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THE SOLUTION MATCHING BY LIAPUNOV THEORY OF BVPS WITH ODD GAPS IN BOUNDARY CONDITIONS FOR nTH ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. We are concerned with the existence and uniqueness of solutions to boundary value problems on an interval $[a, c]$ for the nth order ordinary differential equation $y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$, for $n \geq 3$, by matching solutions on $[a, b]$ with solutions on $[b, c]$ to extend the interval of existence for solutions. In this paper, we consider a general case where the gap in boundary conditions at b is odd. Different from the literature, we use Liapunov theory to deal with the case.

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1. INTRODUCTION

Matching of solutions of boundary value problems is intimately involved with interface problems for which an intermediate boundary point corresponds (BVPs) to a point of interface [1, 24, 28, 31]. For such problems, smooth as possible interfacing is desired. Otherwise, leakage or impulses in transfer rates occur. Most matching results deal with smoothing one possible break in some order derivative. This paper deals with smoothing by solution matching, when gaps in the derivatives at the interface point involve several successive derivatives.

In this paper, we are concerned with the existence and uniqueness of solutions to boundary value problems on an interval $[a, c]$ for the following nth order ordinary differential equation,

$$y^{(n)}(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x)), \quad n \geq 3, \quad x \in [a, c], \tag{1.1}$$

satisfying the three-point boundary conditions,

$$\begin{aligned} &y(a) = y_1,\ y^{(i)}(b) = y_{i+2}, \quad 0 \leq i \leq k_1 - 1, \\ &y^{(i)}(b) = y_{i+1}, \quad k_1 + 1 \leq i \leq k_2 - 1, \\ &y^{(i)}(b) = y_i,\ k_2 + 1 \leq i \leq n - 1, \quad y(c) = y_n, \end{aligned} \tag{1.2}$$

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where $a < b < c$, $y_1, y_2, \ldots, y_n \in \mathbb{R}$, and $k_1, k_2 \in \mathbb{Z}$ such that $0 \le k_1 < k_2 \le n-1$ and $k_2 - k_1$ is odd, which is called an odd gap in this paper.

Also, it is assumed throughout that $f : [a, c] \times \mathbb{R}^n \to \mathbb{R}$ is continuous and that solutions to initial value problems (IVPs) for (1.1) are unique and exist on the entire interval $[a, c]$. Moreover, k_1 and k_2 are fixed.

To apply the solution-matching technique, we consider the following sets of boundary conditions,

$$\begin{aligned}&y(a) = y_1,\ y^{(i)}(b) = y_{i+2},\ 0 \le i \le k_1 - 1,\ y^{(k_1)}(b) = m,\\ &y^{(i)}(b) = y_{i+1},\ k_1 + 1 \le i \le k_2 - 1,\ y^{(i)}(b) = y_i,\ k_2 + 1 \le i \le n - 1,\end{aligned} \tag{1.3}$$

$$\begin{aligned}&y(a) = y_1,\ y^{(i)}(b) = y_{i+2},\ 0 \le i \le k_1 - 1,\ y^{(k_2)}(b) = m,\\ &y^{(i)}(b) = y_{i+1},\ k_1 + 1 \le i \le k_2 - 1,\ y^{(i)}(b) = y_i,\ k_2 + 1 \le i \le n - 1,\end{aligned} \tag{1.4}$$

$$\begin{aligned}&y^{(i)}(b) = y_{i+2},\ 0 \le i \le k_1 - 1,\ y^{(i)}(b) = y_{i+1},\ k_1 + 1 \le i \le k_2 - 1,\\ &y^{(k_1)}(b) = m,\ y^{(i)}(b) = y_i,\ k_2 + 1 \le i \le n - 1,\ y(c) = y_n,\end{aligned} \tag{1.5}$$

$$\begin{aligned}&y^{(i)}(b) = y_{i+2},\ 0 \le i \le k_1 - 1,\ y^{(i)}(b) = y_{i+1},\ k_1 + 1 \le i \le k_2 - 1,\\ &y^{(k_2)}(b) = m,\ y^{(i)}(b) = y_i,\ k_2 + 1 \le i \le n - 1,\ y(c) = y_n,\end{aligned} \tag{1.6}$$

where $m \in \mathbb{R}$. We will match solutions of the BVPs (1.1), (1.3) on $[a, b]$ with solutions of the BVPs (1.1), (1.5) on $[b, c]$, or solutions of (1.1), (1.4) on $[a, b]$ with solutions of (1.1), (1.6) on $[b, c]$, to obtain a desired unique solution of (1.1), (1.2). The condition that $k_2 - k_1$ is odd is key here.

The solution-matching technique was introduced by Bailey *et al.* [2]. They obtained the existence and uniqueness of solutions to two-point BVPs for the second order differential equation $y'' = f(x, y, y')$ by matching solutions of initial value problems. Then, in 1973, Barr and Sherman [3] assumed monotonicity conditions on f and applied the solution-matching technique to third order equations and generalized to equations of arbitrary order. Since then, much work has been done on existence and uniqueness of certain BVPs for third order or higher order differential equations, differential systems or differential equations on time scales by matching solutions. We refer the readers to [4, 6, 7, 8, 9, 12, 14, 15, 16, 17, 18, 19, 20, 23, 25, 26, 27, 29].

Concerning three-point BVPs for nth order differential equations (1.1), (1.2), the special cases of $k_2 = n - 1$ and $k_1 = n - 2$ were discussed in [7, 19]. Recently, Henderson and Liu [16] studied more general cases where $k_2 - k_1$ is only required to be an odd number. In that paper, monotonicity conditions on f are essentially important. Monotonicity conditions on f guarantee that the postulation of the value of the k_1st or k_2nd derivative at b of a solution to (1.1) presupposes a knowledge of the values of all derivatives at b. The parity of the order n of the differential equation also plays a role since the odd or even property of $n - k_1$ will invoke different monotonicity conditions on f. In 2014, Liu [17] studied the even case of BVPs for third order differential equations, that is, $n = 3$, $k_1 = 0$, $k_2 = 2$ such that $k_2 - k_1$ is even.

The Liapunov theory has been used for the existence and uniqueness of solutions of differential equations in the solution matching technique in many works, see [4, 12, 13, 21, 22]. In this paper we will use proper Liapunov functions as control functions to substitute for the monotonicity conditions on f. The Liapunov functions used in this paper is different from the literature since the present paper studies BVPs with odd gaps in the boundary conditions.

In Section 2, we present definition and basic properties of Liapunov functions; In Section 3, we show the fundamental lemmas on the relation between the values of the k_1st order derivative and the k_2nd order derivative at b of two solutions to (1.1) that satisfy the boundary conditions (1.2), respectively, on the interval $[a, b]$ and the interval $[b, c]$; In Section 4, the main results on the existence and uniqueness of solutions to (1.1), (1.2) are given.

2. LIAPUNOV FUNCTIONS

In this paper, we define a Liapunov function as below:

Definition 2.1. A Liapunov function $V(x, u_0, u_1, \ldots, u_{n-1}) : [\alpha, \beta] \times \mathbb{R}^n \to \mathbb{R}$ is continuous in x and locally Lipschitzian with respect to $(u_0, u_1, \ldots, u_{n-1})$ and satisfies:

(a) $V(x, u_0, u_1, \ldots, u_{n-1}) = 0$ if $u_{k_2} = 0$;

(b) $V(x, u_0, u_1, \ldots, u_{n-1}) > 0$ if $u_{k_2} \neq 0$.

Suppose ϕ is a solution of (1.1). Consider the following differential equation:

$$\begin{aligned} w^{(n)}(x) &= F_\phi(x, w(x), w'(x), \ldots, w^{(n-1)}(x)) \\ &:= f(x, \phi(x), \phi'(x), \ldots, \phi^{(n-1)}(x)) \\ &\quad - f(x, \phi(x) - w(x), \phi'(x) - w'(x), \ldots, \phi^{(n-1)}(x) - w^{(n-1)}(x)). \end{aligned} \tag{2.1}$$

Notice that if φ is another solution of (1.1), then, $w := \phi - \varphi$ is a solution of $(2.1)_\phi$.

Suppose ϕ is a solution of (1.1) and w is a solution of $(2.1)_\phi$. Then, we define V'_{F_ϕ}:

$$\begin{aligned} &V'_{F_\phi}(x, w(x), w'(x), \ldots, w^{(n-1)}(x)) \\ &= \overline{\lim_{h \to 0}} \frac{1}{h} \left[V(x + h, w + hw', \ldots, w^{(n-2)} + hw^{(n-1)}, w^{(n-1)} + hF_\phi) \right. \\ &\quad \left. - V(x, w(x), \ldots, w^{(n-1)}(x)) \right] \end{aligned} \tag{2.2}$$

and V':

$$
\begin{aligned}
&V'(x,w(x),w'(x),\ldots,w^{(n-1)}(x)) \\
&\quad = \overline{\lim_{h\to 0}}\frac{1}{h}\left[V(x+h,w(x+h),\ldots,w^{(n-1)}(x+h))\right. \\
&\quad\quad \left. -V(x,w(x),\ldots,w^{(n-1)}(x))\right]
\end{aligned}
\tag{2.3}
$$

As an extension of the case for $n=1$, the proof to the following lemma is similar to that in Yoshizawa[30] on page 4.

Lemma 2.1. Suppose ϕ is a solution of (1.1) and w is a solution of $(2.1)_\phi$ and $V(x,u_0,u_1,\ldots,u_{n-1}):[\alpha,\beta]\times\mathbb{R}^n\to\mathbb{R}$ is a Liapunov function. Then, $V'(x,w(x),\ldots,w^{(n-1)}(x))= V'_{F_\phi}(x,w(x),\ldots,w^{(n-1)}(x))$ and $V(x,w(x),\ldots,w^{(n-1)}(x))$ is nonincreasing (nondecreasing) along the solution w if and only if $V'_{F_\phi}(x,w(x),\ldots,w^{(n-1)}(x))\le 0$ ($V'_{F_\phi}(x,w(x),\ldots,w^{(n-1)}(x))\ge 0$).

3. FUNDAMENTAL LEMMAS

This section presents two fundamental lemmas showing the relation between the values of the k_1st order derivative and the k_2nd order derivative at b of solutions to (1.1) that satisfy the boundary conditions (1.2), respectively, on the interval $[a,b]$ and the interval $[b,c]$.

Lemma 3.1. Assume ϕ, φ are solutions of (1.1) satisfying (1.2) on the interval $[a,b]$. Let $w=\phi-\varphi$. Then, w is a solution to $(2.1)_\phi$ on $[a,b]$ satisfying

$$w(a)=0,\quad w^{(i)}(b)=0,\ i\ne k_1,k_2.$$

Suppose there is a Liapunov function $V(x,u_0,u_1,\ldots,u_{n-1}):[a,b]\times\mathbb{R}^n\to\mathbb{R}$ such that along the solution w, $V'_{F_\phi}(x,w(x),\ldots,w^{(n-1)}(x))\ge 0$ on $[a,b]$. Then, $w^{(k_1)}(b)=0$ if and only if $w^{(k_2)}(b)=0$. Also, $w^{(k_1)}(b)>0$ if and only if $w^{(k_2)}(b)>0$.

Proof. ($\Rightarrow$) The necessity of the equalities.

Suppose $w^{(k_1)}(b)=0$ and $w^{(k_2)}(b)\ne 0$. Without loss of generality, we assume $w^{(k_2)}(b)>0$. Since $w(a)=0$, $w^{(i)}(b)=0$ for $i=0,1,\ldots,k_2-1$, by repeated applications of Rolle's theorem, there is some $r_1\in(a,b)$ such that $w^{(k_2-1)}(r_1)=0$, $w^{(k_2-1)}(x)<0$, for $x\in(r_1,b)$. By the Mean Value Theorem and $w^{(k_2-1)}(b)=0$, there are some $r_1<r_2<r_3<r_4<b$ such that $w^{(k_2)}(r_2)<0$, $w^{(k_2)}(r_3)=0$, $w^{(k_2)}(r_4)>0$, which imply that $V(r_2,w(r_2),\ldots,w^{(n-1)}(r_2))>0$, $V(r_3,w(r_3),\ldots,w^{(n-1)}(r_3))=0$, $V(r_4,w(r_4),\ldots,\ w^{(n-1)}(r_4))>0$, which contradicts the fact that $V(x,w(x),\ldots,w^{(n-1)}(x))$ is nondecreasing on $[a,b]$ by Lemma 2.1. Therefore, $w^{(k_2)}(b)=0$ if $w^{(k_1)}(b)=0$.

($\Leftarrow$) The sufficiency of equalities.

Suppose $w^{(k_2)}(b) = 0$ and $w^{(k_1)}(b) \neq 0$. Without loss of generality, we assume $w^{(k_1)}(b) > 0$.

By $w^{(i)}(b) = 0$, for $0 \leq i \leq n-1$, $i \neq k_1$, $w^{(k_1)}(b) > 0$ and repeated applications of Rolle's Theorem, we have some $s_1 \in [a, b)$ such that $w^{(k_1)}(s_1) = 0$, $w^{(k_1)}(x) > 0$ on $(s_1, b]$, and $(-1)^{k_1-i}w^{(i)}(x) > 0$ on $[s_1, b)$ for $0 \leq i \leq k_1 - 1$, and so by the Mean Value Theorem, there is some $s_2 \in (s_1, b)$ such that $w^{(k_1+1)}(s_2) > 0$.

From $w^{(i)}(b) = 0$, for $k_1+1 \leq i \leq k_2-1$, and the fact that k_2-k_1 is odd, we repeatedly use the Mean Value Theorem to see there is some $s_3 \in [s_2, b)$ such that $w^{(k_2)}(s_3) > 0$. Then, we have $V(s_3, w(s_3), \ldots, w^{(n-1)}(s_3)) > 0$ and $V(b, w(b), \ldots, w^{(n-1)}(b)) = 0$, which contradict the fact that $V(x, w(x), \ldots, w^{(n-1)}(x))$ is nondecreasing on $[a, b]$ by Lemma 2.1. Therefore, $w^{(k_1)}(b) = 0$ if $w^{(k_2)}(b) = 0$.

($\Rightarrow$) The necessity of inequalities.

Suppose $w^{(k_1)}(b) > 0$ and $w^{(k_2)}(b) < 0$.

From $w(a) = 0$, $w^{(i)}(b) = 0$, $0 \leq i \leq k_1 - 1$, and $w^{(k_1)}(b) > 0$, and by repeated applications of Rolle's Theorem, there is some $t_1 \in [a, b)$ such that $(-1)^{k_1-i}w^{(i)}(x) > 0$, for $x \in [t_1, b)$ and $0 \leq i \leq k_1 - 1$, $w^{(k_1)}(t_1) = 0$ and $w^{(k_1)}(x) > 0$, for $x \in (t_1, b]$. Therefore, by the Mean Value Theorem, there is some $t_2 \in (t_1, b)$ such that $w^{(k_1+1)}(t_2) > 0$.

By $w^{(i)}(b) = 0$ for $k_1+1 \leq i \leq k_2-1$ and repeated applications of the Mean Value Theorem, there is some $t_3 \in [t_2, b)$ such that $w^{(k_2)}(t_3) > 0$. Notice $w^{(k_2)}(b) < 0$. Therefore, $w^{(k_2)}(t_4) = 0$ for some $t_4 \in (t_3, b)$. Then, we have the signs of Liapunov function at t_3, t_4, b: $V(t_3, w(t_3), \ldots, w^{(n-1)}(t_3)) > 0$, $V(t_4, w(t_4), \ldots, w^{(n-1)}(t_4)) = 0$ and $V(b, w(b), \ldots, w^{(n-1)}(b)) > 0$, which is against the assumption that $V(x, w(x), \ldots, w^{(n-1)}(x))$ is nondecreasing.

($\Leftarrow$) The sufficiency of inequalities.

We assume that $w^{(k_1)}(b) < 0$ and $w^{(k_2)}(b) > 0$. Then, we are in the similar situation to the proof for necessity of inequalities, which also leads to a contradiction. Hence, the sufficiency is true. $\square$

Lemma 3.2. Assume ϕ, φ are solutions of (1.1) satisfying (1.2) on the interval $[b, c]$. Let $w = \phi - \varphi$. Then, w is a solution to $(2.1)_\phi$ on $[b, c]$ satisfying

$$w^{(i)}(b) = 0, \quad i \neq k_1, k_2, \quad w(c) = 0.$$

Suppose there is a Liapunov function $U(x, u_0, u_1, \ldots, u_{n-1}) : [b, c] \times \mathbb{R}^n \to \mathbb{R}$ such that along the solution w, $U'_{F_\phi}(x, w(x), \ldots, w^{(n-1)}(x)) \leq 0$ on $[b, c]$. Then, $w^{(k_1)}(b) = 0$ if and only if $w^{(k_2)}(b) = 0$. Also, $w^{(k_1)}(b) > 0$ if and only if $w^{(k_2)}(b) < 0$.

Proof. ($\Rightarrow$) The necessity of the equalities.

Suppose $w^{(k_1)}(b) = 0$ and $w^{(k_2)}(b) \neq 0$. Without loss of generality, we assume $w^{(k_2)}(b) > 0$. Since $w^{(i)}(b) = 0$ for $i = 0, 1, \dots, k_2 - 1$, $w(c) = 0$ by repeated applications of Rolle's theorem, there is some $r_1 \in (a, b)$ such that $w^{(k_2-1)}(r_1) = 0$, $w^{(k_2-1)}(x) > 0$, for $x \in (b, r_1)$. By the Mean Value Theorem, there are some $b < r_2 < r_3 < r_4 < r_1$ such that $w^{(k_2)}(r_2) > 0$, $w^{(k_2)}(r_3) = 0$, $w^{(k_2)}(r_4) < 0$, which imply that $U(r_2, w(r_2), \dots, w^{(n-1)}(r_2)) > 0$, $U(r_3, w(r_3), \dots, w^{(n-1)}(r_3)) = 0$, $U(r_4, w(r_4), \dots, w^{(n-1)}(r_4)) > 0$, which contradicts the fact that $U(x, w(x), \dots, w^{(n-1)}(x))$ is nonincreasing on $[b, c]$ by Lemma 2.1. Therefore, $w^{(k_2)}(b) = 0$ if $w^{(k_1)}(b) = 0$.

($\Leftarrow$) The sufficiency of equalities.

Suppose $w^{(k_2)}(b) = 0$ and $w^{(k_1)}(b) \neq 0$. Without loss of generality, we assume $w^{(k_1)}(b) > 0$.

By $w^{(i)}(b) = 0$, for $0 \leq i \leq n-1$, $i \neq k_1$, $w^{(k_1)}(b) > 0$ and repeated applications of Rolle's Theorem, we have some $s_1 \in [b, c)$ such that $w^{(k_1)}(s_1) = 0$, $w^{(k_1)}(x) > 0$ on $[b, s_1)$, and $w^{(i)}(x) > 0$ on $(b, s_1]$ for $0 \leq i \leq k_1 - 1$, and so by the Mean Value Theorem, there is some $s_2 \in (b, s_1)$ such that $w^{(k_1+1)}(s_2) < 0$.

From $w^{(i)}(b) = 0$, for $k_1 + 1 \leq i \leq k_2 - 1$, we repeatedly use the Mean Value Theorem to see there is some $s_3 \in (b, s_2]$ such that $w^{(k_2)}(s_3) < 0$. Then, we have $U(b, w(b), \dots, w^{(n-1)}(b)) = 0$ and $U(s_3, w(s_3), \dots, w^{(n-1)}(s_3)) > 0$, which contradicts the assumption that $U(x, w(x), \dots, w^{(n-1)}(x))$ is nonincreasing on $[b, c]$ by Lemma 2.1. Therefore, $w^{(k_1)}(b) = 0$ if $w^{(k_2)}(b) = 0$.

($\Rightarrow$) The necessity of inequalities.

Suppose $w^{(k_1)}(b) > 0$ and $w^{(k_2)}(b) > 0$. From $w^{(i)}(b) = 0$, $0 \leq i \leq k_1 - 1$, $w^{(k_1)}(b) > 0$, and $w(c) = 0$, by repeated applications of Rolle's Theorem, there is some $t_1 \in (b, c]$ such that $w^{(i)}(x) > 0$, for $x \in (b, t_1]$ and $0 \leq i \leq k_1 - 1$, $w^{(k_1)}(t_1) = 0$ and $w^{(k_1)}(x) > 0$, for $x \in [b, t_1)$. Therefore, by the Mean Value Theorem, there is some $t_2 \in (b, t_1)$ such that $w^{(k_1+1)}(t_2) < 0$.

By $w^{(i)}(b) = 0$ for $k_1 + 1 \leq i \leq k_2 - 1$ and repeated applications of Mean Value Theorem, there is some $t_3 \in (b, t_2]$ such that $w^{(k_2)}(t_3) < 0$. Notice $w^{(k_2)}(b) > 0$. Therefore, $w^{(k_2)}(t_4) = 0$ for some $t_4 \in (b, t_3)$. Then, we have the signs of Liapunov function at t_3, t_4, b: $U(b, w(b), \dots, w^{(n-1)}(b)) > 0$, $U(t_4, w(t_4), \dots, w^{(n-1)}(t_4)) = 0$ and $U(t_3, w(t_3), \dots, w^{(n-1)}(t_3)) > 0$, which is against the assumption that $U(x, w(x), \dots, w^{(n-1)}(x))$ is nonincreasing.

Therefore, $w^{(k_2)}(b) > 0$ if $w^{(k_1)}(b) > 0$

($\Leftarrow$) The sufficiency of inequalities.

Assume the conclusion is not true, that is, $w^{(k_2)}(b) < 0$ and $w^{(k_1)}(b) < 0$. Then, by a very similar proof to that for necessity of inequalities, we can arrive at a contradiction. Hence, the sufficiency is also true. $\square$

3. UNIQUENESS AND EXISTENCE OF SOLUTIONS FOR (1.1), (1.2)

This section presents the uniqueness and existence of solutions for (1.1), (1.2). First, we show the uniqueness of solutions to each of the BVPs for (1.1) satisfying any of (1.3), (1.4), (1.5), or (1.6), respectively.

Lemma 4.1. Suppose for any solution ϕ of (1.1), (1.2) and any solution w of $(2.1)_\phi$ on $[a,b]$ satisfying $w(a)=0$, $w^{(i)}(b)=0$, for $i \neq k_1, k_2$, there exists a Liapunov function $V(x,u_0,u_1,\ldots,u_{n-1}) : [a,b]\times\mathbb{R}^n \to \mathbb{R}$ such that along the solution w, $V'_{F_\phi}(x,w(x),\ldots,w^{(n-1)}(x)) \geq 0$ on $[a,b]$. Then, for every $m \in \mathbb{R}$, each of the BVPs for (1.1) satisfying one of (1.3) and (1.4) has at most one solution.

Proof. We show the proof for (1.1), (1.3). The proof for (1.1), (1.4) is similar and will be omitted.

Suppose for some $m \in \mathbb{R}$, there are two solutions ϕ, φ satisfying (1.1), (1.3). Let $w := \phi-\varphi$. Then, w is a solution of $(2.1)_\phi$ on $[a,b]$ satisfying $w(a)=0$, $w^{(i)}(b)=0$, for $i \neq k_2$. Then by $w^{(k_1)}(b)=0$ and Lemma 3.1, we have $w^{(k_2)}(b)=0$. The uniqueness of solutions of initial value problems for (1.1) implies that $\phi\equiv\varphi$ on $[a,b]$. □

Lemma 4.2. Suppose for any solution ϕ of (1.1), (1.2) and any solution w of $(2.1)_\phi$ on $[b,c]$ satisfying $w^{(i)}(b)=0$, for $i\neq k_1,k_2$, $w(c)=0$, there exists a Liapunov function $U(x,u_0,u_1,\ldots,u_{n-1}) : [b,c]\times\mathbb{R}^n \to \mathbb{R}$ such that along the solution w, $U'_{F_\phi}(x,w(x),\ldots,w^{(n-1)}(x)) \leq 0$ on $[b,c]$. Then, for every $m\in\mathbb{R}$, each of the BVPs for (1.1) satisfying one of (1.5) and (1.6) has at most one solution.

Proof. The proof is similar to Lemma 4.1. □

Lemma 4.3. With the same assumptions as Lemma 4.1 and Lemma 4.2, we have that the BVP (1.1), (1.2) has at most one solution.

Proof. Suppose the BVP (1.1), (1.2) has two solutions ϕ, φ. Let $w := \phi-\varphi$. Then, w is a solution of $(2.1)_\phi$ on $[a,b]$ satisfying $w(a)=0$, $w^{(i)}(b)=0$, for $i\neq k_1,k_2$, $w(c)=0$. Then, by Lemma 3.1 or Lemma 3.2, we have that $w^{(k_1)}(b)\neq 0$ and $w^{(k_2)}(b)\neq 0$. Suppose $w^{(k_1)}(b)>0$. By Lemma 3.1, $w^{(k_2)}(b)>0$; however, by Lemma 3.2, $w^{(k_2)}(b)<0$. The contradiction and the uniqueness of solutions of BVP (1.1), (1.2) imply that $\phi\equiv\varphi$ on $[a,c]$. □

For notation purposes, given any $m\in\mathbb{R}$, let $\alpha(x,m)$, $u(x,m)$, $\beta(x,m)$, $v(x,m)$ denote the solutions, when they exist, of the BVPs of (1.1) satisfying (1.3), (1.4), (1.5), or (1.6), respectively. Next, we show that $\alpha^{(k_2)}(b,m)$, $u^{(k_1)}(b,m)$, $\beta^{(k_2)}(b,m)$, $v^{(k_1)}(b,m)$, respectively, are strictly monotone functions of m.

Lemma 4.4. We assume the same assumptions as in Lemma 4.1 and Lemma 4.2, and for each $m\in\mathbb{R}$, there exist solutions of (1.1) satisfying each of the conditions (1.3), (1.4), (1.5), (1.6), respectively. Then, $\alpha^{(k_2)}(b,m)$ and $\beta^{(k_2)}(b,m)$ are, respectively, strictly increasing and strictly decreasing functions of m with ranges all of $\mathbb{R}$.

Proof. The monotonicity of $\alpha^{(k_2)}(b,m)$ and $\beta^{(k_2)}(b,m)$ is from Lemma 3.1 and Lemma 3.2.

The proof that $\{\alpha^{(k_2)}(b,m)|m\in\mathbb{R}\}=\mathbb{R}$ is the same as that in [7, Theorem 2.4]. □

The next lemma states the monotonicity of $u^{(k_1)}(b,m)$ and $v^{(k_1)}(b,m)$ with respect to m. Its proof follows much along the lines of Lemma 4.4, and so is omitted.

Lemma 4.5. Assume the same hypotheses of Lemma 4.4. Then, $u^{(k_1)}(b,m)$ and $v^{(k_1)}(b,m)$ are, respectively, strictly increasing and strictly decreasing functions of m with ranges all of $\mathbb{R}$.

Now, we establish our existence and uniqueness result for (1.1), (1.2), which is obtained by matching solutions.

Theorem 4.6. *Assume the same hypotheses of Lemma* 4.4. *Then,* (1.1)*,* (1.2) *has a unique solution.*

Proof. We can get the conclusion by either Lemma 4.4 or Lemma 4.5. Here we use Lemma 4.5.

By Lemma 4.5, $u^{(k_1)}(b,m)$ and $v^{(k_1)}(b,m)$ are, respectively, strictly increasing and strictly decreasing functions of m with ranges all of $\mathbb{R}$. Then there is a unique $m_0\in\mathbb{R}$ such that $u^{(k_1)}(b,m_0)=v^{(k_1)}(b,m_0)$. Then, we define

$$y(x)=\begin{cases} u(x,m_0), & a\le x\le b,\\ v(x,m_0), & b\le x\le c,\end{cases}$$

which is the unique solution of (1.1), (1.2) by Lemma 4.3. □

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