ *provided* $y \in Q(\gamma, \beta, d', c')$ *with* $\psi(Ty) < h'$.

Then T has at least three fixed points $y_1, y_2, y_3 \in \overline{P(\gamma, c')}$ such that $\beta(y_1) < d'$, $a' < \alpha(y_2)$ and $d' < \beta(y_3)$ with $\alpha(y_3) < a'$.

Let $B = E \times E \times E$, where $E = \{y : y \in C^{(p)}[a, b]\}$, $p = \max\{m, n, l\}$, be the Banach space equipped with the norm $\|(y_1, y_2, y_3)\| = \|y_1\|_0 + \|y_2\|_0 + \|y_3\|_0$ and

$$\|y_i\|_0 = \max_{t\in[a,b]} |y_i(t)|, \text{ for } i = 1, 2, 3.$$

Let

$$\eta = \min\left\{\frac{1}{4^{m-1}}, \frac{1}{4^{n-1}}, \frac{1}{4^{l-1}}\right\}. \tag{3.1}$$

Define the cone $P \subset B$ by

$$P = \left\{ \begin{array}{c} (y_1, y_2, y_3) \in B,\ y_i(t) \geq 0,\ t \in [a, b],\ i = 1, 2, 3 \text{ and} \\ \min_{t\in I} \left\{\sum_{i=1}^{3} |y_i(t)|\right\} \geq \eta\|(y_1, y_2, y_3)\| \end{array} \right\}.$$

Let $I_1 = \left[\frac{2a+b}{3}, \frac{a+2b}{3}\right]$ and define the nonnegative continuous concave functionals α, ψ and the nonnegative continuous convex functionals β, θ, γ on P by

$$\alpha(y_1, y_2, y_3) = \min_{t\in I}\left\{\sum_{i=1}^{3} |y_i(t)|\right\},\ \psi(y_1, y_2, y_3) = \min_{t\in I_1}\left\{\sum_{i=1}^{3} |y_i(t)|\right\},$$

$$\gamma(y_1, y_2, y_3) = \max_{t\in[a,b]}\left\{\sum_{i=1}^{3} |y_i(t)|\right\},\ \beta(y_1, y_2, y_3) = \max_{t\in I_1}\left\{\sum_{i=1}^{3} |y_i(t)|\right\},$$

and

$$\theta(y_1, y_2, y_3) = \max_{t\in I}\left\{\sum_{i=1}^{3} |y_i(t)|\right\}.$$

We observe that, for any $(y_1, y_2, y_3) \in P$, we have

$$\alpha(y_1, y_2, y_3) \leq \beta(y_1, y_2, y_3), \tag{3.2}$$

and

$$\|(y_1, y_2, y_3)\| \le \frac{1}{\eta}\gamma(y_1, y_2, y_3). \tag{3.3}$$

Let

$$L = \min\left\{\int_a^b G_m(b,s)ds,\ \int_a^b G_n(b,s)ds,\ \int_a^b G_l(b,s)ds\right\}$$

and

$$M = \max\left\{\int_a^b G_m(b,s)ds,\ \int_a^b G_n(b,s)ds,\ \int_a^b G_l(b,s)ds\right\}.$$

We denote the operators $T_m : P \to E$, $T_n : P \to E$, $T_l : P \to E$, and defined by

$$T_m(y_1, y_2, y_3)(t) = \int_a^b G_m(t,s) f_1(s, y_1(s), y_2(s), y_3(s))ds,$$

$$T_n(y_1, y_2, y_3)(t) = \int_a^b G_n(t,s) f_2(s, y_1(s), y_2(s), y_3(s))ds,$$

$$T_l(y_1, y_2, y_3)(t) = \int_a^b G_l(t,s) f_3(s, y_1(s), y_2(s), y_3(s))ds.$$

Theorem 3.2. *Suppose there exist* $0 < a' < b' < \frac{b'}{\eta} < c'$ *such that* f_i, *satisfies the following conditions for* $i = 1, 2, 3,$

$$(A1)\ f_i(t, y_1, y_2, y_3) < \frac{a'}{3M},\ t \in [a,b] \text{ and } \sum_{i=1}^{3} |y_i| \in [\eta a', a'],$$

$$(A2)\ f_i(t, y_1, y_2, y_3) > \frac{b'}{3\eta L},\ t \in I \text{ and } \sum_{i=1}^{3} |y_i| \in \left[b', \frac{b'}{\eta}\right],$$

$$(A3)\ f_i(t, y_1, y_2, y_3) < \frac{c'}{3M},\ t \in [a,b] \text{ and } \sum_{i=1}^{3} |y_i| \in [0, c'].$$

Then the BVP (1.1)–(1.2) *has at least three positive solutions* (u_1, u_2, u_3),(v_1, v_2, v_3) *and* (w_1, w_2, w_3) *such that* $\beta(u_1, u_2, u_3) < a'$, $b' < \alpha(v_1, v_2, v_3)$ *and* $a' < \beta(w_1, w_2, w_3)$ *with* $\alpha(w_1, w_2, w_3) < b'$.

Proof. Define the operator $T : P \to B$ by

$$T(y_1, y_2, y_3)(t) = (T_m(y_1, y_2, y_3)(t), T_n(y_1, y_2, y_3)(t), T_l(y_1, y_2, y_3)(t)).$$

It is obvious that the fixed point of T is the solution of the BVP (1.1)–(1.2). We seek three fixed points of T. First, we show that $T : P \to P$. Let $(y_1, y_2, y_3) \in P$. Clearly, $T_m(y_1, y_2, y_3) \ge 0, T_n(y_1, y_2, y_3) \ge 0$ and $T_l(y_1, y_2, y_3) \ge 0$, for $t \in [a,b]$. Also, for $(y_1, y_2, y_3) \in P$,

$$\begin{aligned} T_m(y_1, y_2, y_3)(t) &= \int_a^b G_m(t,s) f_1(s, y_1(s), y_2(s), y_3(s))ds \\ &\le \int_a^b G_m(b,s) f_1(s, y_1(s), y_2(s), y_3(s))ds \end{aligned}$$

so that

$$\|T_m(y_1, y_2, y_3)\|_0 \le \int_a^b G_m(b, s) f_1(s, y_1(s), y_2(s), y_3(s)) ds.$$

Next, if $(y_1, y_2, y_3) \in P$, then by Lemma 2.2, we have

$$\begin{aligned}\min_{t\in I} T_m(y_1, y_2, y_3)(t) &= \min_{t\in I} \int_a^b G_m(t, s) f_1(s, y_1(s), y_2(s), y_3(s)) ds \\ &\ge \eta \int_a^b G_m(b, s) f_1(s, y_1(s), y_2(s), y_3(s)) ds \\ &\ge \eta \|T_m(y_1, y_2, y_3)\|_0.\end{aligned}$$

Similarly,

$$\min_{t\in I} T_n(y_1, y_2, y_3)(t) \ge \eta \|T_n(y_1, y_2, y_3)\|_0$$

and

$$\min_{t\in I} T_l(y_1, y_2, y_3)(t) \ge \eta \|T_l(y_1, y_2, y_3)\|_0.$$

Therefore,

$$\begin{aligned}&\min_{t\in I}\{T_m(y_1, y_2, y_3)(t) + T_n(y_1, y_2, y_3)(t) + T_l(y_1, y_2, y_3)(t)\} \\ &\qquad \ge \eta\|T_m(y_1, y_2, y_3)\|_0 + \eta\|T_n(y_1, y_2, y_3)\|_0 + \eta\|T_l(y_1, y_2, y_3)\|_0 \\ &\qquad = \eta\|(T_m(y_1, y_2, y_3), T_n(y_1, y_2, y_3), T_l(y_1, y_2, y_3))\| \\ &\qquad = \eta\|T(y_1, y_2, y_3)\|.\end{aligned}$$

Hence, $T(y_1, y_2, y_3) \in P$ and so $T : P \to P$. Moreover, T is completely continuous operator. From (3.2) and (3.3), for each $(y_1, y_2, y_3) \in P$, we have $\alpha(y_1, y_2, y_3) \le \beta(y_1, y_2, y_3)$ and $\|(y_1, y_2, y_3)\| \le \frac{1}{\eta}\gamma(y_1, y_2, y_3)$. To show that $T : \overline{P(\gamma, c')} \to \overline{P(\gamma, c')}$. Let $(y_1, y_2, y_3) \in \overline{P(\gamma, c')}$. Then $0 \le \sum_{i=1}^{3} |y_i(t)| \le c'$. We may use condition $(A3)$ to obtain

$$\begin{aligned}\gamma(T(y_1, y_2, y_3)(t)) = \max_{t\in[a,b]}\Big\{ &\int_a^b G_m(t, s) f_1(s, y_1(s), y_2(s), y_3(s)) ds \\ &+ \int_a^b G_n(t, s) f_2(s, y_1(s), y_2(s), y_3(s)) ds \\ &+ \int_a^b G_l(t, s) f_3(s, y_1(s), y_2(s), y_3(s)) ds\Big\} \\ < \frac{c'}{3M}\int_a^b G_m(b, s) ds &+ \frac{c'}{3M}\int_a^b G_n(b, s) ds + \frac{c'}{3M}\int_a^b G_l(b, s) ds \\ \le c'.\end{aligned}$$

Therefore $T : \overline{P(\gamma, c')} \to \overline{P(\gamma, c')}$.

Now we verify that the conditions $(B1)$, $(B2)$ of Theorem 3.1 are satisfied. It is obvious that

$$\left\{(y_1, y_2, y_3) \in P\left(\gamma, \theta, \alpha, b', \frac{b'}{\eta}, c'\right) : \alpha(y_1, y_2, y_3) > b'\right\} \ne \emptyset$$

and

$$\{(y_1, y_2, y_3) \in Q(\gamma, \beta, \psi, \eta a', a', c') : \beta(y_1, y_2, y_3) < a'\} \neq \emptyset.$$

Next, let $(y_1, y_2, y_3) \in P(\gamma, \theta, \alpha, b', \frac{b'}{\eta}, c')$ or $(y_1, y_2, y_3) \in Q(\gamma, \beta, \psi, \eta a', a', c')$. Then, $b' \leq \sum_{i=1}^{3} |y_i(t)| \leq \frac{b'}{\eta}$ and $\eta a' \leq \sum_{i=1}^{3} |y_i(t)| \leq a'$.

Now, we may apply condition $(A2)$ to get

$$\begin{aligned}
\alpha(T(y_1, y_2, y_3)(t)) = \min_{t \in I} \Big\{ & \int_a^b G_m(t,s) f_1(s, y_1(s), y_2(s), y_3(s)) ds \\
& + \int_a^b G_n(t,s) f_2(s, y_1(s, y_2(s), y_3(s)) ds \\
& + \int_a^b G_l(t,s) f_3(s, y_1(s), y_2(s), y_3(s)) ds \Big\} \\
\geq \eta \Big\{ & \int_a^b G_m(b,s) f_1(s, y_1(s), y_2(s), y_3(s)) ds \\
& + \int_a^b G_n(b,s) f_2(s, y_1(s), y_2(s), y_3(s)) ds \\
& + \int_a^b G_l(b,s) f_3(s, y_1(s), y_2(s), y_3(s)) ds \Big\} \\
> \frac{b'}{3L} & \int_a^b G_m(b,s) ds + \frac{b'}{3L} \int_a^b G_n(b,s) ds + \frac{b'}{3L} \int_a^b G_l(b,s) ds \\
\geq b'. &
\end{aligned}$$

Clearly, by condition $(A1)$, we have

$$\begin{aligned}
\beta(T(y_1, y_2, y_3)(t)) = \max_{t \in I_1} \Big\{ & \int_a^b G_m(t,s) f_1(s, y_1(s), y_2(s), y_3(s)) ds \\
& + \int_a^b G_n(t,s) f_2(s, y_1(s), y_2(s), y_3(s)) ds \\
& + \int_a^b G_l(t,s) f_3(s, y_1(s), y_2(s), y_3(s)) ds \Big\} \\
\leq & \int_a^b G_m(b,s) f_1(s, y_1(s), y_2(s), y_3(s)) ds \\
& + \int_a^b G_n(b,s) f_2(s, y_1(s), y_2(s), y_3(s)) ds \\
& + \int_a^b G_l(b,s) f_3(s, y_1(s), y_2(s), y_3(s)) ds \\
< \frac{a'}{3M} & \int_a^b G_m(b,s) ds + \frac{a'}{3M} \int_a^b G_n(b,s) ds + \frac{a'}{3M} \int_a^b G_l(b,s) ds \\
\leq a'. &
\end{aligned}$$

To see that $(B3)$ is satisfied, let $(y_1, y_2, y_3) \in P(\gamma, \alpha, b', c')$ with $\theta(T(y_1, y_2, y_3)) > \frac{b'}{\eta}$. Then, we have

$$\begin{aligned}
\alpha(T(y_1,y_2,y_3)(t)) &= \min_{t\in I}\Big\{\int_a^b G_m(t,s)f_1(s,y_1(s),y_2(s),y_3(s))ds \\
&\qquad + \int_a^b G_n(t,s)f_2(s,y_1(s),y_2(s),y_3(s))ds \\
&\qquad + \int_a^b G_l(t,s)f_3(s,y_1(s),y_2(s),y_3(s))ds\Big\} \\
&\geq \eta\Big\{\max_{t\in[a,b]}\int_a^b G_m(t,s)f_1(s,y_1(s),y_2(s),y_3(s))ds \\
&\qquad + \max_{t\in[a,b]}\int_a^b G_n(t,s)f_2(s,y_1(s),y_2(s),y_3(s))ds \\
&\qquad + \max_{t\in[a,b]}\int_a^b G_l(t,s)f_3(s,y_1(s),y_2(s),y_3(s))ds\Big\} \\
&\geq \eta\Big\{\max_{t\in I}\int_a^b G_m(t,s)f_1(s,y_1(s),y_2(s),y_3(s))ds \\
&\qquad + \max_{t\in I}\int_a^b G_n(t,s)f_2(s,y_1(s),y_2(s),y_3(s))ds \\
&\qquad + \max_{t\in I}\int_a^b G_l(t,s)f_3(s,y_1(s),y_2(s),y_3(s))ds\Big\} \\
&= \eta\theta(T(y_1,y_2,y_3)) \geq b'.
\end{aligned}$$

Finally, we show that (B4) holds. Let $(y_1, y_2, y_3) \in Q(\gamma, \beta, a', c')$ with $\psi(T(y_1, y_2, y_3)) < \eta a'$. Then, we have

$$\begin{aligned}
\beta(T(y_1,y_2,y_3)(t)) &= \max_{t\in I_1}\Big\{\int_a^b G_m(t,s)f_1(s,y_1(s),y_2(s),y_3(s))ds \\
&\qquad + \int_a^b G_n(t,s)f_2(s,y_1(s),y_2(s),y_3(s))ds \\
&\qquad + \int_a^b G_l(t,s)f_3(s,y_1(s),y_2(s),y_3(s))ds\Big\} \\
&\leq \max_{t\in[a,b]}\int_a^b G_m(t,s)f_1(s,y_1(s),y_2(s),y_3(s))ds \\
&\qquad + \max_{t\in[a,b]}\int_a^b G_n(t,s)f_2(s,y_1(s),y_2(s),y_3(s))ds \\
&\qquad + \max_{t\in[a,b]}\int_a^b G_l(t,s)f_3(s,y_1(s),y_2(s),y_3(s))ds \\
&\leq \int_a^b G_m(b,s)f_1(s,y_1(s),y_2(s),y_3(s))ds \\
&\qquad + \int_a^b G_n(b,s)f_2(s,y_1(s),y_2(s),y_3(s))ds
\end{aligned}$$

$$
\begin{aligned}
&+ \int_a^b G_l(b,s) f_3(s, y_1(s), y_2(s), y_3(s)) ds \\
\leq &\frac{1}{\eta}\Big\{ \min_{t\in I} \int_a^b G_m(t,s) f_1(s, y_1(s), y_2(s), y_3(s)) ds \\
&+ \min_{t\in I} \int_a^b G_n(t,s) f_2(s, y_1(s), y_2(s), y_3(s)) ds \\
&+ \min_{t\in I} \int_a^b G_l(t,s) f_3(s, y_1(s), y_2(s), y_3(s)) ds \Big\} \\
\leq &\frac{1}{\eta}\Big\{ \min_{t\in I_1} \int_a^b G_m(t,s) f_1(s, y_1(s), y_2(s), y_3(s)) ds \\
&+ \min_{t\in I_1} \int_a^b G_n(t,s) f_2(s, y_1(s), y_2(s), y_3(s)) ds \\
&+ \min_{t\in I_1} \int_a^b G_l(t,s) f_3(s, y_1(s), y_2(s), y_3(s)) ds \Big\} \\
= &\frac{1}{\eta}\psi(T(y_1, y_2, y_3)) \leq a'.
\end{aligned}
$$

We have proved that all the conditions of Theorem 3.1 are satisfied. Therefore, the BVP (1.1)–(1.2) has at least three positive solutions, $(u_1, u_2, u_3), (v_1, v_2, v_3)$ and (w_1, w_2, w_3). ☐

Now, we establish the existence of at least $2k-1$ positive solutions for the BVP (1.1)–(1.2), by using induction on k.

Theorem 3.3. *Let k be an arbitrary positive integer. Assume that there exist numbers a_r $(r = 1, 2, \ldots, k)$ and b_s $(s = 1, 2, \ldots, k-1)$ with $0 < a_1 < b_1 < \frac{b_1}{\eta} < a_2 < b_2 < \frac{b_2}{\eta} < \cdots < a_{k-1} < b_{k-1} < \frac{b_{k-1}}{\eta} < a_k$ such that f_i, for $i = 1, 2, 3$ satisfies the following conditions:*

$$
f_i(t, y_1, y_2, y_3) < \frac{a_r}{3M}, \text{ for all } t \in [a,b] \text{ and } \sum_{i=1}^{3} |y_i| \in [\eta a_r, a_r],\ r = 1, 2, \ldots, k, \tag{3.4}
$$

$$
f_i(t, y_1, y_2, y_3) > \frac{b_s}{3\eta L}, \text{ for all } t \in I \text{ and} \sum_{i=1}^{3} |y_i| \in \left[b_s, \frac{b_s}{\eta}\right],\ s = 1, 2, \ldots, k-1. \tag{3.5}
$$

Then the BVP (1.1)–(1.2) has at least $2k-1$ positive solutions in $\overline{P}_{a_k}$.

Proof. We use induction on k. First, for $k = 1$, we know from (3.4) that $T : \overline{P}_{a_1} \to P_{a_1}$, then it follows from Schauder fixed point theorem that the BVP (1.1)–(1.2) has at least one positive solution in $\overline{P}_{a_1}$. Next, we assume that this conclusion holds for $k = l$. In order to prove that this conclusion holds for $k = l + 1$, we suppose that there exist numbers $a_r (r = 1, 2, \ldots, l + 1)$ and $b_s (s = 1, 2, \ldots, l)$ with $0 < a_1 < b_1 < \frac{b_1}{\eta} < a_2 < b_2 < \frac{b_2}{\eta} < \cdots < a_l < b_l < \frac{b_l}{\eta} < a_{l+1}$ such that f_i, for

$i = 1, 2, 3$, satisfies

$$f_i(t, y_1, y_2, y_3) < \frac{a_r}{3M}, \text{ for all } t \in [a, b] \text{ and } \sum_{i=1}^{3} |y_i| \in [\eta a_r, a_r], \; r = 1, 2, \ldots, l+1, \tag{3.6}$$

$$f_i(t, y_1, y_2, y_3) > \frac{b_s}{3\eta L}, \text{ for all } t \in I \text{ and } \sum_{i=1}^{3} |y_i| \in \left[b_s, \frac{b_s}{\eta}\right], \; s = 1, 2, \ldots, l. \tag{3.7}$$

By assumption, the BVP (1.1)–(1.2) has at least $2l - 1$ positive solutions (u_i, v_i, w_i), $i = 1, 2, \ldots, 2l - 1$ in $\overline{P}_{a_l}$. At the same time, it follows from Theorem (3.3), (3.6) and (3.7) that the BVP (1.1)–(1.2) has at least three positive solutions (u_1^*, v_1^*, w_1^*), (u_2^*, v_2^*, w_2^*) and (u_3^*, v_3^*, w_3^*) in $\overline{P}_{a_{l+1}}$ such that $\beta(u_1^*, v_1^*, w_1^*) < a_l$, $b_l < \alpha(u_2^*, v_2^*, w_2^*)$ and $a_l < \beta(u_3^*, v_3^*, w_3^*)$ with $\alpha(u_3^*, v_3^*, w_3^*) < b_l$. Obviously, (u_2^*, v_2^*, w_2^*) and (u_3^*, v_3^*, w_3^*) are different from (u_i, v_i, w_i), $i = 1, 2, \ldots, 2l - 1$. Therefore, the BVP(1.1)–(1.2) has at least $2l + 1$ positive solutions in $\overline{P}_{a_{l+1}}$, which shows that this conclusion also holds for $k = l + 1$. □

4. EXAMPLE

Consider the system of differential equations,

$$y_1''(t) + f_1(t, y_1(t), y_2(t), y_3(t)) = 0, \; t \in [0, 1],$$
$$y_2'''(t) + f_2(t, y_1(t), y_2(t), y_3(t)) = 0, \; t \in [0, 1],$$
$$y_3''''(t) + f_3(t, y_1(t), y_2(t), y_3(t)) = 0, \; t \in [0, 1],$$

subject to boundary conditions,

$$y_1(0) = 0, \; y_1'(1) = 0,$$

$$y_2(0) = 0, \; y_2'(0) = 0, \; y_2'(1) = 0,$$

$$y_3(0) = 0, \; y_3'(0) = 0, \; y_3''(0) = 0, \; y_3''(1) = 0,$$

where

$$f_1(t, y_1, y_2, y_3) = \begin{cases} \frac{e^{(y_1+y_2+y_3)}}{100} + \frac{(y_1+y_2+y_3)^2}{75} + \frac{1}{2}, \; 0 \le y_1 + y_2 + y_3 \le 15, \\ \frac{e^{(y_1+y_2+y_3)}}{101} + \frac{7}{2}, \; y_1 + y_2 + y_3 \ge 15, \end{cases}$$

$$f_2(t, y_1, y_2, y_3) = \begin{cases} \frac{e^{(y_1+y_2+y_3)}}{101} + \frac{2(y_1+y_2+y_3)^2}{87} + \frac{1}{3}, \; 0 \le y_1 + y_2 + y_3 \le 15, \\ \frac{e^{(y_1+y_2+y_3)}}{101} + \frac{479}{87}, \; y_1 + y_2 + y_3 \ge 15, \end{cases}$$

$$f_3(t, y_1, y_2, y_3) = \begin{cases} \frac{e^{(y_1+y_2+y_3)}}{99} + \frac{3(y_1+y_2+y_3)^2}{95} + \frac{1}{5}, \; 0 \le y_1 + y_2 + y_3 \le 15, \\ \frac{e^{(y_1+y_2+y_3)}}{99} + \frac{694}{95}, \; y_1 + y_2 + y_3 \ge 15. \end{cases}$$

The Green's functions $G_1(t,s)$, $G_2(t,s)$ and $G_3(t,s)$ are given by

$$G_1(t,s)=\begin{cases} t, \ t\le s,\\ s, \ s\le t,\end{cases}$$

$$G_2(t,s)=\begin{cases} \frac{t^2(1-s)}{2!}, & t\le s,\\ \frac{t^2(1-s)}{2!}-\frac{(t-s)^2}{2!}, & s\le t,\end{cases}$$

$$G_3(t,s)=\begin{cases} \frac{t^3(1-s)}{3!}, & t\le s,\\ \frac{t^3(1-s)}{3!}-\frac{(t-s)^3}{3!}, & s\le t.\end{cases}$$

Clearly, the Green's function $G_i(t,s)$, i=1,2,3 is positive. By direct calculations, we obtain $\eta = 0.015625$, $L = 0.04167$ and $M = 0.5$. Clearly, f_i are continuous and increasing on $[0,\infty)$, for $i = 1,2,3$. If we choose $a' = 1.01$, $b' = 15$ and $c' = 50000$, then $0 < a' < b' < \frac{b'}{\eta} \le c'$ and f_i satisfies

(i) $f_i(t,y_1,y_2,y_3) < 0.67333 = \frac{a'}{3M}$, $t\in[0,1]$ and $y_1+y_2+y_3\in[0.01578125, 1.01]$,

(ii) $f_i(t,y_1,y_2,y_3) > 7679.385649 = \frac{b'}{3\eta L}$, $t\in[\frac{1}{4},\frac{3}{4}]$ and $y_1+y_2+y_3\in[15,960]$,

(iii) $f_i(t,y_1,y_2,y_3) < 33333.333 = \frac{c'}{3M}$, $t\in[0,1]$ and $y_1+y_2+y_3\in[0,50000]$,

for $i = 1,2,3$. Then all the conditions of Theorem 3.2 are satisfied. Thus, by Theorem 3.2, the BVP has at least three positive solutions (u_1,u_2,u_3), (v_1,v_2,v_3) and (w_1,w_2,w_3) satisfying $\beta(u_1,u_2,u_3) < 1.01$, $15 < \alpha(v_1,v_2,v_3)$ and $1.01 < \beta(w_1,w_2,w_3)$ with $\alpha(w_1,w_2,w_3) < 15$.

ACKNOWLEDGMENTS

The authors thank the referees for their valuable suggestions and comments.

REFERENCES

[1] R. P. Agarwal, D. O'Regan and P. J. Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations,* Kluwer Academic Publishers, Dordrecht, The Netherlands, 1999.

[2] R. I. Avery, A generalization of the Leggett–Williams fixed point theorem, *Math. Sci. Res. Hot-line*, **3** (1999), 9–14.

[3] X. Cheng and C. Zhang, Existence of positive solutions for a second order differential system, *J. Math. Anal. Appl.*, **312** (2005), 14–23.

[4] K. Deimling, *Nonlinear Functional Analysis,* Springer, New York, 1985.

[5] P. W. Eloe and J. Henderson, Positive solutions for $(n-1,1)$ conjugate boundary value problems, *Nonlinear Anal.*, **28** (1997), 1669–1680.

[6] L. H. Erbe and H. Wang, On the existence of positive solutions of ordinary differential equations, *Proc. Amer. Math. Soc.*, **120** (1994), 743–748.

[7] J. Henderson and S. K. Ntouyas, Positive solutions for system of nth order three-point nonlocal boundary value problems, *Elec. J. Qual. Theory Diff. Equ.*, **2007** (2007), no. 18, 1–12.

[8] J. Henderson and S. K. Ntouyas, Positive solutions for systems of three-point nonlinear boundary value problems, *Aust. J. Math. Anal. Appl.*, **5** (2008), 1–9.

[9] J. Henderson and S. K. Ntouyas, Positive solutions for systems of nonlinear boundary value problems, *Nonlinear Stud.*, **15** (2008), 51–60.

[10] J. Henderson, S. K. Ntouyas and I. K. Purnaras, Positive solutions for systems of generalized three-point nonlinear boundary value problems, *Comment. Math. Univ. Carolin.*, **49** (2008), 79–91.

[11] L. Hu and L. Wang, Multiple positive solutions of boundary value problems for systems of nonlinear second-order differential equations, *J. Math. Anal. Appl.*, **335** (2007), no. 2, 1052–1060.

[12] R. W. Leggett and L. R. Williams, Multiple positive fixed points of nonlinear operator on order Banach spaces, *Indiana Univer. Math. J.*, **28** (1979), 673–688.

[13] L. Liu, P. Kang and Y. Wu, Positive solutions of singular boundary value problems for systems of nonlinear fourth order differential equations, *Nonlinear Anal.*, **68** (2008), 485–498.

[14] Y. Liu and B. Yan, Multiple solutions of singular boundary value problems for differential systems, *J. Math. Appl. Anal.*, **287** (2003), 40–55.

[15] R. Ma, Multiple nonnegative solutions of second order systems of boundary value problems, *Nonlinear Anal.*, **42**(2000), 1003–1010.

[16] K. R. Prasad, P. Murali and A. K. Rao, Multiple positive solutions for the system of higher order two-point boundary value problems, *Pan Amer. Math. J.*, **19** (2009), no. 2, 63–73.

[17] J. Xu and Z. Yang, Positive solutions for a system of nth-order nonlinear boundary value problems, *Elec. J. Qual. Theory Diff. Equ.*, **2011** (2011), no. 4, 1–16.