Communications in Applied Analysis **19** (2015), 453–462

AN ORDERING ON GREEN'S FUNCTIONS FOR A FAMILY OF TWO-POINT BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS

PAUL ELOE[1], JEFFREY W. LYONS[2], AND JEFFREY T. NEUGEBAUER[3]

[1]Department of Mathematics, University of Dayton
Dayton, OH 45469 USA
E-mail: peloe1@udayton.edu

[2]Division of Math, Science, and Technology, Nova Southeastern University
Fort Lauderdale, FL 33314 USA
E-mail: jlyons@nova.edu

[3]Department of Mathematics and Statistics, Eastern Kentucky University
Richmond, KY 40475 USA
E-mail: Jeffrey.Neugebauer@eku.edu

ABSTRACT. Let $2 \le n$ denote an integer and let $n-1 < \alpha \le n$. For each $0 < b$, $0 \le \beta \le n-1$, the authors will construct the Green's function, $G(b, \beta; t, s)$, of the two-point boundary value problem for the fractional differential equation

$$D_{0+}^{\alpha} u + h(t) = 0, \quad 0 < t < b,$$

$$u^{(i)}(0) = 0, \quad i = 0, \dots n-2, \quad D_{0+}^{\beta} u(b) = 0,$$

where D_{0+}^{α} and D_{0+}^{β} denote the standard Riemann-Liouville derivatives. The authors will compare Green's functions, $G(b_1, \beta; t, s)$ and $G(b_2, \beta; t, s)$ or $G(b, \beta_1; t, s)$ and $G(b, \beta_2; t, s)$, and the authors will show the existence of a unique limiting function as $b \to \infty$. An application to a nonlinear problem will be given.

AMS (MOS) Subject Classification. 26A33, 34A08, 34A40.

1. INTRODUCTION

Let $2 \le n$ denote an integer and let $n-1 < \alpha \le n$. For each $0 < b$, $0 \le \beta \le n-1$, we shall consider a boundary value problem (BVP) consisting of a fractional differential equation of the form

$$D_{0+}^{\alpha} u + h(t) = 0, \quad 0 < t < b, \tag{1.1}$$

with two-point boundary conditions of the form

$$u^{(i)}(0) = 0, \quad i = 0, \dots n-2, \quad D_{0+}^{\beta} u(b) = 0, \tag{1.2}$$

Received December 15, 2014

where D_{0+}^{α} and D_{0+}^{β} denote the standard Riemann-Liouville derivatives which are defined in Section 2. As special cases, the boundary conditions (1.2) contain

$$u^{(i)}(0) = 0, \quad i = 0, \dots n-2, \quad u^{(j)}(b) = 0, \quad j \in \{0, \dots, n-1\}, \tag{1.3}$$

or

$$u^{(i)}(0) = 0, \quad i = 0, \dots n-2, \quad D_{0+}^{\alpha-(n-1)+j} u(b) = 0, \quad j \in \{0, \dots, n-2\}. \tag{1.4}$$

For simplicity, $h(t)$ is assumed to be continuous $[0, \infty)$.

We shall construct the corresponding Green's function, $G(b, \beta; t, s)$, of the BVP, (1.1), (1.2). We shall show the Green's functions are all positive on $(0, b) \times (0, b)$, and we shall order the Green's functions with respect to $b_1 < b_2$ or $\beta_1 < \beta_2$. Moreover, we shall show the existence of a unique limiting function, a Green's function, as $b \to \infty$. In the case of ordinary differential equations ($\alpha = n$), with ordinary boundary conditions (1.3), it is known that the right focal problem ($\beta = n-1$) is the dominating BVP (see [6], [7], or [9]). It will be particularly interesting to note that for the fractional differential equation, the dominating BVP for (1.1), (1.2) is in the case $\beta = \alpha - 1$.

Many researchers are currently applying fixed point theorems to study boundary value problems for the fractional differential operator D_{0+}^{α}, and so, we assume the construction we employ and the explicit Green's functions we exhibit are not new. See for example, [3], [12], or [17] . The ordering of the family of Green's functions and the exhibition of a limiting Green's function represent the contributions of this article.

In the case of ordinary differential equations, the concept and existence of unique limiting Green's functions is known. For one example, if the differential operator is of limit-point type at a singular point, [4], then the existence of a unique limiting Green's function is well-known. More in line with the study in this paper, the inequalities and properties obtained here are motivated by Elias', [6] or [7], extensive work for the two-term right disfocal differential operator with two point boundary conditions. Disconjugacy theory or disfocality theory for fractional differential equations is almost nonexistent (see [1] or [10]) and so, for now, we consider the very simple equation (1.1). Because of this, we exhibit an analytic expression for $G(b, \beta; t, s)$. The analytic expression can be used to obtain many of the results we exhibit. We shall also develop some qualitative arguments.

In the next section, we provide the fundamental definitions and remind the reader of some basic results. We shall then construct the Green's functions, obtain some qualitative properties and show the existence of a unique limiting Green's function. In the third section, we shall provide an immediate application to a nonlinear boundary value problem in which we employ monotone methods.

2. CONSTRUCTIONS OF GREEN'S FUNCTIONS

Let $0 < \nu$ and recall the Riemann-Liouville fractional integral of a function, [5], u is defined by

$$I_{0+}^{\nu} u(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} u(s) ds, \tag{2.1}$$

provided the right-hand side exists. Moreover, let n denote a positive integer and assume $n-1 < \alpha \le n$. The α-th Riemann-Liouville fractional derivative of the function $u : [0,\infty) \to \mathbb{R}$, denoted $D_{0+}^{\alpha} u$, is defined as

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s) ds = D^n I_{0+}^{n-\alpha} u(t),$$

provided the right-hand side exists. We shall employ a standard notation, D_{0+}^{α}, to denote fractional derivatives and D^j to denote classical derivatives in the case j is a nonnegative integer.

We only require a few well known properties in fractional calculus to construct and analyze the family of Green's functions. There are many good presentations on fractional calculus, [13, 14, 15, 16], for example; we refer the reader to [5]. Recall

$$I_{0+}^{\nu_1} I_{0+}^{\nu_2} u(t) = I_{0+}^{\nu_1+\nu_2} u(t) = I_{0+}^{\nu_2} I_{0+}^{\nu_1} u(t), \quad \nu_1, \nu_2 > 0, \text{ if } u \in L_1[0,b], \tag{2.2}$$

$$D_{0+}^{\nu_1} I_{0+}^{\nu_2} u(t) = I_{0+}^{\nu_2-\nu_1} u(t), \quad \text{ if } 0 \le \nu_1 \le \nu_2, \text{ if } u \in L_1[0,b], \tag{2.3}$$

$$D_{0+}^{\alpha} I_{0+}^{\alpha} u(t) = u(t), \ 0 < t, \text{ if } u \in L_1[0,b],$$

and

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + \sum_{i=1}^{n} c_i t^{\alpha-n+(i-1)}, \quad \text{ if } D_{0+}^{\alpha} u \in L_1[0,b]. \tag{2.4}$$

The property (2.2) is referred to as the semigroup property for the fractional integral.

We also require the power rule, and so, we recall [5]

$$I_{0+}^{\nu_2} t^{\nu_1} = \frac{\Gamma(\nu_1+1)}{\Gamma(\nu_2+\nu_1+1)} t^{\nu_2+\nu_1}, \quad \nu_1 > -1, \nu_2 \ge 0,$$

and

$$D_{0+}^{\nu_2} t^{\nu_1} = \frac{\Gamma(\nu_1+1)}{\Gamma(\nu_1+1-\nu_2)} t^{\nu_1-\nu_2}, \quad \nu_1 > -1, \nu_2 \ge 0, \tag{2.5}$$

where it is assumed that $\nu_2 - \nu_1$ is not a positive integer. If $\nu_2 - \nu_1$ is a positive integer, then the right hand side of (2.5) vanishes. To see this, one can appeal to the convention that $\frac{1}{\Gamma(\nu_1+1-\nu_2)} = 0$ if $\nu_2 - \nu_1$ is a positive integer, or one can perform the calculation on the left hand side and calculate

$$D^n t^{n-(\nu_2-\nu_1)} = 0.$$

To construct the Green's function, $G(b,\beta;t,s)$, apply (2.4) to (1.1), and a solution u of (1.1) has the form

$$u(t)+\sum_{i=1}^{n-1} c_i t^{\alpha-n+(i-1)} + c_n t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds = 0.$$

The boundary conditions, $u^{(i)}(0)=0,\ i=0,\dots n-2$, imply $c_i=0,\ i=1,\dots n-1$, and now, the solution u of (1.1) has the form

$$u(t)+c_n t^{\alpha-1} + I_{0+}^{\alpha}(h)(t) = 0.$$

Apply the boundary condition $D_{0+}^{\beta}u(b)=0$, (2.3), and (2.5); solve for c_n to obtain

$$c_n = -\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\frac{I_{0+}^{\alpha-\beta}(h)(b)}{b^{\alpha-1-\beta}}.$$

Thus, the Green's function for the BVP, (1.1), (1.2), has the form

$$G(b,\beta;t,s)=\begin{cases} \frac{t^{\alpha-1}(b-s)^{\alpha-1-\beta}}{b^{\alpha-1-\beta}\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0\le s\le t<b, \\ \frac{t^{\alpha-1}(b-s)^{\alpha-1-\beta}}{b^{\alpha-1-\beta}\Gamma(\alpha)}, & 0\le t\le s<b, \end{cases} \tag{2.6}$$

and the solution u of (1.1) has the form

$$u(t)=\int_0^b G(b,\beta;t,s)h(s)ds, \quad 0\le t\le b. \tag{2.7}$$

Note that in the case $\alpha-1<\beta\le n-1$, the Green's function does not extend to $[0,b]\times[0,b]$. The continuity of h is sufficient to imply that u, given by (2.7), is $n-2$ times differentiable on $[0,b]$.

Remark 2.1. If $\beta=j$ is an integer, then (2.6) reduces to

$$G(b,j;t,s)=\begin{cases} \frac{t^{\alpha-1}(b-s)^{\alpha-1-j}}{b^{\alpha-1-j}\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0\le s\le t\le b, \\ \frac{t^{\alpha-1}(b-s)^{\alpha-1-j}}{b^{\alpha-1-j}\Gamma(\alpha)}, & 0\le t\le s\le b, \end{cases} \tag{2.8}$$

an expression that is known and employed in the literature, [3], [12]. If $\alpha-\beta$ is a positive integer, the form is also known, [2].

Theorem 2.2. *If* $0\le\beta_1<\beta_2\le n-1$,

$$0<G(b,\beta_1;t,s)<G(b,\beta_2;t,s), \quad (t,s)\in(0,b)\times(0,b). \tag{2.9}$$

Proof. To see that $0<G(b,\beta_1;t,s)$, note that the inequality is clear for $t\le s$. For $s\le t$, write

$$\frac{t^{\alpha-1}(b-s)^{\alpha-1-\beta}}{b^{\alpha-1-\beta}} - (t-s)^{\alpha-1} = t^{\alpha-1}\left(1-\frac{s}{b}\right)^{\alpha-1-\beta} - t^{\alpha-1}\left(1-\frac{s}{t}\right)^{\alpha-1}. \tag{2.10}$$

Note that for $0\le s\le t\le b$,

$$\left(1-\frac{s}{b}\right)^{\alpha-1-\beta} \ge \left(1-\frac{s}{b}\right)^{\alpha-1} \ge \left(1-\frac{s}{t}\right)^{\alpha-1}$$

is valid in each case, $0\le\beta\le\alpha-1$, or $\alpha-1<\beta\le n-1$.

To see that $G(b,\beta_1;t,s) < G(b,\beta_2;t,s)$ on $(0,b)\times(0,b)$ if $\beta_1<\beta_2$, note that

$$G(b,\beta_2;t,s) - G(b,\beta_1;t,s) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\left[\left(1-\frac{s}{b}\right)^{\alpha-1-\beta_2} - \left(1-\frac{s}{b}\right)^{\alpha-1-\beta_1}\right] \tag{2.11}$$

and write

$$\left[\left(1-\frac{s}{b}\right)^{\alpha-1-\beta_2} - \left(1-\frac{s}{b}\right)^{\alpha-1-\beta_1}\right] = \left(1-\frac{s}{b}\right)^{\alpha-1-\beta_2}\left[1-\left(1-\frac{s}{b}\right)^{\beta_2-\beta_1}\right].$$

□

The proof of Theorem (2.2) is easily modified to obtain a related result.

Theorem 2.3. *Let $j\in\{0,\dots,n-2\}$, $i\in\{0,\dots,j\}$. If $j\le\beta_1<\beta_2\le n-1$,*

$$0 < \left(\frac{\partial^i}{\partial t^i}\right)G(b,\beta_1;t,s) < \left(\frac{\partial^i}{\partial t^i}\right)G(b,\beta_2;t,s), \quad (t,s)\in(0,b)\times(0,b). \tag{2.12}$$

Proof. Note that

$$\left(\frac{\partial^i}{\partial t^i}\right)G(b,\beta;t,s) = \begin{cases} \frac{\Gamma(\alpha)}{\Gamma(\alpha-i)}\left(\frac{t^{\alpha-1-i}(b-s)^{\alpha-1-\beta}}{b^{\alpha-1-\beta}\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1-i}}{\Gamma(\alpha)}\right), & 0\le s\le t<b, \\ \frac{\Gamma(\alpha)}{\Gamma(\alpha-i)}\frac{t^{\alpha-1-i}(b-s)^{\alpha-1-\beta}}{b^{\alpha-1-\beta}\Gamma(\alpha)}, & 0\le t\le s<b. \end{cases}$$

Replace the right hand sides of (2.10) and (2.11) respectively by

$$t^{\alpha-1-i}\left(1-\frac{s}{b}\right)^{\alpha-1-\beta} - t^{\alpha-1-i}\left(1-\frac{s}{t}\right)^{\alpha-1-i}$$

and

$$\frac{t^{\alpha-1-i}}{\Gamma(\alpha)}\left[\left(1-\frac{s}{b}\right)^{\alpha-1-\beta_2} - \left(1-\frac{s}{b}\right)^{\alpha-1-\beta_1}\right].$$

□

We shall use Theorem 2.3 in Section 3 for a straight-forward application to a nonlinear problem.

Theorem 2.4. *Assume $0<b_1<b_2$. If $0\le\beta<\alpha-1$, then*

$$0 < G(b_1,\beta;t,s) < G(b_2,\beta;t,s), \quad (t,s)\in(0,b_1)\times(0,b_1), \tag{2.13}$$

and if $\alpha-1<\beta\le n-1$, then

$$G(b_1,\beta;t,s) > G(b_2,\beta;t,s) > 0, \quad (t,s)\in(0,b_1)\times(0,b_1). \tag{2.14}$$

If $\beta=\alpha-1$, then $G(b,\alpha-1;t,s)$ is independent of b on $(0,b)\times(0,b)$.

Proof. Note that

$$H(b,\beta;t,s) := \frac{\partial}{\partial b}G(b,\beta;t,s) = \frac{\alpha-1-\beta}{\Gamma(\alpha)}\left(1-\frac{s}{b}\right)^{\alpha-2-\beta}\left(\frac{s}{b^2}\right)t^{\alpha-1}.$$

So, H changes sign at $\alpha-1-\beta=0$. □

We point out one more property of $G(b,\beta;t,s)$ that is analogous to properties of Green's functions for boundary value problems of ordinary differential equations, [8, 10, 11]. H is the solution of the BVP,

$$D^{\alpha}_{0+}u = 0, \quad 0 < t < b,$$

$$u^{(i)}(0) = 0, \quad i = 0, \dots n-2, \quad D^{\beta}_{0+}u(b) = -D^{\beta+1}_{0+}G(b,\beta;b,s).$$

This is shown with a direct calculation.

Finally, since G is given explicitly in (2.6), we can exhibit a uniquely determined limiting function in the following sense. Extend, in any way, $G(b,\beta;t,s)$ to $G(b,\beta;t,s)$ defined on $[0,\infty)\times[0,\infty)$. The following observation is clear.

Theorem 2.5. *Let $0 \le \beta \le \alpha - 1$. Then*

$$\lim_{b\to\infty} G(b,\beta;t,s) = G(\alpha-1;t,s),$$

where

$$G(\alpha-1;t,s) = \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t < \infty, \\ \frac{t^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s < \infty. \end{cases} \tag{2.15}$$

The convergence is monotone increasing and uniform on compact domains. If $\alpha - 1 \le \beta \le n-1$, then

$$\lim_{b\to\infty} G(b,\beta;t,s) = G(\alpha-1;t,s),$$

where the convergence is monotone decreasing and uniform on compact domains.

In particular, $G(b,\alpha-1;t,s) = G(\alpha-1;t,s)$ agree on $[0,b]\times[0,b]$.

3. AN APPLICATION TO A NONLINEAR PROBLEM

In this section, we consider a family of nonlinear BVPs for a specific set of boundary conditions and apply the method of upper and lower solutions to obtain sufficient conditions for existence of solutions. Moreover, monotone methods can be applied to generate sequences of approximate solutions that converge uniformly to a solution of the nonlinear problem.

Let $2 \le n$, $0 \le j \le n-2$ denote integers and let $n-1 < \alpha \le n$. Let $0 < b$, $0 \le j \le \beta \le n-1$, and consider the boundary value problem

$$D^{\alpha}_{0+}u + f(t,u(t),u'(t),\dots,u^{(j)}(t)) = 0, \quad 0 < t < b, \tag{3.1}$$

with the boundary conditions conditions (1.2) where $f : [0,b]\times\mathbb{R}^{(j+1)} \to \mathbb{R}$ is continuous and

$$f(t,u_0,\dots,u_j) \ge f(t,v_0,\dots,v_j) \text{ if } u_i \ge v_i, \quad i = 0,\dots,j. \tag{3.2}$$

We shall say that $w \in C^{(j)}[0,b]$ is a lower solution of the BVP, (3.1), (1.2), if w satisfies the boundary conditions (1.2) and

$$D_{0+}^{\alpha} w + f(t, w(t), w'(t), \ldots, w^{(j)}(t)) \geq 0, \quad 0 < t < b, \tag{3.3}$$

and that $z \in C^{(j)}[0,b]$ is an upper solution of the BVP, (3.1), (1.2), if z satisfies the boundary conditions (1.2) and

$$D_{0+}^{\alpha} z + f(t, z(t), z'(t), \ldots, z^{(j)}(t)) \leq 0, \quad 0 < t < b. \tag{3.4}$$

Assume the existence of lower and upper solutions, w_0 and z_0 respectively such that

$$w_0^{(i)}(t) \leq z_0^{(i)}(t), \quad 0 \leq t \leq b, \quad i = 0, 1, \ldots, j.$$

Define sequences, $\{w_{k+1}\}, \{v_{k+1}\}$, respectively, by

$$w_{k+1}(t) = Tw_k(t), \quad z_{k+1}(t) = Tz_k(t), \quad 0 \leq t \leq b, \tag{3.5}$$

where

$$Ty(t) = \int_0^b G(b, \beta; t, s) f(s, y(s), y'(s), \ldots, y^{(j)}(s)) ds, \quad 0 \leq t \leq b. \tag{3.6}$$

Apply Theorem 2.3 and (3.2) to obtain

$$x^{(i)}(t) \leq y^{(i)}(t), \quad 0 \leq t \leq b, \quad i = 0, 1, \ldots, j,$$

implies

$$(Tx)^{(i)}(t) \leq (Ty)^{(i)}(t), \quad 0 \leq t \leq b, \quad i = 0, 1, \ldots, j. \tag{3.7}$$

Also, note that Theorem 2.3, (3.3), and (3.4) imply for $0 \leq t \leq b,\ i = 0, 1, \ldots, j$,

$$w_0^{(i)}(t) = \left(\int_0^b G(b, \beta; t, s)(-D_{0+}^{\alpha} w_0(s)) ds \right)^{(i)} (t) \leq (Tw_0)^{(i)}(t) = w_1^{(i)}(t),$$

and

$$z_0^{(i)}(t) = \left(\int_0^b G(b, \beta; t, s)(-D_{0+}^{\alpha} z_0(s)) ds \right)^{(i)} (t) \geq (Tz_0)^{(i)}(t) = z_1^{(i)}(t).$$

In particular,

$$w_0^{(i)}(t) \leq w_1^{(i)}(t) \leq z_1^{(i)}(t) \leq z_0^{(i)}(t), \quad 0 \leq t \leq b. \tag{3.8}$$

Apply T iteratively to (3.8), and it follows from (3.7) that

$$w_k^{(i)}(t) \leq w_{k+1}^{(i)}(t) \leq z_{k+1}^{(i)}(t) \leq z_k^{(i)}(t), \quad 0 \leq t \leq b, \quad k = 0, 1, \ldots.$$

Theorem 3.1. *Let $2 \leq n$ denote an integer, $n-1 < \alpha \leq n$, and let $j \in \{0, \ldots, n-2\}$. Assume $f : [0,b] \times \mathbb{R}^{(j+1)} \to \mathbb{R}$ is continuous and satisfies*

$$f(t, u_0, \ldots, u_j) \geq f(t, v_0, \ldots, v_j) \text{ if } u_i \geq v_i, \quad i = 0, \ldots, j.$$

Assume there exists a lower solution, $w_0(t) \in C^j[0,b]$ and an upper solution, $z_0(t) \in C^j[0,b]$ of the BVP, (3.1), (1.2), satisfying

$$w_0^{(i)}(t) \le z_0^{(i)}(t), \quad 0 \le t \le b, \quad i = 0, 1, \ldots, j.$$

Define sequences, $\{w_{k+1}\}, \{v_{k+1}\}$, by

$$w_{k+1}(t) = Tw_k(t), \quad z_{k+1}(t) = Tz_k(t), \quad 0 \le t \le b,$$

where

$$Ty(t) = \int_0^b G(b, \beta; t, s) f(s, y(s), y'(s), \ldots, y^{(j)}(s)) ds, \quad 0 \le t \le b.$$

Then there exists a solution u of the BVP, (3.1), (1.2), satisfying

$$w_k^{(i)}(t) \le w_{k+1}^{(i)}(t) \le u^{(i)}(t) \le v_{k+1}^{(i)}(t) \le v_k^{(i)}(t), \quad 0 \le t \le b, \quad i = 0, 1, \ldots, j,$$

for each $k = 0, 1, \ldots$. Moreover, the sequences, $\{w_k\}, \{v_k\}$, converge in $C^j[0,b]$ to w, v, respectively, where each of w and v are solutions of the BVP, (3.1), (1.2), and

$$w^{(i)}(t) \le u^{(i)}(t) \le v^{(i)}(t), \quad 0 \le t \le b, \quad i = 0, 1, \ldots, j.$$

As a first example to illustrate the theorem, consider the BVP,

$$D_{0+}^{\frac{5}{2}} u(t) + 1 + u^3(t) + \frac{1}{2}(u')^3(t) = 0, \ \ 0 < t < 1,$$
$$u(0) = u'(0) = 0, \quad u'(1) = 0.$$

Set $w_0 \equiv 0$. Set $z_0(t) = t^{\frac{3}{2}} - \frac{3}{5} t^{\frac{5}{2}}$. Each of w_0, z_0 satisfy the boundary conditions and it is clear that w_0 is a lower solution. Direct calculations give that

$$D_{0+}^{\frac{5}{2}} z_0(t) = -\frac{3}{5} \Gamma\left(\frac{7}{2}\right) = -\frac{9}{8}\sqrt{\pi},$$

$$0 \le z_0(t) \le \frac{2}{5}, \quad 0 \le z_0'(t) \le 1, \quad 0 \le t \le 1.$$

Thus, z_0 is an upper solution. Since (3.2) is valid on all of $[0,1] \times \mathbb{R}^2$, Theorem 3.1 applies, and there exists a solution u such that

$$0 \le u^{(i)}(t) \le z_0^{(i)}(t), \quad i = 0, 1, \quad 0 \le t \le 1.$$

As a second example, we note that (3.2) is too strong. If the lower and upper solutions w_0 and z_0, respectively, exist satisfying

$$z_0^{(i)}(t) \ge w_0^{(i)}(t), \quad 0 \le t \le b,$$

then one only needs to assume

$$f(t, u_0, \ldots, u_j) \ge f(t, v_0, \ldots, v_j), \text{ if } z_0^{(i)}(t) \ge u_i \ge v_i \ge w_0^{(i)}(t), \tag{3.9}$$

and $i = 0, \ldots, j$, $0 \le t \le b$.

Consider the BVP,

$$D_{0+}^{\frac{5}{2}} u + 1 + u(t)u'(t) = 0, \quad 0 < t < 1,$$

$$u(0) = u'(0) = 0, \quad u'(1) = 0.$$

Again $w_0 \equiv 0$ and $z_0(t) = t^{\frac{3}{2}} - \frac{3}{5}t^{\frac{5}{2}}$ serve as lower and upper solutions, respectively, and now, (3.9) is valid. Thus, there exists a solution u of this BVP satisfying

$$0 \le u^{(i)}(t) \le z_0^{(i)}(t), \quad i = 0, 1, \quad 0 \le t \le 1.$$

REFERENCES

[1] T. Abdeljawad, D. Băleanu, F. Jarad, O. Mustafa, and J. J. Trujillo, A Fite type result for sequential fractional differential equations, *Dynam. Systems Appl.* **19** (2010), no. 2, 383–394.

[2] B. Ahmad and J. Nieto, Riemann-Liouville fractional differential equations with fractional boundary conditions, *Fixed Point Theory* **13** (2012), no. 2, 329–336.

[3] Z. Bai and H. Lu, Positive solutions for boundary value problems of nonlinear fractional differential equations, *J. Math. Anal. Appl.* **311** (2005), 495–505.

[4] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[5] K. Diethelm, *The Analysis of Fractional Differential Equations. An Application-oriented Exposition Using Differential Operators of Caputo Type*. Lecture Notes in Mathematics, 2004, Springer-Verlag, Berlin, 2010.

[6] U. Elias, Green's functions for a non-disconjugate differential operator, *J. Differential Equations* **37** (1980), 318–350.

[7] U. Elias, *Oscillation Theory of Two-Term Differential Equations*, Kluwer, Dordrecht, 1997.

[8] P. Eloe and J. Henderson, Focal points and comparison theorems for a class of two point boundary value problems, *J. Differential Equations* **103** (1993), 375–386.

[9] P. Eloe and E. Kaufmann, A singulary boundary value problem for a right disfocal linear differential operator, *Comput. Math. Appl.*, **28** (1994), 93–99.

[10] P. Eloe and J. Neugebauer, Conjugate points for fractional differential equations, *Fract. Calc. Appl. Anal.*, **17** (2014), no. 3, 855-871.

[11] P. Hartman, *Ordinary Differential Equations*, Birkhäuser, Boston, 1964.

[12] E.R. Kaufmann and E. Mboumi, Positive solutions of a boundary value problem for a nonlinear fractional differential equation, *Electron. J. Qual. Theory Differ. Equ.* (2008), no. 3, 11 pp.

[13] A.A. Kibas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam, 2006.

[14] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, Inc., New York, 1993.

[15] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[16] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[17] Y. Su and Z. Feng, Existence theory for an arbitrary order fractional differential equation with deviating argument, *Acta Appl. Math.* **118** (2012), 81–105.