Communications in Applied Analysis **19** (2015), 463–486

HILLE AND NEHARI TYPE CRITERIA FOR HIGHER ORDER FUNCTIONAL DYNAMIC EQUATIONS

LYNN ERBE[1] AND TAHER S. HASSAN[2]

[1]Department of Mathematics, University of Nebraska
Lincoln, NE 68588 USA
E-mail: lerbe@unl.edu

[2] Department of Mathematics, Mansoura University
Mansoura, 35516 Egypt
E-mail: tshassan@mans.edu.eg

ABSTRACT. In this paper, we study the higher order functional dynamic equation

$$\left\{r_{n-1}(t)\left(r_{n-2}(t)\left(\cdots(r_1(t)x^{\Delta}(t))^{\Delta}\cdots\right)^{\Delta}\right)^{\Delta}\right\}^{\Delta}+p\,(t)\,x\,(g\,(t))=0,$$

on a time scale $\mathbb{T}$, which is unbounded above, and where $n \geq 2$. We will extend the so-called Hille and Nehari type criteria to higher order dynamic equations on time scales. Our results are essentially new even for higher order differential and difference equations. Therefore, the results obtained extend and improve several known results in the literature on second-order and third-order dynamic equations. We illustrate these new results by means of several examples.

AMS (MOS) Subject Classification. 34K11, 39A10, 39A99.

1. INTRODUCTION

In this paper we study the asymptotic behavior of solutions of the following higher order functional dynamic equation:

$$\left\{r_{n-1}(t)\left(r_{n-2}(t)\left(\cdots(r_1(t)x^{\Delta}(t))^{\Delta}\cdots\right)^{\Delta}\right)^{\Delta}\right\}^{\Delta}+p\,(t)\,x\,(g\,(t))=0, \tag{1.1}$$

on a time scale $\mathbb{T}$, which is unbounded above, and where $n \geq 2$ and r_i, $i = 1, 2, \ldots, n-1$ are positive $rd-$continuous functions on $\mathbb{T}$ such that for $t_0 \in \mathbb{T}$,

$$\int_{t_0}^{\infty}\frac{\Delta s}{r_i(s)}=\infty, \quad i=1,2,\ldots,n-1; \tag{1.2}$$

We assume further that p is a nonnegative $rd-$continuous function on $\mathbb{T}$ such that $p \not\equiv 0$ and the function $g : \mathbb{T} \to \mathbb{T}$ such that $\lim_{t\to\infty} g(t) = \infty$ and $\tau(t) := \inf\{\sigma(t), g(t)\}$ is nondecreasing on $\mathbb{T}$, where the forward jump operator $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. Since we are interested in the oscillatory and asymptotic behavior of solutions near

Received December 15, 2014

infinity, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. Throughout this paper, we let

$$x^{[i]} := r_i \, \left(x^{[i-1]}\right)^{\Delta}, \quad i = 1, 2, \ldots, n \text{ with } r_n = 1 \text{ and } x^{[0]} = x.$$

We will assume that the reader is familiar with the basic facts of time scales and time scale notation. For an excellent introduction to the calculus on time scales, see Bohner and Peterson [5, 6]. By a solution of Eq. (1.1) we mean a nontrivial real-valued function $x \in C^1_{rd}[T_x, \infty)_{\mathbb{T}}$ for some $T_x \geq t_0$ such that $x^{[i]} \in C^1_{rd}[T_x, \infty)_{\mathbb{T}}$, $i = 1, 2, \ldots, n-1$ and $x(t)$ satisfies Eq. (1.1) on $[T_x, \infty)_{\mathbb{T}}$, where C_{rd} is the space of right-dense continuous functions. In the following, we state some oscillation results for differential and difference equations that will be related to our oscillation results for (1.1) on time scales and explain the important contributions of this paper. In 1918, Fite [23] studied the oscillatory behavior of solutions of the second order linear differential equation

$$x''(t) + p(t)x(t) = 0, \tag{1.3}$$

and showed that if

$$\int_{t_0}^{\infty} p(s)ds = \infty, \tag{1.4}$$

then every solution of equation (1.3) oscillates. In 1948, Hille [15] improved the condition (1.4) and showed that if

$$\liminf_{t\to\infty} t \int_t^{\infty} p(s)ds > \frac{1}{4}, \tag{1.5}$$

then every solution of (1.3) oscillates. In 1957, Nehari [42] proved that if

$$\liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} s^2 p(s)ds > \frac{1}{4}, \tag{1.6}$$

then every solution of (1.3) oscillates. We note that the inequalities (1.5) and (1.6) are 'sharp' and can not be weakened. Indeed, let $p(t) = 1/4t^2$ for $t \geq 1$. Then we have

$$\liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} s^2 p(s)ds = \liminf_{t\to\infty} t \int_t^{\infty} p(s)ds = \frac{1}{4},$$

and the second-order Euler-Cauchy differential equation

$$x''(t) + \frac{1}{4t^2} x(t) = 0, \quad t \geq 1, \tag{1.7}$$

has a non-oscillatory solution $x(t) = \sqrt{t}$. In other words the constant $1/4$ is the lower bound for oscillation for all solutions of (1.7). In 1971 Wong [48] generalized the Hille-type condition (1.5) for the delay equation

$$x''(t) + p(t)x(\tau(t)) = 0, \tag{1.8}$$

where $\tau(t) \geq \alpha t$ with $0 < \alpha < 1$, and proved that if

$$\liminf_{t\to\infty} t \int_t^\infty p(s)ds > \frac{1}{4\alpha}, \tag{1.9}$$

then every solution of (1.8) is oscillatory. In 1973 Erbe [9] improved the condition (1.9) and proved that if

$$\liminf_{t\to\infty} t \int_t^\infty p(s)\frac{\tau(s)}{s}ds > \frac{1}{4}, \tag{1.10}$$

then every solution of (1.8) oscillates where $\tau(t) \leq t$. In 1984 Ohriska [43] proved that, if

$$\limsup_{t\to\infty} t \int_t^\infty p(s)\left(\frac{\tau(s)}{s}\right)ds > 1, \tag{1.11}$$

then every solution of (1.8) oscillates. Note that when $p(t) = \frac{\lambda}{t\tau(t)}$, with $\tau(t) \leq t$, (1.8) reduces to the second-order delay differential equation

$$x''(t) + \frac{\lambda}{t\tau(t)}x(\tau(t)) = 0, \quad t \geq t_0. \tag{1.12}$$

From (1.10) we see that every solution of (1.12) is oscillatory if $\lambda > \frac{1}{4}$ and non-oscillatory if $\lambda \leq \frac{1}{4}$, with oscillation constant $1/4$ (see [1]).

Erbe, Hassan, Peterson, and Saker in [16, 17] extended the Hille and Nehari oscillation criteria to the second order superlinear half-linear delay dynamic equation

$$(r(t)(x^\Delta(t))^\gamma)^\Delta + p(t)x^\gamma(\tau(t)) = 0,$$

where γ is a quotient of odd positive integers, $\tau(t) \leq t$ and

$$r^\Delta(t) \geq 0 \qquad \text{and} \qquad \int_{t_0}^\infty \tau^\gamma(t)p(t)\Delta t = \infty. \tag{1.13}$$

Also, in [13, 14, 15] the authors extended Hille and Nehari type oscillation criteria to more general second order dynamic equations. Erbe, Peterson, and Saker [20] established Hille and Nehari oscillation criteria for the third order dynamic equation

$$x^{\Delta\Delta\Delta}(t) + p(t)x(t) = 0,$$

and their work was further extended by Erbe, Hassan, and Peterson [12] to the equation

$$\left(r_2(t)\left[(r_1(t)x^\Delta(t))^\Delta\right]^\gamma\right)^\Delta + p\,(t)\,x^\gamma\,(g\,(t)) = 0,$$

where γ is a quotient of odd positive integers, $g(t) \leq t$, $g^\Delta(t) \geq 0$ and where r_1 is a Δ-differentiable function with $r_1^\Delta(t) \geq 0$. Wang and Xu in [46] considered the ordinary dynamic equation

$$\left(r_2(t)\left[(r_1(t)x^\Delta(t))^\Delta\right]^\gamma\right)^\Delta + p\,(t)\,x^\gamma\,(t) = 0,$$

where $\gamma \geq 1$ is a quotient of odd positive integers and with the condition

$$\lim_{t\to\infty} \frac{R^\sigma(t)}{R(t)} = 1, \tag{1.14}$$

where

$$\delta_1(t,t_1) := \int_{t_1}^{t} \frac{\Delta s}{r_2^{1/\gamma}(s)}, \quad \delta_2(t,t_1) := \int_{t_1}^{t} \frac{\delta_1(s,t_1)}{r_1(s)}\Delta s,$$

$$r(t) := \frac{\gamma}{r_1(t)}\delta_1(t,t_1)\delta_2^{\gamma-1}(t,t_1) \text{ and } R(t) := \int_{t_1}^{t} r(s)\Delta s.$$

Very recently, Agarwal, Bohner, Li and Zhang [2] extended the Hille and Nehari oscillation criteria to the third order delay dynamic equation

$$\left(r_2(t)(r_1(t)x^{\Delta}(t))^{\Delta}\right)^{\Delta} + p\,(t)\,x\,(\tau(t)) = 0,$$

where $\tau(t) \le t$ on $[t_0,\infty)_{\mathbb{T}}$. The results in [2] included the results which were established in [20] and without condition (1.14). For more results on dynamic equations, we refer the reader to the papers [4, 7, 8, 10, 11, 18, 19, 21, 22, 24, 26, 27, 30, 31, 32, 33, 34, 35, 39, 44, 45, 49, 25].

The purpose of this paper is to extend the Hille and Nehari oscillation criteria to higher order dynamic equation (1.1) without assuming the conditions (1.13) and (1.14). The results in this paper improve the results in [20] and [2] for higher order dynamic equations.

To simplify the discussion below, we introduce the following notation: For any $u, v \in \mathbb{T}$, define

$$R_i(v,u) := \int_u^v \frac{\Delta s}{r_i(s)}, \quad i = 1, 2, \ldots, n-1, \tag{1.15}$$

and for a fixed $m \in \{0,\ldots,n-1\}$, define the functions $\bar{R}_{m,i}(v,u)$, $i = 1, 2\ldots, m$, and $p_i(t)$, $i = 1,\ldots,n$, by the following recurrence formulas:

$$\bar{R}_{m,i}(v,u) := \begin{cases} \int_u^v \bar{R}_{m,i-1}(s,u)\big/\, r_{m-i+1}\,(s)\,\Delta s, & i = 1,\ldots,m, \\ 1 & i = 0; \end{cases} \tag{1.16}$$

and

$$p_i(t) := \begin{cases} \dfrac{1}{r_{n-i}(t)} \int_t^{\infty} p_{i-1}(s)\Delta s, & i = 1,\ldots,n-1, \\ p\,(t)\,, & i = 0, \end{cases} \tag{1.17}$$

provided the improper integrals involved are convergent. Note that for $i = 1,\ldots,m$, $\bar{R}_{m,i}(v,u) \ge 0$ if $u \le v$, and $(-1)^{n-i-1}\bar{R}_{m,i}(v,u) \ge 0$ if $u \ge v$.

2. MAIN RESULTS

Lemma 2.1. *Assume Eq.* (1.1) *has an eventually positive solution* $x(t)$. *Then there exists an integer* $m \in \{0,\ldots,n-1\}$ *with* $m+n$ *odd such that*

$$x^{[k]}(t) > 0 \quad \textit{for} \quad k = 0, 1, \ldots, m, \tag{2.1}$$

and

$$(-1)^{m+k}\, x^{[k]}(t) > 0 \quad \textit{for} \quad k = m+1, m+2, \ldots, n, \tag{2.2}$$

eventually.

Proof. Since $x(t)$ is an eventually positive solution of Eq. (1.1) then without loss of generality, we may assume $x(t) > 0$ and $x(g(t)) > 0$ on $[t_0, \infty)_{\mathbb{T}}$. From (1.1), we have that for $t \in [t_0, \infty)_{\mathbb{T}}$,

$$x^{[n]}(t) = -p(t)\, x(g(t)) < 0. \tag{2.3}$$

This implies that $x^{[i]}(t)$, $i = 1, 2, \dots, n-1$, are eventually monotone and hence are of one sign. There are two possibilities:

(a) $x^{[k]}(t)$ and $x^{[k-1]}(t)$ have opposite signs eventually for $k = 1, 2, \dots, n$; or

(b) there exists a largest $m \in \{1, 2, \dots, n-1\}$ such that $x^{[m]}(t)x^{[m-1]}(t) > 0$ eventually.

If (a) holds, then (2.1) and (2.2) hold with $m = 0$.

Assume (b) holds with $x^{[m]}(t) < 0$ and $x^{[m-1]}(t) < 0$ for $t \ge t_1$, where $t_1 \in [t_0, \infty)_{\mathbb{T}}$. Then

$$\begin{aligned} x^{[m-2]}(t) &= x^{[m-2]}(t_1) + \int_{t_1}^{t} \frac{x^{[m-1]}(s)}{r_{m-1}(s)} \Delta s \\ &< x^{[m-2]}(t_1) + x^{[m-1]}(t_1) \int_{t_1}^{t} \frac{\Delta s}{r_{m-1}(s)}. \end{aligned}$$

By (1.2) with $i = m-1$, $\lim_{t\to\infty} x^{[m-2]}(t) = -\infty$. Hence $x^{[m-2]}(t) < 0$ eventually. By the same reasoning, we see that $x^{[k]}(t) < 0$ eventually for $k = m-2, m-3, \dots, 0$. This contradicts the assumption that $x(t)$ is eventually positive.

Assume (b) holds with $x^{[m]}(t) > 0$ and $x^{[m-1]}(t) > 0$ eventually. By (2.3) we note that $m + n$ must be an odd number. Using a similar argument as above, we see that $x^{[k]}(t) > 0$ eventually for $k = m-2, m-3, \dots, 0$. Therefore, (2.1) and (2.2) hold with this m. □

Lemma 2.2. *Assume Eq.* (1.1) *has an eventually positive solution* $x(t)$ *and* $m \in \{1, \dots, n-1\}$ *is as given in Lemma* 2.1 *such that* (2.1) *and* (2.2) *hold for* $t \ge t_1 \in [t_0, \infty)_{\mathbb{T}}$. *Then the following inequalities hold for* $t \in (t_1, \infty)_{\mathbb{T}}$*:*

(a)

$$\left[x^{[m-1]}(t) / R_m(t, t_1)\right]^{\Delta} < 0; \tag{2.4}$$

(b) for $i = 0, 1, \dots, m-1$

$$x^{[i]}(t) \ge x^{[m-1]}(t)\, \bar{R}_{m,m-i}(t, t_1) / R_m(t, t_1). \tag{2.5}$$

Proof. **(a)** From (2.1) and (2.2), we get for $t \in [t_1, \infty)_{\mathbb{T}}$

$$\begin{aligned} x^{[m-1]}(t) &= x^{[m-1]}(t_1) + \int_{t_1}^{t} \frac{x^{[m]}(s)}{r_m(s)} \Delta s \\ &\ge x^{[m]}(t) \int_{t_1}^{t} \frac{\Delta s}{r_m(s)} \\ &= x^{[m]}(t)\, R_m(t, t_1). \end{aligned}$$

Noting that

$$\left[\frac{x^{[m-1]}(t)}{R_m(t,t_1)}\right]^{\Delta}=\frac{1/r_m(t)}{R_m(t,t_1)R_m(\sigma(t),t_1)}\left[R_m(t,t_1)x^{[m]}(t)-x^{[m-1]}(t)\right],$$

we have

$$\left[\frac{x^{[m-1]}(t)}{R_m(t,t_1)}\right]^{\Delta}<0 \qquad \text{for } t\in(t_1,\infty)_{\mathbb{T}}.$$

(b) It is easy to show, using (1.16) and (2.2) that (2.5) holds for $i=m-1$ since $\bar{R}_{m,1}=R_m$. By (2.1) and the fact that $x^{[m-1]}(t)/R_m(t,t_1)$ is decreasing on $(t_1,\infty)_{\mathbb{T}}$, we have for $t\in(t_1,\infty)_{\mathbb{T}}$

$$\begin{aligned}
x^{[m-2]}(t) &\geq x^{[m-2]}(t)-x^{[m-2]}(t_1)=\int_{t_1}^{t}\left(x^{[m-2]}(s)\right)^{\Delta}\Delta s\\
&= \int_{t_1}^{t}\frac{x^{[m-1]}(s)}{R_m(s,t_1)}\frac{R_m(s,t_1)}{r_{m-1}(s)}\Delta s\\
&\geq \frac{x^{[m-i]}(t)}{R_m(t,t_1)}\int_{t_1}^{t}\frac{R_m(s,t_1)}{r_{m-i}(s)}\Delta s\\
&= \frac{x^{[m-i]}(t)}{R_m(t,t_1)}\int_{t_1}^{t}\frac{\bar{R}_{m,i}(s,t_1)}{r_{m-i}(s)}\Delta s\\
&= \frac{x^{[m-i]}(t)}{R_m(t,t_1)}\bar{R}_{m,i+1}(t,t_1).
\end{aligned}$$

This shows that (2.5) holds for $i=m-2$. Next assume (2.5) holds for some $i\in\{1,\dots,m-1\}$. Then for $t\in(t_1,\infty)_{\mathbb{T}}$, we have

$$x^{[i]}(t)\geq\frac{x^{[m-1]}(t)}{R_m(t,t_1)}\bar{R}_{m,m-i}(t,t_1),$$

which implies

$$\left(x^{[i-1]}(t)\right)^{\Delta}\geq\frac{x^{[m-1]}(t)}{R_m(t,t_1)}\frac{\bar{R}_{m,m-i}(t,t_1)}{r_i(t)}.$$

Replacing t by s in the above inequality and then integrating it from t_1 to $t\in(t_1,\infty)_{\mathbb{T}}$, we have

$$\begin{aligned}
x^{[i-1]}(t) &\geq x^{[i-1]}(t)-x^{[i-1]}(t_1)\geq\int_{t_1}^{t}\frac{x^{[m-1]}(s)}{R_m(s,t_1)}\frac{\bar{R}_{m,m-i}(s,t_1)}{r_i(s)}\Delta s\\
&\geq \frac{x^{[m-1]}(t)}{R_m(t,t_1)}\int_{t_1}^{t}\frac{\bar{R}_{m,m-i}(s,t_1)}{r_i(s)}\Delta s\\
&= \frac{x^{[m-1]}(t)}{R_m(t,t_1)}\bar{R}_{m,m-i+1}(t,t_1).
\end{aligned}$$

This shows that (2.5) holds for $i-1$. Therefore, by induction, (2.5) holds for all $i=0,1,\dots,m-1$. □

Lemma 2.3. *Assume Eq.* (1.1) *has an eventually positive solution* $x(t)$ *and let* m *be as given in Lemma 2.1 such that* $m\in\{0,\dots,n-1\}$ *and* (2.1) *and* (2.2) *hold for* $t\geq t_1\in[t_0,\infty)_{\mathbb{T}}$. *Then the following inequalities hold for* $t\in[t_1,\infty)_{\mathbb{T}}$*:*

(a) If $m \geq 1$, then for $i = m, \ldots, n-1$, $\int_t^\infty p_{n-i-1}(s)\Delta s < \infty$, and

$$(-1)^{m+i} x^{[i]}(t) \geq x(\tau(t)) \int_t^\infty p_{n-i-1}(s)\Delta s; \tag{2.6}$$

(b) for $i = 0, 1, \ldots, m$,

$$x^{[i]}(t) \geq x^{[m]}(t)\, \bar{R}_{m,m-i}(t, t_1) \tag{2.7}$$

Proof. **(a)** Note that $m \in \{1, \ldots, n-1\}$ implies that $x^{[1]}(t) > 0$ and $x^{[n-1]}(t) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. This implies that $x(t)$ is strictly increasing on $[t_1, \infty)_{\mathbb{T}}$. Replacing t by s in Eq. (1.1), integrating from $t \geq t_1$ to $v \in [t, \infty)_{\mathbb{T}}$, and using the fact that τ is nondecreasing, we have

$$\begin{aligned} x^{[n-1]}(t) &\geq -x^{[n-1]}(v) + x^{[n-1]}(t) = \int_t^v p(s)\, x(g(s))\, \Delta s \\ &\geq \int_t^v p(s)\, x(\tau(s))\, \Delta s \geq x(\tau(t)) \int_t^v p(s)\, \Delta s \\ &= x(\tau(t)) \int_t^v p_0(s)\, \Delta s. \end{aligned}$$

Taking limits as $v \to \infty$ we obtain that

$$x^{[n-1]}(t) \geq x(\tau(t)) \int_t^\infty p_0(s)\, \Delta s \qquad \text{for } t \in [t_1, \infty)_{\mathbb{T}}.$$

This shows that $\int_t^\infty p_0(s)\Delta s < \infty$ and (2.6) holds for $i = n-1$. Next, we assume $\int_t^\infty p_{i+1}(s)\Delta s < \infty$ and that (2.6) holds for some $i \in \{m+1, \ldots, n-1\}$. Then

$$(-1)^{m+i} x^{[i]}(t) \geq x(\tau(t)) \int_t^\infty p_{n-i-1}(s)\Delta s \qquad \text{for } t \in [t_1, \infty)_{\mathbb{T}}.$$

It follows that

$$(-1)^{m+i} \left(x^{[i-1]}(t)\right)^\Delta \geq x(\tau(t)) \frac{1}{r_i(t)} \int_t^\infty p_{n-i-1}(s)\Delta s = x(\tau(t))\, p_{n-i}(t).$$

Replacing t by s in the above inequality and then integrating it from $t \geq t_1$ to $v \in [t, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned} (-1)^{m+i-1} x^{[i-1]}(t) &\geq (-1)^{m+i}(x^{[i-1]}(v) - x^{[i-1]}(t)) \\ &= \int_t^v x(\tau(s))\, p_{n-i}(s)\Delta s \\ &\geq x(\tau(t)) \int_t^v p_{n-i}(s)\Delta s. \end{aligned}$$

Taking limits as $v \to \infty$ we obtain that

$$(-1)^{m+i-1} x^{[i-1]}(t) \geq x(\tau(t)) \int_t^\infty p_{n-i}(s)\Delta s.$$

This shows that $\int_t^\infty p_{n-i}(s)\Delta s < \infty$ and (2.6) holds for $i-1$. Therefore, the conclusion follows by induction.

(b) It is easy to show, using (1.16) and (2.2), that (2.7) holds for $i = m$ since $\bar{R}_{m,0} = 1$. Also by (2.2) and $x^{[m+1]}(t) < 0$ and hence $x^{[m]}(t)$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$. Let $t \in [t_1, \infty)_{\mathbb{T}}$. Then

$$\begin{aligned} x^{[m-1]}(t) &= x^{[m-1]}(t_1) + \int_{t_1}^{t} \left(x^{[m-1]}(s)\right)^{\Delta} \Delta s \\ &= x^{[m-1]}(t_1) + \int_{t_1}^{t} \frac{x^{[m]}(s)}{r_m(s)} \Delta s \\ &\geq x^{[m]}(t) \int_{t_1}^{t} \frac{\Delta s}{r_m(s)} = x^{[m]}(t)\, \bar{R}_{m,1}(t, t_1). \end{aligned}$$

This shows that (2.7) holds for $i = m-1$. Assume (2.7) holds for some $i \in \{1, \ldots, m\}$. Then

$$x^{[i]}(t) \geq x^{[m]}(t)\, \bar{R}_{m,m-i}(t, t_1) \qquad \text{for } t \in [t_1, \infty)_{\mathbb{T}}.$$

which implies

$$\left(x^{[i-1]}(t)\right)^{\Delta} \geq x^{[m]}(t)\, \bar{R}_{m,m-i}(t, t_1) / r_i(t).$$

Replacing t by s in the above inequality and then integrating it for s from t_1 to t with $t \geq t_1$, we have

$$\begin{aligned} x^{[i-1]}(t) &\geq x^{[i-1]}(t_1) + \int_{t_1}^{t} x^{[m]}(s)\, \bar{R}_{m,m-i}(s, t_1) / r_i(s) \Delta s \\ &\geq x^{[m]}(t) \int_{t_1}^{t} \bar{R}_{m,m-i}(s, t_1) / r_i(s) \Delta s \\ &= x^{[m]}(t)\, \bar{R}_{m,m-i+1}(t, t_1). \end{aligned}$$

This shows that (2.7) holds for $i - 1$. By induction, it follows that (2.7) holds for all $i = 0, 1, \ldots, m$. □

3. CRITERIA FOR EVEN ORDER EQUATIONS

In this section, we establish Hille, Nehari, Ohriska and Fite-Wintner type criteria for the even order dynamic equation (1.1) when n is even. It follows from Lemma 2.1 that there exists an odd $m \in \{1, \ldots, n-1\}$ such that (2.1) and (2.2) hold eventually.

Theorem 3.1. *Assume that, for every odd number $i \in \{1, \ldots, n-1\}$ and for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$, we have*

$$\liminf_{t \to \infty} R_i(t, T) \int_{t}^{\infty} P_i(s, T)\, \Delta s > \frac{1}{4}, \tag{3.1}$$

where

$$P_i(s, T) := p_{n-i-1}(s) \bar{R}_{i,i}(\tau(s), T) / R_1(\sigma(s), T). \tag{3.2}$$

Then every solution of Eq. (1.1) is oscillatory.

Proof. Assume Eq. (1.1) has a non-oscillatory solution $x(t)$. Then without loss of generality, we may assume $x(t) > 0$ and $x(g(t)) > 0$ on $[t_0, \infty)_{\mathbb{T}}$. It follows from Lemma 2.1 that there exists an odd $m \in \{1, \ldots, n-1\}$ such that (2.1) and (2.2) hold for $t \geq t_1 \in [t_0, \infty)_{\mathbb{T}}$. Define

$$z_m(t) := \frac{x^{[m]}(t)}{x^{[m-1]}(t)}. \tag{3.3}$$

By the quotient rule, we have

$$\begin{aligned} z_m^{\Delta}(t) &= \frac{x^{[m-1]}(t)\left(x^{[m]}(t)\right)^{\Delta} - x^{[m]}(t)\left[x^{[m-1]}(t)\right]^{\Delta}}{x^{[m-1]}(t)\, x^{[m-1]}(\sigma(t))} \\ &= \frac{\left(x^{[m]}(t)\right)^{\Delta}}{x^{[m-1]}(\sigma(t))} - \frac{\left[x^{[m-1]}(t)\right]^{\Delta}}{x^{[m-1]}(\sigma(t))} z_m(t). \end{aligned} \tag{3.4}$$

By Lemma 2.3, Part (a) we have that for $i = m+1$

$$-x^{[m+1]}(t) \geq x(\tau(t)) \int_t^{\infty} p_{n-m-2}(s)\Delta s,$$

which, together with (1.17), implies that for $t \in [t_1, \infty)_{\mathbb{T}}$

$$-\left(x^{[m]}(t)\right)^{\Delta} \geq x(\tau(t))\, p_{n-m-1}(t). \tag{3.5}$$

(i) Assume $m = 1$. In this case, by (1.15) and (1.16) we see that

$$\bar{R}_{1,1}(\tau(t), t_1) = R_1(\tau(t), t_1) = \int_{t_1}^{\tau(t)} \frac{\Delta s}{r_1(s)}.$$

From (3.5) and Lemma 2.2, Part (a), we have for $\tau(t) \in (t_1, \infty)_{\mathbb{T}}$

$$\begin{aligned} -\left(x^{[1]}(t)\right)^{\Delta} &\geq x(\tau(t))\, p_{n-2}(t) = \frac{x(\tau(t))}{R_1(\tau(t), t_1)} R_1(\tau(t), t_1) p_{n-2}(t) \\ &= \frac{x(\tau(t))}{R_1(\tau(t), t_1)} \bar{R}_{1,1}(\tau(t), t_1) p_{n-2}(t) \\ &\geq \frac{x(\sigma(t))}{R_1(\sigma(t), t_1)} R_1(\tau(t), t_1) p_{n-2}(t) \\ &= p_{n-2}(t) x(\sigma(t))\, \bar{R}_{1,1}(\tau(t), t_1) / R_1(\sigma(t), t_1). \end{aligned}$$

(ii) Assume $m \geq 2$. By Lemma 2.2, Part (b) with $i = 0$, we get

$$x(t) \geq \frac{x^{[m-1]}(t)}{R_m(t, t_1)} \bar{R}_{m,m}(t, t_1). \tag{3.6}$$

Then by Lemma 2.2, Part (a), we see that for $\tau(t) \in (t_1, \infty)_{\mathbb{T}}$

$$\begin{aligned} x(\tau(t)) &\geq \frac{x^{[m-1]}(\tau(t))}{R_m(\tau(t), t_1)} \bar{R}_{m,m}(\tau(t), t_1) \\ &\geq \frac{x^{[m-1]}(\sigma(t))}{R_m(\sigma(t), t_1)} \bar{R}_{m,m}(\tau(t), t_1). \end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.5), we obtain that for $\tau(t) \in (t_1, \infty)_{\mathbb{T}}$

$$-\left(x^{[m]}(t)\right)^{\Delta} \geq p_{n-m-1}(t) x^{[m-1]}(\sigma(t))\ \bar{R}_{m,m}(\tau(t), t_1) / R_m(\sigma(t), t_1).$$

Combining cases (i) and (ii) we see that for $\tau(t) \in (t_1, \infty)_{\mathbb{T}}$

$$-\left(x^{[m]}(t)\right)^{\Delta} \geq x^{[m-1]}(\sigma(t))\, P_m(t, t_1) \quad \text{for } t \in [t_2, \infty)_{\mathbb{T}}. \tag{3.8}$$

Substituting (3.8) into (3.4) we get

$$\begin{aligned} z_m^{\Delta}(t) &\leq -P_m(t, t_1) - \frac{\left[x^{[m-1]}(t)\right]^{\Delta}}{x^{[m-1]}(\sigma(t))} z_m(t) \\ &\leq -P_m(t, t_2) - \frac{\left[x^{[m-1]}(t)\right]^{\Delta}}{x^{[m-1]}(\sigma(t))} z_m(t) \quad \text{for } t \in [t_2, \infty)_{\mathbb{T}}. \end{aligned} \tag{3.9}$$

Since

$$\frac{\left[x^{[m-1]}(t)\right]^{\Delta}}{x^{[m-1]}(\sigma(t))} = \frac{x^{[m]}(t)}{r_m(t) x^{[m-1]}(\sigma(t))} \geq \frac{x^{[m]}(\sigma(t))}{r_m(t) x^{[m-1]}(\sigma(t))} = \frac{z_m(\sigma(t))}{r_m(t)},$$

we get from (3.9) that

$$-z_m^{\Delta}(t) \geq P_m(t, t_2) + \frac{z_m(t) z_m(\sigma(t))}{r_m(t)} \quad \text{for } t \in [t_2, \infty)_{\mathbb{T}}.$$

Replacing t by s in the above inequality and then integrating it from $t \geq t_2$ to $v \in [t, \infty)_{\mathbb{T}}$ and using the fact that $z_m > 0$, we have

$$z_m(t) \geq -z_m(v) + z_m(t) \geq \int_t^v P_m(s, t_2)\, \Delta s + \int_t^v \frac{z_m(s) z_m(\sigma(s))}{r_m(s)} \Delta s.$$

Taking limits as $v \to \infty$ we obtain that

$$z_m(t) \geq \int_t^{\infty} P_m(s, t_2)\, \Delta s + \int_t^{\infty} \frac{z_m(s) z_m(\sigma(s))}{r_m(s)} \Delta s. \tag{3.10}$$

Multiplying both sides of (3.10) by $R_m(t, t_2)$, we get (suppressing arguments)

$$\begin{aligned} R_m z_m &\geq R_m \int_t^{\infty} P_m\ \Delta s + R_m \int_t^{\infty} \frac{z_m\ z_m^{\sigma}}{r_m} \Delta s \\ &= R_m \int_t^{\infty} P_m\ \Delta s + R_m \int_t^{\infty} \frac{1/r_m}{R_m R_m^{\sigma}} R_m z_m\ R_m^{\sigma} z_m^{\sigma}\ \Delta s \\ &= R_m \int_t^{\infty} P_m\ \Delta s + R_m \int_t^{\infty} \left[\frac{-1}{R_m}\right]^{\Delta} R_m z_m\ R_m^{\sigma} z_m^{\sigma} \Delta s. \end{aligned}$$

Let $\varepsilon > 0$ and pick $t_3 \in [t_2, \infty)_{\mathbb{T}}$, sufficiently large, so that

$$R_m(t, t_2)\, z_m(t) \geq r_m^* - \varepsilon \text{ for } t \in [t_3, \infty)_{\mathbb{T}}, \tag{3.11}$$

where

$$r_m^* := \liminf_{t \to \infty} R_m(t, t_2)\, z_m(t). \tag{3.12}$$

Therefore

$$\begin{aligned} R_m z_m &\geq R_m \int_t^{\infty} P_m\ \Delta s + (r_m^* - \varepsilon)^2 R_m \int_t^{\infty} \left[\frac{-1}{R_m}\right]^{\Delta} \Delta s \\ &\overset{(1.2)}{=} R_m \int_t^{\infty} P_m\ \Delta s + (r_m^* - \varepsilon)^2. \end{aligned}$$

Taking the lim inf of both sides as $t \to \infty$ we get that

$$r_m^* \geq \liminf_{t\to\infty} R_m(t, t_2) \int_t^\infty P_m(s, t_2)\,\Delta s + (r_m^* - \varepsilon)^2 .$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\liminf_{t\to\infty} R_m(t, t_2) \int_t^\infty P_m(s, t_2)\,\Delta s \leq r_m^* - (r_m^*)^2 = \frac{1}{4} - \left(r_m^* - \frac{1}{2}\right)^2 \leq \frac{1}{4}.$$

which contradicts (3.1). This completes the proof. □

Theorem 3.2. *Assume that, for every odd number $i \in \{1, \ldots, n-1\}$ and for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$,*

$$\liminf_{t\to\infty} \frac{1}{R_i(t, T)} \int_T^t R_i^2(\sigma(s), T)\, P_i(s, T)\,\Delta s > \frac{L_m^*}{1 + L_m^*}, \tag{3.13}$$

where P_i is defined by (3.2) *and*

$$L_m^* := \limsup_{t\to\infty} R_i(\sigma(t), T)/\, R_i(t, T) . \tag{3.14}$$

Then every solution of Eq. (1.1) *is oscillatory.*

Proof. Assume Eq. (1.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume $x(t) > 0$ and $x(g(t)) > 0$ on $[t_0, \infty)_{\mathbb{T}}$. It follows from Lemma 2.1 that there exists an odd $m \in \{1, \ldots, n-1\}$ such that (2.1) and (2.2) hold for $t \geq t_1 \in [t_0, \infty)_{\mathbb{T}}$. As shown in the proof of Theorem 3.1

$$z_m^{\Delta}(t) \leq -P_m(t, t_2) - \frac{z_m(t) z_m(\sigma(t))}{r_m(t)} \quad \text{for } t \in [t_2, \infty)_{\mathbb{T}}, \tag{3.15}$$

where z_m is defined by (3.3). Multiplying both sides of (3.15) by $R_m^2(\sigma(t), t_2)$, we get

$$\begin{aligned} R_m^2(\sigma(t), t_2)\, z_m^{\Delta}(t) &\leq -R_m^2(\sigma(t), t_2)\;\; P_m(t, t_2) \\ &\quad - \frac{1}{r_m(t)} R_m(\sigma(t), t_2)\, z_m(t)\;\; R_m(\sigma(t), t_2)\, z_m(\sigma(t)) \\ &\leq -R_m^2(\sigma(t), t_2)\;\; P_m(t, t_2) \\ &\quad - \frac{1}{r_m(t)} R_m(t, t_2)\, z_m(t)\;\; R_m(\sigma(t), t_2)\, z_m(\sigma(t)). \end{aligned}$$

Let $\varepsilon > 0$ and pick $t_3 \in [t_2, \infty)_{\mathbb{T}}$, sufficiently large, so that

$$r_m^* - \varepsilon \leq R_m(t, t_2)\, z_m(t) \leq R_m^* + \varepsilon,$$

and

$$l_m^* - \varepsilon \leq \frac{R_m(\sigma(s), t_2)}{R_m(s, t_2)} \leq L_m^* + \varepsilon.$$

where r_m^* is defined by (3.12),

$$R_m^* := \limsup_{t\to\infty} R_m(t, t_2)\, z_m(t), \tag{3.16}$$

and

$$L_m^* := \limsup_{t\to\infty} \frac{R_m(\sigma(s),t_2)}{R_m(s,t_2)},\ l_m^* := \liminf_{t\to\infty} \frac{R_m(\sigma(s),t_2)}{R_m(s,t_2)}$$

Therefore

$$R_m^2(\sigma(t),t_2)\, z_m^\Delta(t) \le -R_m^2(\sigma(t),t_2)\ \ P_m(t,t_2) - \frac{(r_m^*-\varepsilon)^2}{r_m(t)} \quad \text{for } t\in[t_3,\infty)_{\mathbb{T}}.$$

Integrating the above inequality from t_3 to $t\in[t_3,\infty)_{\mathbb{T}}$, we get

$$\int_{t_3}^t R_m^2(\sigma(s),t_2)\, z_m^\Delta(s)\Delta s \le -\int_{t_3}^t R_m^2(\sigma(s),t_2)\, P_m(s,t_2)\,\Delta s - (r_m^*-\varepsilon)^2 \int_{t_3}^t \frac{\Delta s}{r_m(s)}.$$

Using integration by parts, we obtain

$$\begin{aligned}
R_m^2(t,t_2)\, z_m(t) &\le R_m^2(t_3,t_2)\, z_m(t_3) + \int_{t_3}^t \frac{R_m(\sigma(s),t_2)+R_m(s,t_2)}{r_m(s)} z_m(s)\Delta s \\
&\quad - \int_{t_3}^t R_m^2(\sigma(s),t_2)\, P_m(s,t_2)\,\Delta s - (r_m^*-\varepsilon)^2 R_m(t,t_3) \\
&\le R_m^2(t_3,t_2)\, z_m(t_3) + (1+L_m^*+\varepsilon)(R_m^*+\varepsilon)\int_{t_3}^t \frac{\Delta s}{r_m(s)} \\
&\quad - \int_{t_3}^t R_m^2(\sigma(s),t_2)\, P_m(s,t_2)\,\Delta s - (r_m^*-\varepsilon)^2 R_m(t,t_3) \\
&= R_m^2(t_3,t_2)\, z_m(t_3) + (1+L_m^*+\varepsilon)(R_m^*+\varepsilon) R_m(t,t_3) \\
&\quad - \int_{t_3}^t R_m^2(\sigma(s),t_2)\, P_m(s,t_2)\,\Delta s - (r_m^*-\varepsilon)^2 R_m(t,t_3),
\end{aligned}$$

or

$$\begin{aligned}
R_m^2(t,t_3)\, z_m(t) &\le R_m^2(t_3,t_2)\, z_m(t_3) + (1+L_m^*+\varepsilon)(R_m^*+\varepsilon) R_m(t,t_3) \\
&\quad - \int_{t_3}^t R_m^2(\sigma(s),t_3)\, P_m(s,t_3)\,\Delta s - (r_m^*-\varepsilon)^2 R_m(t,t_3).
\end{aligned}$$

Dividing both sides by $R_m(t,t_3)$ for $t\in(t_3,\infty)_{\mathbb{T}}$, we have

$$\begin{aligned}
R_m(t,t_3)\, z_m(t) &\le \frac{R_m^2(t_3,t_2)}{R_m(t,t_3)} z_m(t_3) + (1+L_m^*+\varepsilon)(R_m^*+\varepsilon) \\
&\quad - \frac{1}{R_m(t,t_3)} \int_{t_3}^t R_m^2(\sigma(s),t_3)\, P_m(s,t_3)\,\Delta s - (r_m^*-\varepsilon)^2 \\
&\le \frac{R_m^2(t_3,t_2)}{R_m(t,t_3)} z_m(t_3) + (1+L_m^*+\varepsilon)(R_m^*+\varepsilon) \\
&\quad - \frac{1}{R_m(t,t_3)} \int_{t_3}^t R_m^2(\sigma(s),t_3)\, P_m(s,t_3)\,\Delta s.
\end{aligned}$$

Taking the $\limsup$ of both sides as $t\to\infty$ and using (1.2) we get that

$$R_* \le (1+L_m^*+\varepsilon)(R_m^*+\varepsilon) - \liminf_{t\to\infty} \frac{1}{R_m(t,t_2)} \int_{t_2}^t R_m^2(\sigma(s),t_2)\, P_m(s,t_2)\,\Delta s.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\liminf_{t\to\infty} \frac{1}{R_m(t,t_2)} \int_{t_2}^{t} R_m^2(\sigma(s),t_2)\, P_m(s,t_2)\, \Delta s \le R_m^* L_m^*. \tag{3.17}$$

Also, we have from the proof of Theorem 3.1,

$$z_m^{\Delta}(t) \le -P_m(t,t_2) - \frac{\left[x^{[m-1]}(t)\right]^{\Delta}}{x^{[m-1]}(\sigma(t))} z_m(t) \quad \text{for } t \in [t_2,\infty)_{\mathbb{T}}.$$

Since

$$\begin{aligned} \frac{\left[x^{[m-1]}(t)\right]^{\Delta}}{x^{[m-1]}(\sigma(t))} &= \frac{\left[x^{[m-1]}(t)\right]^{\Delta}}{x^{[m-1]}(t)} \frac{x^{[m-1]}(t)}{x^{[m-1]}(\sigma(t))} = \frac{z_m(t)}{r_m(t)} \frac{x^{[m-1]}(t)}{x^{[m-1]}(t) + \mu(t)\left(x^{[m-1]}(t)\right)^{\Delta}} \\ &= \frac{z_m(t)}{r_m(t) + \mu(t) z_m(t)}, \end{aligned}$$

we have

$$z_m^{\Delta}(t) \le -P_m(t,t_2) - \frac{z_m^2(t)}{r_m(t) + \mu(t) z_m(t)} \quad \text{for } t \in [t_2,\infty)_{\mathbb{T}}. \tag{3.18}$$

Multiplying both sides of (3.18) by $R_m^2(\sigma(t),t_2)$, and integrating from t_2 to t ($t \ge t_2$) we get

$$\begin{aligned} \int_{t_2}^{t} R_m^2(\sigma(s),t_2)\, z_m^{\Delta}(s) \Delta s &\le -\int_{t_2}^{t} R_m^2(\sigma(s),t_2)\, P_m(s,t_2)\, \Delta s \\ &\quad - \int_{t_2}^{t} \frac{R_m^2(\sigma(s),t_2)\, z_m^2(s)}{r_m(s) + \mu(s) z_m(s)} \Delta s. \end{aligned}$$

Using integration by parts, we obtain

$$\begin{aligned} R_m^2(t,t_2)\, z_m(t) &\le -\int_{t_2}^{t} R_m^2(\sigma(s),t_2)\, P_m(s,t_2)\, \Delta s + \int_{t_2}^{t} \left[R_m^2(s,t_2)\right]^{\Delta} z_m(s) \Delta s \\ &\quad - \int_{t_2}^{t} \frac{R_m^2(\sigma(s),t_2)\, z_m^2(s)}{r_m(s) + \mu(s) z_m(s)} \Delta s \\ &= -\int_{t_2}^{t} R_m^2(\sigma(s),t_2)\, P_m(s,t_2)\, \Delta s \\ &\quad + \int_{t_2}^{t} \left[\frac{2R_m(\sigma(s),t_2)}{r_m(s)} - \frac{\mu(t)}{r_m^2(s)}\right] z_m(s) \Delta s \\ &\quad - \int_{t_2}^{t} \frac{R_m^2(\sigma(s),t_2)\, z_m^2(s)}{r_m(s) + \mu(s) z_m(s)} \Delta s \end{aligned}$$

since

$$\begin{aligned} \left[R_m^2(s,t_2)\right]^{\Delta} &= \left[R_m(\sigma(s),t_2) + R_m(s,t_2)\right] R_m^{\Delta}(s,t_2) \\ &= \frac{2R_m(\sigma(s),t_2)}{r_m(s)} - \frac{\mu(t)}{r_m^2(s)}. \end{aligned}$$

It follows that

$$\begin{aligned} R_m^2(t,t_2)\,z_m(t) &\le -\int_{t_2}^{t} R_m^2(\sigma(s),t_2)\,P_m(s,t_2)\,\Delta s \\ &\quad -\int_{t_2}^{t}\left[\frac{R_m^2(\sigma(s),t_2)\,z_m^2(s)}{r_m(s)+\mu(s)z_m(s)} - \frac{2R_m(\sigma(s),t_2)}{r_m(s)} \right. \\ &\quad \left. +\frac{\mu(t)}{r_m^2(s)}z_m(s)\right]\Delta s \\ &= -\int_{t_2}^{t} R_m^2(\sigma(s),t_2)\,P_m(s,t_2)\,\Delta s \\ &\quad -\int_{t_2}^{t}\left[\frac{R_m^2(\sigma(s),t_2)\,z_m^2(s)}{r_m(s)+\mu(s)z_m(s)} - \frac{2R_m(\sigma(s),t_2)}{r_m(s)} \right. \\ &\quad \left. +\frac{r_m(s)+\mu(t)z_m(s)}{r_m^2(s)} - \frac{1}{r_m(s)}\right]\Delta s \\ &= -\int_{t_2}^{t} R_m^2(\sigma(s),t_2)\,P_m(s,t_2)\,\Delta s \\ &\quad -\int_{t_2}^{t}\left[\left(\frac{R_m(\sigma(s),t_2)\,z_m(s)}{\sqrt{r_m(s)+\mu(s)z_m(s)}} - \frac{\sqrt{r_m(s)+\mu(s)z_m(s)}}{r_m(s)}\right)^2 \right. \\ &\quad \left. -\frac{1}{r_m(s)}\right]\Delta s \\ &\le -\int_{t_2}^{t} R_m^2(\sigma(s),t_2)\,P_m(s,t_2)\,\Delta s + R_m(t,t_2)\,. \end{aligned}$$

Dividing both sides by $R_m(t,t_2)$, we have

$$R_m(t,t_2)\,z_m(t) \le 1 - \frac{1}{R_m(t,t_2)}\int_{t_2}^{t} R_m^2(\sigma(s),t_2)\,P_m(s,t_2)\,\Delta s.$$

Taking the lim sup of both sides as $t \to \infty$ we get that

$$R_m^* \le 1 - \liminf_{t\to\infty}\frac{1}{R_m(t,t_2)}\int_{t_2}^{t} R_m^2(\sigma(s),t_2)\,P_m(s,t_2)\,\Delta s, \tag{3.19}$$

where R_m^* is defined by (3.16). From (3.17) and (3.19), we get

$$\begin{aligned} &\liminf_{t\to\infty}\frac{1}{R_m(t,t_2)}\int_{t_2}^{t} R_m^2(\sigma(s),t_2)\,P_m(s,t_2)\,\Delta s \\ &\quad \le \left(1-\liminf_{t\to\infty}\frac{1}{R_m(t,t_2)}\int_{t_2}^{t} R_m^2(\sigma(s),t_2)\,P_m(s,t_2)\,\Delta s\right)L_m^*, \end{aligned}$$

which implies

$$\liminf_{t\to\infty}\frac{1}{R_m(t,t_2)}\int_{t_2}^{t} R_m^2(\sigma(s),t_2)\,P_m(s,t_2)\,\Delta s \le \frac{L_m^*}{1+L_m^*},$$

which contradicts (3.13). This completes the proof. □

Theorem 3.3. *Assume that, for every odd number $i \in \{1, \ldots, n-1\}$ and for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$,*

$$\limsup_{t\to\infty} \bar{R}_{i,i}(\tau(t), T) \int_t^\infty p_{n-i-1}(s)\Delta s > 1. \tag{3.20}$$

Then every solution of Eq. (1.1) is oscillatory.

Proof. Assume Eq. (1.1) has a nonoscillatory solution $x(t)$. Then without loss of generality, assume $x(t) > 0$ and $x(g(t)) > 0$ on $[t_0, \infty)_{\mathbb{T}}$. It follows from Lemma 2.1 that there exists an odd $m \in \{1, \ldots, n-1\}$ such that (2.1) and (2.2) hold for $t \geq t_1 \in [t_0, \infty)_{\mathbb{T}}$. By Lemma 2.3, Part (a) we have that for $i = m$

$$x^{[m]}(t) \geq x(\tau(t)) \int_t^\infty p_{n-m-1}(s)\Delta s \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}.$$

Since $\left(x^{[m]}\right)^\Delta < 0$ on $[t_1, \infty)_{\mathbb{T}}$, we have

$$x^{[m]}(\tau(t)) \geq x(\tau(t)) \int_t^\infty p_{n-m-1}(s)\Delta s \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}. \tag{3.21}$$

By Lemma 2.3, Part (b) we have that for $i = 0$

$$x(t) \geq x^{[m]}(t)\, \bar{R}_{m,m}(t, t_1) \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}.$$

Then

$$x(\tau(t)) \geq x^{[m]}(\tau(t))\, \bar{R}_{m,m}(\tau(t), t_1) \quad \text{for } \tau(t) \in [t_1, \infty)_{\mathbb{T}}. \tag{3.22}$$

From (3.21) and (3.22), we obtain

$$\bar{R}_{m,m}(\tau(t), t_1) \int_t^\infty p_{n-m-1}(s)\Delta s \leq 1 \quad \text{for } \tau(t) \in [t_1, \infty)_{\mathbb{T}},$$

which implies

$$\limsup_{t\to\infty} \bar{R}_{m,m}(\tau(t), t_1) \int_t^\infty p_{n-m-1}(s)\Delta s \leq 1.$$

This leads to a contradiction to (3.20). This completes the proof. ☐

The following result is a Fite-Wintner type oscillation criterion.

Theorem 3.4. *Assume that*

$$\int_{t_0}^\infty p(s)\, \Delta s = \infty. \tag{3.23}$$

Then every solution of Eq. (1.1) is oscillatory.

Proof. Assume Eq. (1.1) has a non-oscillatory solution $x(t)$. Then without loss of generality, assume $x(t) > 0$ and $x(g(t)) > 0$ on $[t_0, \infty)_{\mathbb{T}}$. It follows from Lemma 2.1 that there exists an odd integer $m \in \{1, \ldots, n-1\}$ such that (2.1) and (2.2) hold for $t \geq t_1 \in [t_0, \infty)_{\mathbb{T}}$. This implies $x(t)$ is strictly increasing on $[t_1, \infty)_{\mathbb{T}}$. Then for sufficiently large $t_2 \in [t_1, \infty)_{\mathbb{T}}$, we have $x(g(t)) \geq l$ for $t \geq t_2$. Eq. (1.1) becomes

$$-\left(x^{[n-1]}(t)\right)^\Delta = p(t)\, x(g(t)) \geq l\, p(t) \quad \text{for } t \in [t_2, \infty)_{\mathbb{T}}. \tag{3.24}$$

Replacing t by s in (3.24), integrating from t_2 to $t \in [t_2, \infty)_{\mathbb{T}}$, we obtain

$$-x^{[n-1]}(t) + x^{[n-1]}(t_2) \geq l \int_{t_2}^{t} p(s)\,\Delta s$$

Hence by (3.23), we have $\lim_{t\to\infty} x^{[n-1]}(t) = -\infty$, which contradicts the fact that $x^{[n-1]}(t) > 0$ eventually. This completes the proof. □

As a direct consequence of Theorems 3.1–3.4, we obtain oscillation criteria for Eq. (1.1) with $n = 2$, namely, for the equation

$$\left(r_1(t)x^{\Delta}(t)\right)^{\Delta} + p(t)\,x(g(t)) = 0, \ t \in [t_0, \infty)_{\mathbb{T}}. \tag{3.25}$$

Corollary 3.5. *Assume that, for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$,*

$$\liminf_{t\to\infty} R_1(t,T) \int_t^{\infty} p(s)\,\frac{R_1(\tau(s),T)}{R_1(\sigma(s),T)}\Delta s > \frac{1}{4},$$

Then every solution of Eq. (3.25) *is oscillatory.*

Corollary 3.6. *Assume that, for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$,*

$$\liminf_{t\to\infty} \frac{1}{R_1(t,T)} \int_T^{t} p(s)\,R_1(\sigma(s),T)\,R_1(\tau(s),T)\Delta s > \frac{L^*}{1+L^*},$$

where

$$L^* := \limsup R_1(\sigma(s),T)/\,R_1(s,T)\,.$$

Then every solution of Eq. (3.25) *is oscillatory.*

Corollary 3.7. *Assume that, for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$,*

$$\limsup_{t\to\infty} R_1(\tau(t),T) \int_t^{\infty} p(s)\Delta s > 1.$$

Then every solution of Eq. (3.25) *is oscillatory.*

Corollary 3.8. *Assume that* (3.23) *holds. Then every solution of Eq.* (3.25) *is oscillatory.*

Example 3.9. Consider the second order nonlinear dynamic equation (3.25) with $p(t) = \frac{\beta}{r_1(t)}$, where β is a positive constant such that (1.2) holds. We see that

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{R_1(t,T)} \int_T^{t} p(s)\,R_1(\sigma(s),T)\,R_1(\tau(s),T)\Delta s \\
&\geq \beta \liminf_{t\to\infty} \frac{1}{R_1(t,T)} \int_T^{t} \frac{1}{r_1(s)}\Delta s \\
&= \beta \liminf_{t\to\infty} \frac{1}{R_1(t,T)} \int_T^{t} \left[R_1(s,t_0)\right]^{\Delta} \Delta s \\
&= \beta,
\end{aligned}$$

since $\lim_{t\to\infty} \frac{R_1(t,t_0)}{R_1(t,T)} = 1$. Then, by Corollary 3.6, we get that (3.26) is oscillatory if $\beta > \frac{L^*}{1+L^*}$.

Example 3.10. Consider the second order nonlinear dynamic equation

$$\left(\frac{1}{t}x^{\Delta}(t)\right)^{\Delta}+\frac{\eta t}{R_1^2(t,t_0)}x(g(t))=0, \quad g(t)\geq\sigma(t), \tag{3.26}$$

where η is a positive constant. Here $r_1(t)=\frac{1}{t}$, $p(t)=\dfrac{\eta t}{R_1^2(t,t_0)}$. Note that

$$\int_{t_0}^{\infty}\frac{\Delta t}{r_1(t)}=\int_{t_0}^{\infty}t\ \Delta t=\infty,$$

and

$$\begin{aligned}
&\liminf_{t\to\infty}R_1(t,T)\int_t^{\infty}p(s)\frac{R_1(\tau(s),T)}{R_1(\sigma(s),T)}\Delta s\\
&=\eta\liminf_{t\to\infty}R_1(t,T)\int_t^{\infty}\frac{s}{R_1^2(s,t_0)}\Delta s\\
&\geq\eta\liminf_{t\to\infty}R_1(t,T)\int_t^{\infty}\frac{s}{R_1(s,t_0)R_1(\sigma(s),t_0)}\Delta s\\
&=\eta\liminf_{t\to\infty}R_1(t,T)\int_t^{\infty}\left[\frac{-1}{R_1(s,t_0)}\right]^{\Delta}\Delta s\\
&=\eta,
\end{aligned}$$

since $\lim_{t\to\infty}\frac{R_1(t,T)}{R_1(t,t_0)}=1$. Then, by Corollary 3.5, we get that (3.26) is oscillatory if $\eta>\frac{1}{4}$.

For Eq. (1.1) with an even $n\geq 4$, we have further criteria for oscillation as shown below.

Theorem 3.11. *Assume that*

$$\int_{t_0}^{\infty}p_2(t)\Delta t=\infty, \tag{3.27}$$

and

$$\liminf_{t\to\infty}R_{n-1}(t,T)\int_t^{\infty}p(s)\frac{\bar{R}_{n-1,n-1}(\tau(s),T)}{R_{n-1}(\sigma(s),T)}\Delta s>\frac{1}{4}, \tag{3.28}$$

for sufficiently large $T\in[t_0,\infty)_{\mathbb{T}}$. Then every solution of Eq. (1.1) *is oscillatory.*

Theorem 3.12. *Assume that* (3.27) *and*

$$\liminf_{t\to\infty}\frac{1}{R_{n-1}(t,T)}\int_T^t p(s)\bar{R}_{n-1,n-1}(\tau(s),T)R_{n-1}(\sigma(s),T)\,\Delta s>\frac{L_m^*}{1+L_m^*}, \tag{3.29}$$

for sufficiently large $T\in[t_0,\infty)_{\mathbb{T}}$ and where

$$L_m^*:=\limsup R_{n-1}(\sigma(s),T)/\,R_{n-1}(s,T)\,.$$

Then every solution of Eq. (1.1) *is oscillatory.*

Theorem 3.13. *Assume that* (3.27) *and*

$$\limsup_{t\to\infty} \bar{R}_{n-1,n-1}(\tau(t), T) \int_t^\infty p(s)\Delta s > 1, \tag{3.30}$$

for sufficiently large $T \in [t_0,\infty)_{\mathbb{T}}$. *Then every solution of Eq.* (1.1) *is oscillatory.*

Proofs of Theorems 3.11–3.13. Assume Eq. (1.1) has a non-oscillatory solution $x(t)$. Then without loss of generality, assume $x(t) > 0$ and $x(g(t)) > 0$ on $[t_0,\infty)_{\mathbb{T}}$. It follows from Lemma 2.1 that there exists an odd $m \in \{1,\ldots,n-1\}$ such that (2.1) and (2.2) hold for $t \geq t_1 \in [t_0,\infty)_{\mathbb{T}}$. We claim that (3.27) implies that $m = n-1$. In fact, if $1 \leq m \leq n-3$, then for $t \geq t_1$

$$x^{[n]}(t) < 0,\; x^{[n-1]}(t) > 0,\; x^{[n-2]}(t) < 0,\; x^{[n-3]}(t) > 0. \tag{3.31}$$

By Lemma 2.3, Part (a) we have that for $i = n-2$

$$-x^{[n-2]}(t) \geq x(\tau(t)) \int_t^\infty p_1(s)\Delta s \quad \text{for } t \in [t_1,\infty)_{\mathbb{T}}.$$

By using the fact that $x(t)$ is strictly increasing on $[t_1,\infty)_{\mathbb{T}}$. Then for sufficiently large $t_2 \in [t_1,\infty)_{\mathbb{T}}$, we have $x(\tau(t)) \geq l$ for $t \geq t_2$. Then

$$-x^{[n-2]}(t) \geq l \int_t^\infty p_1(s)\Delta s \quad \text{for } t \in [t_2,\infty)_{\mathbb{T}}.$$

It follows that

$$-\left(x^{[n-3]}(t)\right)^\Delta \geq l \frac{1}{r_{n-2}(t)} \int_t^\infty p_1(s)\Delta s = l\, p_2(t).$$

Integrating above inequality from t_2 to $t \in [t_2,\infty)_{\mathbb{T}}$ and noting that $x^{[n-3]} > 0$ eventually, we get

$$x^{[n-3]}(t_2) - x^{[n-3]}(t) \geq l \int_{t_2}^t p_2(s)\Delta s.$$

As a result, $\lim_{t\to\infty} x^{[n-3]}(t) = -\infty$, which contradicts the fact that $x^{[n-3]} > 0$ on $[t_2,\infty)_{\mathbb{T}}$. This shows that if (3.27) holds, then $m = n-1$. The rest of the proof of Theorems 3.11–3.13 is similar to the proof of Theorems 3.1–3.3 with $m = n-1$ respectively and hence can be omitted. □

Remark 3.14. The conclusion of Theorems 3.11–3.13 remains intact if assumption (3.27) is replaced by one of the following conditions

$$\text{either} \quad \int_{t_0}^\infty p_0(t)\Delta t = \infty \quad \text{or} \quad \int_{t_0}^\infty p_1(t)\Delta t = \infty. \tag{3.32}$$

4. CRITERIA FOR ODD ORDER EQUATIONS

In this section, we establish Hille, Nehari, Ohriska and Fite-Wintner type criteria for odd order dynamic equation (1.1). It follows from Lemma 2.1 that there exists an even integer $m \in \{0, \dots, n-1\}$ such that (2.1) and (2.2) hold eventually.

Theorem 4.1. *Assume that* (3.1) *holds, for every even number* $i \in \{2, \dots, n-1\}$ *and*

$$\int_{t_0}^{\infty} p_{n-1}(t)\Delta t = \infty. \tag{4.1}$$

Then every solution of Eq. (1.1) *is either oscillatory or tends to zero monotonically.*

Theorem 4.2. *Assume that* (3.13)*, for every even number* $i \in \{2, \dots, n-1\}$ *and* (4.1) *hold. Then every solution of Eq.* (1.1) *is either oscillatory or tends to zero monotonically.*

Theorem 4.3. *Assume that* (3.20)*, for every even number* $i \in \{2, \dots, n-1\}$ *and* (4.1) *hold. Then every solution of Eq.* (1.1) *is either oscillatory or tends to zero monotonically.*

Proofs of Theorems 4.1–4.3. Assume Eq. (1.1) has a non-oscillatory solution $x(t)$. Then without loss of generality, assume $x(t) > 0$ and $x(g(t)) > 0$ on $[t_0, \infty)_{\mathbb{T}}$. It follows from Lemma 2.1 that there exists an even $m \in \{0, \dots, n-1\}$ such that (2.1) and (2.2) hold for $t \geq t_1 \in [t_0, \infty)_{\mathbb{T}}$.

(I) We show that if $m = 0$, then $\lim_{t\to\infty} x(t) = 0$. In this case

$$(-1)^k x^{[k]} > 0 \quad \text{for} \quad k = 0, 1, \dots, n.$$

This implies that $x(t)$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$. Then $\lim_{t\to\infty} x(t) = l \geq 0$. Assume $l > 0$. Then for sufficiently large $t_2 \in [t_1, \infty)_{\mathbb{T}}$, we have $x(g(t)) \geq l$ for $t \geq t_2$. Then from (1.1), we obtain

$$-\left(x^{[n-1]}(t)\right)^{\Delta} = p(t)\, x(g(t)) \geq l\; p(t) \quad \text{for } t \in [t_2, \infty)_{\mathbb{T}}.$$

Replacing t by s in above inequality and integrating from t to $v \in [t, \infty)_{\mathbb{T}}$, we get

$$-x^{[n-1]}(v) + x^{[n-1]}(t) \geq l \int_t^v p(s)\,\Delta s = l \int_t^v p_0(s)\,\Delta s.$$

and by (2.2) we see that $x^{[n-1]}(v) > 0$. Hence by taking limits as $v \to \infty$ we have

$$x^{[n-1]}(t) \geq l \int_t^{\infty} p_0(s)\,\Delta s,$$

which implies

$$\left(x^{[n-2]}(t)\right)^{\Delta} \geq l\, \frac{1}{r_{n-1}(t)} \int_t^{\infty} p_0(s)\,\Delta s = l\; p_1(t).$$

Integrating from t to $v \in [t,\infty)_{\mathbb{T}}$ and letting $v \to \infty$ and using (2.2), we get

$$-x^{[n-2]}(t) \geq l \int_t^\infty p_1(s)\ \Delta s.$$

Continuing this process, we get

$$-x^{[1]}(t) \geq l\ \int_t^\infty p_{n-2}(s)\ \Delta s,$$

which implies

$$-x^{\Delta}(t) \geq l\ \frac{1}{r_1(t)} \int_t^\infty p_{n-2}(s)\Delta s = l\ p_{n-1}(t).$$

Again integrating the above inequality from t_2 to $t \in [t_2,\infty)_{\mathbb{T}}$, we get

$$-x(t) + x(t_2) \geq l\ \int_{t_2}^t p_{n-1}(s)\ \Delta s$$

Hence by (4.1), we have $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the fact that $x > 0$ eventually. This shows that if $m = 0$, then $\lim_{t\to\infty} x(t) = 0$.

(II) Assume $m \geq 2$. The same argument holds as in the proof of Theorems 3.1–3.3 respectively and hence is omitted. This completes the proof. □

Example 4.4. Consider the third order nonlinear dynamic equation

$$\left(tx^{\Delta\Delta}(t)\right)^{\Delta} + \frac{\beta}{t^2}x(g(t)) = 0, \quad g\,(t) \geq \sigma(t), \tag{4.2}$$

for $t \in [t_0,\infty)_{\mathbb{T}}$, where β is a positive constant. Here

$$n = 3, \quad i = 2, \quad r_2(t) = 1, \quad r_1(t) = t \text{ and } p(t) = \frac{\beta}{t^2}.$$

It is clear that condition (1.2) holds. Note that

$$p_1(t) = \frac{1}{r_2(t)} \int_t^\infty p(s)\Delta s = \int_t^\infty \frac{\beta}{s^2}\Delta s \geq \beta \int_t^\infty \left(\frac{-1}{s}\right)^{\Delta} \Delta s \geq \frac{\beta}{t},$$

and

$$\int_{t_0}^\infty p_1(s)\Delta s \geq \beta \int_{t_0}^\infty \frac{\Delta s}{s} = \infty,$$

To apply Theorem 4.1, it remains to prove that condition (3.1) holds. To see this note that

$$\liminf_{t\to\infty} R_2\,(t,T) \int_t^\infty P_2\,(s,T)\,\Delta s \geq \beta \liminf_{t\to\infty}\,(t-T) \int_t^\infty \frac{\Delta s}{s^2} \geq \beta \liminf_{t\to\infty} \frac{t-T}{t} = \beta.$$

So if $\beta > \frac{1}{4}$, then (3.1) holds and we have by Theorem 4.1 that (4.2) is oscillatory or every solution tends to zero if $\beta > \frac{1}{4}$.

Theorem 4.5. *Assume that* (3.23) *holds. Then every solution of Eq.* (1.1) *is either oscillatory or tends to zero monotonically.*

Proof. Assume Eq. (1.1) has a non-oscillatory solution $x(t)$. Then without loss of generality, assume $x(t) > 0$ and $x(g(t)) > 0$ on $[t_0, \infty)_{\mathbb{T}}$. It follows from Lemma 2.1 that there exists an even $m \in \{0, \ldots, n-1\}$ such that (2.1) and (2.2) hold for $t \geq t_1 \in [t_0, \infty)_{\mathbb{T}}$.

(I) We show that if $m = 0$, then $\lim_{t\to\infty} x(t) = 0$. In this case

$$(-1)^k x^{[k]} > 0 \quad \text{for} \quad k = 0, 1, \ldots, n.$$

This implies that $x(t)$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$. Then $\lim_{t\to\infty} x(t) = l \geq 0$. Assume $l > 0$. Then for sufficiently large $t_2 \in [t_1, \infty)_{\mathbb{T}}$, we have $x(g(t)) \geq l$ for $t \geq t_2$. Then from (1.1), we obtain

$$-\left(x^{[n-1]}(t)\right)^{\Delta} = p(t)\, x(g(t)) \geq l\; p(t) \quad \text{for } t \in [t_2, \infty)_{\mathbb{T}}.$$

Replacing t by s in the above inequality and integrating from t_2 to $t \in [t_2, \infty)_{\mathbb{T}}$, we get

$$-x^{[n-1]}(t) + x^{[n-1]}(t_2) \geq l \int_{t_2}^{t} p(s)\, \Delta s.$$

Hence by (3.23), we have $\lim_{t\to\infty} x^{[n-1]}(t) = -\infty$, which contradicts the fact that $x^{[n-1]}(t) > 0$ eventually. This shows that if $m = 0$, then $\lim_{t\to\infty} x(t) = 0$.

(II) Assume $m \geq 2$. The same argument as in the proof of Theorem 3.4 and hence is omitted. This completes the proof. □

Example 4.6. Consider the higher order nonlinear dynamic equation (1.1) with

$$r_i(t) = t^{1/i}, \quad i = 1, 2, \ldots, n-1 \text{ and } p(t) = \frac{1}{t^{\beta}},$$

where $\beta \in (-\infty, 1]$ is a constant. Using [5, Example 5.60], we have

$$\int_{t_0}^{\infty} \frac{\Delta s}{s^{1/i}} = \int_{t_0}^{\infty} \frac{\Delta s}{s^{\beta}} = \infty, \quad i = 1, 2, \ldots, n-1.$$

Then by Theorems 3.4 and 4.5:

1. if $n \in 2\mathbb{N}$, then every solution of Eq. (1.1) is oscillatory.
2. if $n \in 2\mathbb{N} - 1$, then every solution of Eq. (1.1) is either oscillatory or tends to zero monotonically.

At the end of this paper, we establish parallel results to Theorems 3.11–3.13 under the assumption that (3.27) holds.

Theorem 4.7. *Assume that* (3.27) *and* (3.28) *hold. Then every solution of Eq.* (1.1) *is either oscillatory or tends to zero monotonically.*

Theorem 4.8. *Assume that* (3.27) *and* (3.29) *hold. Then every solution of Eq.* (1.1) *is either oscillatory or tends to zero monotonically.*

Theorem 4.9. *Assume that* (3.27) *and* (3.30) *hold. Then every solution of Eq.* (1.1) *is either oscillatory or tends to zero monotonically.*

Proofs of Theorems 4.7–4.9. Assume Eq. (1.1) has a non-oscillatory solution $x(t)$. Then without loss of generality, assume $x(t) > 0$ and $x(g(t)) > 0$ on $[t_0, \infty)_{\mathbb{T}}$. It follows from Lemma 2.1 that there exists an even $m \in \{0, \ldots, n-1\}$ such that (2.1) and (2.2) hold for $t \geq t_1 \in [t_0, \infty)_{\mathbb{T}}$.

(I) We show that if $m = 0$, then $\lim_{t\to\infty} x(t) = 0$. In this case

$$(-1)^k x^{[k]} > 0 \quad \text{for} \quad k = 0, 1, \ldots, n.$$

This implies that $x(t)$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$. Then $\lim_{t\to\infty} x(t) = l \geq 0$. Assume $l > 0$. Then for sufficiently large $t_2 \in [t_1, \infty)_{\mathbb{T}}$, we have $x(g(t)) \geq l$ for $t \geq t_2$. As in the proof of Theorems 4.1–4.3, we have

$$-x^{[n-2]}(t) \geq l \int_t^\infty p_1(s)\ \Delta s,$$

which implies

$$-\left(x^{[n-3]}(t)\right)^\Delta \geq l \frac{1}{r_{n-2}(t)} \int_t^\infty p_1(s)\Delta s = l\ p_2(t).$$

Integrating above inequality from t_2 to $t \in [t_2, \infty)_{\mathbb{T}}$ and noting that $x^{[n-3]} > 0$ eventually, we get

$$x^{[n-3]}(t_2) - x^{[n-3]}(t) \geq l \int_{t_2}^t p_2(s)\Delta s.$$

As a result, $\lim_{t\to\infty} x^{[n-3]}(t) = -\infty$, which contradicts the fact that $x^{[n-3]} > 0$ on $[t_2, \infty)_{\mathbb{T}}$. This shows that if $m = 0$, then $\lim_{t\to\infty} x(t) = 0$.

(II) Assume $m \geq 2$. The same argument holds as in the proof of Theorems 3.11–3.13, respectively, and hence is omitted. This completes the proof. □

Remark 4.10. **(a)** The conclusion of Theorems 4.1–4.3 remains intact if assumption (4.1) is replaced by one of the following conditions

$$\int_{t_0}^\infty p_3(t)\Delta t = \infty, \quad \int_{t_0}^\infty p_4(t)\Delta t = \infty, \ldots, \int_{t_0}^\infty p_{n-2}(t)\Delta t = \infty.$$

(b) The conclusion of Theorems 4.7–4.9 remains intact if assumption (3.27)is replaced by (3.32).

REFERENCES

[1] R. P. Agarwal, M. Bohner and S. H. Saker, Oscillation of second order delay dynamic equation, *Canadian Appl. Math. Quart.*, **13** (2005) 1–17.

[2] R. P. Agarwal, M. Bohner, T. Li and C. Zhang, Hille and Nehari type criteria for third order delay dynamic equations, *J. Difference Equ. Appl.* **19** 2013. no. 10, 1563–1579.

[3] E. F. Beckenbach, R. Bellman, *Inequalities*, Springer, Berlin, 1961.

[4] M. Bohner and T. S. Hassan, Oscillation and boundedness of solutions to first and second order forced functional dynamic equations with mixed nonlinearities, *Appl. Anal. Discrete Math.* **3** (2009), no. 2, 242–252.

[5] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.

[6] M. Bohner and A. Peterson, editors, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[7] D. Chen, Oscillation and asymptotic behavior of solutions of certain third-order nonlinear delay dynamic equations, *Theoretical Mathematics & Applications* **3** (2013), 19–33.

[8] E. M. Elabbasy and T. S. Hassan, Oscillation criteria for third order functional dynamic equations, *Electron. J. Diff. Equ.* **2010** (2010), no. 131 1–14.

[9] L. Erbe, Oscillation criteria for second order nonlinear delay equations, *Canad. Math. Bull.* **16** (1973), 49–56.

[10] L. Erbe, J. Baoguo and A. Peterson, Oscillation of nth-order superlinear dynamic equations on time scales, *Rocky Mountain J. Math.* **41** (2011), no. 2, 471–491.

[11] L. Erbe and T. S. Hassan, New oscillation criteria for second order sublinear dynamic equations, *Dynam. Systems Appl.* **22** (2013), no. 1, 49–64.

[12] L. Erbe, T. S. Hassan and A. Peterson, Oscillation of third order nonlinear functional dynamic equations on time scales, *Differ. Equ. Dyn. Syst.* **18** (2010), no. 1-2, 199–227.

[13] L. Erbe, T. S. Hassan and A. Peterson, Oscillation criteria for nonlinear damped dynamic equations on time scales, *Appl. Math. Comput.* **203**, (2008), no. 1, 343–357.

[14] L. Erbe, T. S. Hassan and A. Peterson, Oscillation criteria for nonlinear functional neutral dynamic equations on time scales, *J. Differerence Equ. Appl.* **15** (2009), no. 11-12, 1097–1116.

[15] L. Erbe, T S. Hassan and A. Peterson, Oscillation of second order functional dynamic equations, *Int. J. Differerence Equ.* **5**, (2010), no. 2, 175–193.

[16] L. Erbe, T. S. Hassan, A. Peterson and S. H. Saker, Oscillation criteria for half-linear delay dynamic equations on time scales, *Nonlinear Dyn. Sys. Theory* **9** (2009), no. 1, 51–68.

[17] L. Erbe, T. S. Hassan, A. Peterson and S. H. Saker, Oscillation criteria for sublinear half-linear delay dynamic equations on time scales, *Int. J. Difference Equ.* **3** (2008), no. 2, 227-245.

[18] L. Erbe, A. Peterson and S. H. Saker, Asymptotic behavior of solutions of a third-order nonlinear dynamic equation on time scales, *J. Comput. Appl. Math.* **181** (2005), no. 1, 92–102.

[19] L. Erbe, A. Peterson and S. H. Saker, Oscillation criteria for second-order nonlinear delay dynamic equations on time scales, *J. Math. Anal. Appl.* **333** (2007), no. 1, 505–522.

[20] L. Erbe, A. Peterson and S. H. Saker, Hille and Nehari type criteria for third order dynamic equations, *J. Math. Anal. Appl.* **329** (2007), no. 1, 112–131.

[21] L. Erbe, B. Karapuz and A. Peterson, Kamenev-type oscillation criteria for higher order neutral delay dynamic equations, *Int. J. Differnce Equ.* **6** (2011), no. 1, 1–16.

[22] L. Erbe, R. Mert, A. Peterson and A. Zafer, Oscillation of even order nonlinear delay dynamic equations on time scales, *Czechoslovak Math. J.* **63(138)** (2013), no. 1, 265–279.

[23] W. B. Fite, Concerning the zeros of the solutions of certain differential equations, *Trans. Amer. Math. Soc.* **19** (1918), no. 4, 341–352.

[24] M. Gera, J. R. Graef, M. Gregus, On oscillatory and asymptotic properties of solutions of certain nonlinear third order differential equations, *Nonlinear Anal.* **32** (1998), no. 3, 417–425.

[25] S. R. Grace and T. S. Hassan, Oscillation criteria for higher order nonlinear dynamic equations, *Math. Nachr.* to appear.

[26] S. R. Grace, R. P. Agarwal, M. Bohner and D. O'Regan, Oscillation of second order strongly superlinear and strongly sublinear dynamic equations, *Commun. Nonlinear Sci. Numer. Simul.* **14** (2009), no. 8, 3463–3471.

[27] S. R. Grace, R. P. Agarwal and A. Zafer, Oscillation of higher order nonlinear dynamic equations on time scale, *Adv. Difference Equ.* **2012**, 2012:67, 18 pp.

[28] G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities, second ed.*, Cambridge University Press, Cambridge, 1988.

[29] Z. Han, T. Li, S. Sun and M. Zhang, Oscillation behavior of solutions of third-order nonlinear delay dynamic equations on time scales, *Commun. Korean Math. Soc.* **26** (2011), no. 3, 499–513.

[30] T. S. Hassan, Oscillation of third order nonlinear delay dynamic equations on time scales, *Math. Comput. Modelling* **49** (2009), no. 7-8, 1573–1586.

[31] T. S. Hassan, Oscillation criteria for half-linear dynamic equations on time scales, *J. Math. Anal. Appl.* **345** (2008), no. 1, 176–185.

[32] T. S. Hassan, Kamenev-type oscillation criteria for second order nonlinear dynamic equations on time scales, *Appl. Math. Comput.* **217** (2011), no. 12, 5285–5297.

[33] T. S. Hassan, Oscillation criteria for second order nonlinear dynamic equations, *Adv. Difference Equ.* **2012** 2012:171, 13 pp.

[34] T. S. Hassan, L. Erbe and A. Peterson, Oscillation of second order superlinear dynamic equations with damping on time scales, *Comput. Math. Appl.* **59** (2010), no. 1, 550–558.

[35] T. S. Hassan and Q. Kong, Oscillation criteria for second order nonlinear dynamic equations with p-Laplacian and damping, *Acta Math. Sci. Ser. B Engl. Ed.* **33** (2013), no. 4, 975–988.

[36] S. Hilger, Analysis on measure chains — a unified approach to continuous and discrete calculus, *Results Math.* **18** (1990), no. 1-2, 18–56.

[37] E. Hille, Non-oscillation theorems, *Trans. Amer. Math. Soc.* **64** (1948), 234–252.

[38] V. Kac and P. Chueng, *Quantum Calculus*, Universitext, 2002.

[39] B. Karapuz, Unbounded oscillation of higher-order nonlinear delay dynamic equations of neutral type with oscillating coefficients, *Electron J. Qual. Theory Diff. Equ.* **2009** (2009), no. 34, 1–14.

[40] I. T. Kiguradze, On oscillatory solutions of some ordinary differential equations, *Soviet Math. Dokl.* **144** (1962), 33–36.

[41] W. Leighton, The detection of the oscillation of solutions of a second order linear differential equation, *Duke Math. J.*, **17** (1950), 57–62.

[42] Z. Nehari, Oscillation criteria for second-order linear differential equations, *Trans. Amer. Math. Soc.* **85** (1957), 428–445.

[43] J. Ohriska, Oscillation of second order delay and ordinary differential equations, *Czechoslovak Math. J.* **34** (1984), no. 1, 107–112.

[44] S. H. Saker, Oscillation criteria of second-order half-linear dynamic equations on time scales, *J. Comput. Appl. Math.* **177** (2005), no. 2, 375–387.

[45] T. Sun, W. Yu and H. Xi, Oscillatory behavior and comparison for higher order nonlinear dynamic equations on time scales, *J. Appl. Math. Inform.* **30** (2012), no. 1-2, 289–304.

[46] Y. Wang and Z. Xu, Asymptotic properties of solutions of certain third order dynamic equations, *J. Comput. Appl. Math.* **236** (2012), no. 9, 2354–2366.

[47] A. Wintner, On the nonexistence of conjugate points, *Amer. J. Math.* **73** (1951), 368–380.

[48] J. S. W. Wong, Second order oscillation with retarded arguments, in *Ordinary differential equations*, 581–596, Washington, 1971, Academic press, New York and London, 1972.

[49] Z. Yu and Q. Wang, Asymptotic behavior of solutions of third-order nonlinear dynamic equations, *J. Comp. Appl. Math.* **225** (2009), no. 2, 531–540.